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On Cosets in Segre's Product of Partial Linear Spaces

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Summary. This paper is a continuation of [12]. We prove that the family of cosets in the Segre's product of partial linear spaces remains invariant under automorphisms.

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The terminology and notation used in this paper are introduced in the following articles: [13], [20], [1], [3], [4], [7], [6], [2], [18], [12], [15], [11], [14], [5], [10], [21], [16], [19], [17], [9], and [8].

1. Preliminaries on Finite Sequences

Let D be a set, let p be a finite sequence of elements of D, and let i, j be natural numbers. The functor $\mathrm{Del}(p,i,j)$ yields a finite sequence of elements of D and is defined by:

(Def. 1) $\mathrm{Del}(p,i,j) = (p\restriction(i -' 1)) ^\frown (p_{\restriction j})$.

We now state several propositions:

(1) For every set D and for every finite sequence p of elements of D and for all natural numbers i, j holds $\mathrm{rng}\,\mathrm{Del}(p,i,j) \subseteq \mathrm{rng}\,p$.

(2) Let D be a set, p be a finite sequence of elements of D, and i, j be natural numbers. If $i \in \mathrm{dom}\,p$ and $j \in \mathrm{dom}\,p$, then $\mathrm{len}\,\mathrm{Del}(p,i,j) = ((\mathrm{len}\,p - j) + i) - 1$.

(3) Let D be a set, p be a finite sequence of elements of D, and i, j be natural numbers. If $i \in \mathrm{dom}\,p$ and $j \in \mathrm{dom}\,p$, then if $\mathrm{len}\,\mathrm{Del}(p,i,j) = 0$, then $i = 1$ and $j = \mathrm{len}\,p$.

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(4) Let D be a set, p be a finite sequence of elements of D, and i, j, k be natural numbers. If $i \in \operatorname{dom} p$ and $1 \leqslant k$ and $k \leqslant i - 1$, then $(\mathrm{Del}(p, i, j))(k) = p(k)$.

(5) For all finite sequences p, q and for every natural number k such that $\operatorname{len} p + 1 \leqslant k$ holds $(p \frown q)(k) = q(k - \operatorname{len} p)$.

(6) Let D be a set, p be a finite sequence of elements of D, and i, j, k be natural numbers. Suppose $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $i \leqslant j$ and $i \leqslant k$ and $k \leqslant ((\operatorname{len} p - j) + i) - 1$. Then $(\mathrm{Del}(p, i, j))(k) = p((j -' i) + k + 1)$.

The scheme *FinSeqOneToOne* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, a finite sequence $\mathcal{D}$ of elements of $\mathcal{C}$, and a binary predicate $\mathcal{P}$, and states that:

> There exists an one-to-one finite sequence g of elements of $\mathcal{C}$ such that $\mathcal{A} = g(1)$ and $\mathcal{B} = g(\operatorname{len} g)$ and $\operatorname{rng} g \subseteq \operatorname{rng} \mathcal{D}$ and for every natural number j such that $1 \leqslant j$ and $j < \operatorname{len} g$ holds $\mathcal{P}[g(j), g(j+1)]$

provided the following requirements are met:

- $\mathcal{A} = \mathcal{D}(1)$ and $\mathcal{B} = \mathcal{D}(\operatorname{len} \mathcal{D})$, and
- For every natural number i and for all sets d_1, d_2 such that $1 \leqslant i$ and $i < \operatorname{len} \mathcal{D}$ and $d_1 = \mathcal{D}(i)$ and $d_2 = \mathcal{D}(i + 1)$ holds $\mathcal{P}[d_1, d_2]$.

2. Segre Cosets

Next we state the proposition

(7) Let I be a non empty set, A be a 1-sorted yielding many sorted set indexed by I, L be a many sorted subset indexed by the support of A, i be an element of I, and S be a subset of the carrier of $A(i)$. Then $L{+\cdot}(i, S)$ is a many sorted subset indexed by the support of A.

Let I be a non empty set and let A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I. A subset of Segre_Product A is called a Segre-Coset of A if it satisfies the condition (Def. 2).

(Def. 2) There exists a Segre-like non trivial-yielding many sorted subset L indexed by the support of A such that it $= \prod L$ and $L(\mathrm{index}(L)) = \Omega_{A(\mathrm{index}(L))}$.

The following proposition is true

(8) Let I be a non empty set, A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I, and B_1, B_2 be Segre-Cosets of A. If $2 \subseteq \overline{\overline{B_1 \cap B_2}}$, then $B_1 = B_2$.

Let S be a topological structure and let X, Y be subsets of the carrier of S. We say that X and Y are joinable if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exists a finite sequence f of elements of $2^{\text{the carrier of } S}$ such that

(i) $X = f(1)$,

(ii) $Y = f(\operatorname{len} f)$,

(iii) for every subset W of the carrier of S such that $W \in \operatorname{rng} f$ holds W is closed under lines and strong, and

(iv) for every natural number i such that $1 \leqslant i$ and $i < \operatorname{len} f$ holds $2 \subseteq \overline{\overline{f(i) \cap f(i+1)}}$.

One can prove the following three propositions:

(9) Let S be a topological structure and X, Y be subsets of the carrier of S. Suppose X and Y are joinable. Then there exists an one-to-one finite sequence f of elements of $2^{\text{the carrier of } S}$ such that

(i) $X = f(1)$,

(ii) $Y = f(\operatorname{len} f)$,

(iii) for every subset W of the carrier of S such that $W \in \operatorname{rng} f$ holds W is closed under lines and strong, and

(iv) for every natural number i such that $1 \leqslant i$ and $i < \operatorname{len} f$ holds $2 \subseteq \overline{\overline{f(i) \cap f(i+1)}}$.

(10) Let S be a topological structure and X be a subset of the carrier of S. If X is closed under lines and strong, then X and X are joinable.

(11) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, and X, Y be subsets of the carrier of Segre_Product A. Suppose that

(i) X is non trivial, closed under lines, and strong,

(ii) Y is non trivial, closed under lines, and strong, and

(iii) X and Y are joinable.

Let X_1, Y_1 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. Suppose $X = \prod X_1$ and $Y = \prod Y_1$. Then $\operatorname{index}(X_1) = \operatorname{index}(Y_1)$ and for every set i such that $i \neq \operatorname{index}(X_1)$ holds $X_1(i) = Y_1(i)$.

3. Collineations of Segre Product

One can prove the following proposition

(12) Let S be a 1-sorted structure, T be a non empty 1-sorted structure, and f be a map from S into T. If f is bijective, then f^{-1} is bijective.

Let S, T be topological structures and let f be a map from S into T. We say that f is isomorphic if and only if:

(Def. 4) f is bijective and open and f^{-1} is bijective and open.

Let S be a non empty topological structure. Observe that there exists a map from S into S which is isomorphic.

Let S be a non empty topological structure. A collineation of S is an isomorphic map from S into S.

Let S be a non empty non void topological structure, let f be a collineation of S, and let l be a block of S. Then $f^\circ l$ is a block of S.

Let S be a non empty non void topological structure, let f be a collineation of S, and let l be a block of S. Then $f^{-1}(l)$ is a block of S.

Next we state a number of propositions:

(13) For every non empty topological structure S and for every collineation f of S holds f^{-1} is a collineation of S.

(14) Let S be a non empty topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is non trivial, then $f^\circ X$ is non trivial.

(15) Let S be a non empty topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is non trivial, then $f^{-1}(X)$ is non trivial.

(16) Let S be a non empty non void topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is strong, then $f^\circ X$ is strong.

(17) Let S be a non empty non void topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is strong, then $f^{-1}(X)$ is strong.

(18) Let S be a non empty non void topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is closed under lines, then $f^\circ X$ is closed under lines.

(19) Let S be a non empty non void topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is closed under lines, then $f^{-1}(X)$ is closed under lines.

(20) Let S be a non empty non void topological structure, f be a collineation of S, and X, Y be subsets of the carrier of S. Suppose X is non trivial and Y is non trivial and X and Y are joinable. Then $f^\circ X$ and $f^\circ Y$ are joinable.

(21) Let S be a non empty non void topological structure, f be a collineation of S, and X, Y be subsets of the carrier of S. Suppose X is non trivial and Y is non trivial and X and Y are joinable. Then $f^{-1}(X)$ and $f^{-1}(Y)$ are joinable.

(22) Let I be a non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let W be a subset of the carrier of Segre_Product A. Suppose W is non trivial, strong, and closed under lines. Then $\bigcup\{Y; Y$ ranges over subsets of the carrier of Segre_Product A : Y is non trivial, strong, and

closed under lines $\wedge$ W and Y are joinable$\}$ is a Segre-Coset of A.

(23) Let I be a non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let B be a set. Then B is a Segre-Coset of A if and only if there exists a subset W of the carrier of Segre_Product A such that W is non trivial, strong, and closed under lines and $B = \bigcup\{Y; Y$ ranges over subsets of the carrier of Segre_Product A : Y is non trivial, strong, and closed under lines $\wedge$ W and Y are joinable$\}$.

(24) Let I be a non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let B be a Segre-Coset of A and f be a collineation of Segre_Product A. Then $f^{\circ}B$ is a Segre-Coset of A.

(25) Let I be a non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let B be a Segre-Coset of A and f be a collineation of Segre_Product A. Then $f^{-1}(B)$ is a Segre-Coset of A.

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