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Compactness of Lim-inf Topology

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Summary. Formalization of [10], chapter III, section 3 (3.4–3.6).

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The papers [15], [9], [1], [18], [21], [14], [22], [17], [12], [8], [20], [6], [16], [3], [4], [13], [7], [2], [11], [23], [19], and [5] provide the notation and terminology for this paper.

Let L be a non empty poset, let X be a non empty subset of L, and let F be a filter of $2^X_{\subseteq}$. The functor $\liminf F$ yielding an element of L is defined by:

(Def. 1) $\liminf F = \bigsqcup_L\{\inf B; B$ ranges over subsets of L: $B \in F\}$.

One can prove the following proposition

(1) Let L_1, L_2 be complete lattices. Suppose the relational structure of $L_1 =$ the relational structure of L_2. Let X_1 be a non empty subset of L_1, X_2 be a non empty subset of L_2, F_1 be a filter of $2^{X_1}_{\subseteq}$, and F_2 be a filter of $2^{X_2}_{\subseteq}$. If $F_1 = F_2$, then $\liminf F_1 = \liminf F_2$.

Let L be a non empty FR-structure. We say that L is lim-inf if and only if:

(Def. 2) The topology of $L = \xi(L)$.

Let us note that every non empty FR-structure which is lim-inf is also topological space-like.

One can check that every top-lattice which is trivial is also lim-inf.

One can check that there exists a top-lattice which is lim-inf, continuous, and complete.

We now state several propositions:

(2) Let L_1, L_2 be non empty 1-sorted structures. Suppose the carrier of $L_1 =$ the carrier of L_2. Let N_1 be a net structure over L_1. Then there exists a strict net structure N_2 over L_2 such that

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(i) the relational structure of N_1 = the relational structure of N_2, and
(ii) the mapping of N_1 = the mapping of N_2.

(3) Let L_1, L_2 be non empty 1-sorted structures. Suppose the carrier of L_1 = the carrier of L_2. Let N_1 be a net structure over L_1. Suppose $N_1 \in \mathrm{NetUniv}(L_1)$. Then there exists a strict net N_2 in L_2 such that
(i) $N_2 \in \mathrm{NetUniv}(L_2)$,
(ii) the relational structure of N_1 = the relational structure of N_2, and
(iii) the mapping of N_1 = the mapping of N_2.

(4) Let L_1, L_2 be inf-complete up-complete semilattices. Suppose the relational structure of L_1 = the relational structure of L_2. Let N_1 be a net in L_1 and N_2 be a net in L_2. Suppose that
(i) the relational structure of N_1 = the relational structure of N_2, and
(ii) the mapping of N_1 = the mapping of N_2.
Then $\liminf N_1 = \liminf N_2$.

(5) Let L_1, L_2 be non empty 1-sorted structures. Suppose the carrier of L_1 = the carrier of L_2. Let N_1 be a net in L_1 and N_2 be a net in L_2. Suppose that
(i) the relational structure of N_1 = the relational structure of N_2, and
(ii) the mapping of N_1 = the mapping of N_2.
Let S_1 be a subnet of N_1. Then there exists a strict subnet S_2 of N_2 such that
(iii) the relational structure of S_1 = the relational structure of S_2, and
(iv) the mapping of S_1 = the mapping of S_2.

(6) Let L_1, L_2 be inf-complete up-complete semilattices. Suppose the relational structure of L_1 = the relational structure of L_2. Let N_1 be a net structure over L_1 and a be a set. Suppose $\langle N_1, a\rangle \in$ the lim inf convergence of L_1. Then there exists a strict net N_2 in L_2 such that
(i) $\langle N_2, a\rangle \in$ the lim inf convergence of L_2,
(ii) the relational structure of N_1 = the relational structure of N_2, and
(iii) the mapping of N_1 = the mapping of N_2.

(7) Let L_1, L_2 be non empty 1-sorted structures, N_1 be a non empty net structure over L_1, and N_2 be a non empty net structure over L_2. Suppose that
(i) the relational structure of N_1 = the relational structure of N_2, and
(ii) the mapping of N_1 = the mapping of N_2.
Let X be a set. If N_1 is eventually in X, then N_2 is eventually in X.

(8) Let L_1, L_2 be inf-complete up-complete semilattices. Suppose the relational structure of L_1 = the relational structure of L_2. Then ConvergenceSpace(the lim inf convergence of L_1) = ConvergenceSpace(the lim inf convergence of L_2).

(9) Let L_1, L_2 be inf-complete up-complete semilattices. Suppose the relational structure of L_1 = the relational structure of L_2. Then $\xi(L_1) = \xi(L_2)$.

Let R be an inf-complete non empty reflexive relational structure. Note that every topological augmentation of R is inf-complete.

Let R be a semilattice. One can verify that every topological augmentation of R has g.l.b.'s.

Let L be an inf-complete up-complete semilattice. One can check that there exists a topological augmentation of L which is strict and lim-inf.

The following proposition is true

(10) Let L be an inf-complete up-complete semilattice and X be a lim-inf topological augmentation of L. Then $\xi(L)$ = the topology of X.

Let L be an inf-complete up-complete semilattice. The functor $\Xi(L)$ yielding a strict topological augmentation of L is defined by:

(Def. 3) $\Xi(L)$ is lim-inf.

Let L be an inf-complete up-complete semilattice. One can check that $\Xi(L)$ is lim-inf.

Next we state a number of propositions:

(11) For every complete lattice L and for every net N in L holds $\liminf N = \bigsqcup_L \{\inf(N{\restriction}i) : i$ ranges over elements of $N\}$.

(12) Let L be a complete lattice, F be a proper filter of $2^{\Omega_L}_{\subseteq}$, and f be a subset of L. Suppose $f \in F$. Let i be an element of the net of F. If $i_{\mathbf{2}} = f$, then $\inf f = \inf(($the net of $F){\restriction}i)$.

(13) For every complete lattice L and for every proper filter F of $2^{\Omega_L}_{\subseteq}$ holds $\liminf F = \liminf$ (the net of F).

(14) For every complete lattice L and for every proper filter F of $2^{\Omega_L}_{\subseteq}$ holds the net of $F \in \mathrm{NetUniv}(L)$.

(15) Let L be a complete lattice, F be an ultra filter of $2^{\Omega_L}_{\subseteq}$, and p be a greater or equal to id map from the net of F into the net of F. Then $\liminf F \geqslant \inf(($the net of $F) \cdot p)$.

(16) Let L be a complete lattice, F be an ultra filter of $2^{\Omega_L}_{\subseteq}$, and M be a subnet of the net of F. Then $\liminf F = \liminf M$.

(17) Let L be a non empty 1-sorted structure, N be a net in L, and A be a set. Suppose N is often in A. Then there exists a strict subnet N' of N such that rng (the mapping of N') $\subseteq A$ and N' is a structure of a subnet of N.

(18) Let L be a complete lim-inf top-lattice and A be a non empty subset of L. Then A is closed if and only if for every ultra filter F of $2^{\Omega_L}_{\subseteq}$ such that $A \in F$ holds $\liminf F \in A$.

(19) For every non empty reflexive relational structure L holds $\sigma(L) \subseteq \xi(L)$.

(20) Let T_1, T_2 be non empty topological spaces and B be a prebasis of T_1. Suppose $B \subseteq$ the topology of T_2 and the carrier of $T_1 \in$ the topology of T_2. Then the topology of $T_1 \subseteq$ the topology of T_2.

(21) For every complete lattice L holds $\omega(L) \subseteq \xi(L)$.

(22) Let T_1, T_2 be topological spaces and T be a non empty topological space. Suppose T is a topological extension of T_1 and a topological extension of T_2. Let R be a refinement of T_1 and T_2. Then T is a topological extension of R.

(23) Let T_1 be a topological space, T_2 be a topological extension of T_1, and A be a subset of T_1. Then

(i) if A is open, then A is an open subset of T_2, and

(ii) if A is closed, then A is a closed subset of T_2.

(24) For every complete lattice L holds $\lambda(L) \subseteq \xi(L)$.

(25) Let L be a complete lattice, T be a lim-inf topological augmentation of L, and S be a Lawson correct topological augmentation of L. Then T is a topological extension of S.

(26) For every complete lim-inf top-lattice L and for every ultra filter F of $2_{\subseteq}^{\Omega_L}$ holds $\liminf F$ is a convergence point of F, L.

(27) Every complete lim-inf top-lattice is compact and T_1.

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