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On Ordering of Bags

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Summary. We present a Mizar formalization of chapter 4.4 of [6] devoted to special orderings in additive monoids to be used for ordering terms in multivariate polynomials. We have extended the treatment to the case of infinite number of variables. It turns out that in such case admissible orderings are not necessarily well orderings.

MML Identifier: `BAGORDER`.

The notation and terminology used here are introduced in the following papers: [21], [33], [18], [26], [7], [5], [11], [29], [8], [9], [1], [22], [30], [31], [2], [28], [24], [25], [20], [14], [34], [36], [35], [17], [16], [32], [27], [19], [4], [12], [23], [3], [15], [13], and [10].

1. Preliminaries

The following propositions are true:

(1) For all sets x, y, z such that $z \in x$ and $z \in y$ holds $x \setminus \{z\} = y \setminus \{z\}$ iff $x = y$.

(2) For all natural numbers n, k holds $k \in \mathrm{Seg}\, n$ iff $k-1$ is a natural number and $k - 1 < n$.

Let f be a natural-yielding function and let X be a set. One can verify that $f{\restriction}X$ is natural-yielding.

Let f be a finite-support function and let X be a set. One can check that $f{\restriction}X$ is finite-support.

Next we state three propositions:

(3) For every function f and for every set x such that $x \in \mathrm{dom}\, f$ holds $f \cdot \langle x \rangle = \langle f(x) \rangle$.

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(4) Let f, g, h be functions. Suppose $\mathrm{dom}\, f = \mathrm{dom}\, g$ and $\mathrm{rng}\, f \subseteq \mathrm{dom}\, h$ and $\mathrm{rng}\, g \subseteq \mathrm{dom}\, h$ and f and g are fiberwise equipotent. Then $h \cdot f$ and $h \cdot g$ are fiberwise equipotent.

(5) For every finite sequence f_1 of elements of $\mathbb{N}$ holds $\sum f_1 = 0$ iff $f_1 = \mathrm{len}\, f_1 \mapsto 0$.

Let n, i, j be natural numbers and let b be a many sorted set indexed by n. The functor $\langle b(i), \ldots, b(j) \rangle$ yields a many sorted set indexed by $j -' i$ and is defined by:

(Def. 1) For every natural number k such that $k \in j -' i$ holds $\langle b(i), \ldots, b(j) \rangle(k) = b(i + k)$.

Let n, i, j be natural numbers and let b be a natural-yielding many sorted set indexed by n. One can verify that $\langle b(i), \ldots, b(j) \rangle$ is natural-yielding.

Let n, i, j be natural numbers and let b be a finite-support many sorted set indexed by n. Note that $\langle b(i), \ldots, b(j) \rangle$ is finite-support.

One can prove the following proposition

(6) Let n, i be natural numbers and a, b be many sorted sets indexed by n. Then $a = b$ if and only if the following conditions are satisfied:
   (i) $\langle a(0), \ldots, a(i + 1) \rangle = \langle b(0), \ldots, b(i + 1) \rangle$, and
   (ii) $\langle a(i + 1), \ldots, a(n) \rangle = \langle b(i + 1), \ldots, b(n) \rangle$.

Let x be a non empty set and let n be a non empty natural number. The functor $\mathrm{Fin}(x, n)$ is defined as follows:

(Def. 2) $\mathrm{Fin}(x, n) = \{y; y$ ranges over elements of 2^x: y is finite $\wedge$ y is non empty $\wedge$ $\overline{\overline{y}} \leqslant n\}$.

Let x be a non empty set and let n be a non empty natural number. Observe that $\mathrm{Fin}(x, n)$ is non empty.

One can prove the following propositions:

(7) Let R be an antisymmetric transitive non empty relational structure and X be a finite subset of the carrier of R. Suppose $X \neq \emptyset$. Then there exists an element x of R such that $x \in X$ and x is maximal w.r.t. X, the internal relation of R.

(8) Let R be an antisymmetric transitive non empty relational structure and X be a finite subset of the carrier of R. Suppose $X \neq \emptyset$. Then there exists an element x of R such that $x \in X$ and x is minimal w.r.t. X, the internal relation of R.

(9) Let R be a non empty antisymmetric transitive relational structure and f be a sequence of R. Suppose f is descending. Let j, i be natural numbers. If $i < j$, then $f(i) \neq f(j)$ and $\langle f(j), f(i) \rangle \in$ the internal relation of R.

Let R be a non empty relational structure and let s be a sequence of R. We say that s is non-increasing if and only if:

(Def. 3) For every natural number i holds $\langle s(i+1), s(i)\rangle \in$ the internal relation of R.

We now state three propositions:

(10) Let R be a non empty transitive relational structure and f be a sequence of R. Suppose f is non-increasing. Let j, i be natural numbers. If $i < j$, then $\langle f(j), f(i)\rangle \in$ the internal relation of R.

(11) Let R be a non empty transitive relational structure and s be a sequence of R. Suppose R is well founded and s is non-increasing. Then there exists a natural number p such that for every natural number r if $p \leqslant r$, then $s(p) = s(r)$.

(12) Let X be a set, a be an element of X, A be a finite subset of X, and R be an order in X. If $A = \{a\}$ and R linearly orders A, then $\mathrm{SgmX}(R, A) = \langle a\rangle$.

2. More About Bags

Let n be an ordinal number and let b be a bag of n. The functor $\mathrm{TotDegree}\, b$ yielding a natural number is defined by:

(Def. 4) There exists a finite sequence f of elements of $\mathbb{N}$ such that $\mathrm{TotDegree}\, b = \sum f$ and $f = b \cdot \mathrm{SgmX}(\subseteq_n, \mathrm{support}\, b)$.

The following propositions are true:

(13) Let n be an ordinal number, b be a bag of n, s be a finite subset of n, and f, g be finite sequences of elements of $\mathbb{N}$. If $f = b \cdot \mathrm{SgmX}(\subseteq_n, \mathrm{support}\, b)$ and $g = b \cdot \mathrm{SgmX}(\subseteq_n, \mathrm{support}\, b \cup s)$, then $\sum f = \sum g$.

(14) For every ordinal number n and for all bags a, b of n holds $\mathrm{TotDegree}(a+b) = \mathrm{TotDegree}\, a + \mathrm{TotDegree}\, b$.

(15) For every ordinal number n and for all bags a, b of n such that $b \mid a$ holds $\mathrm{TotDegree}(a -' b) = \mathrm{TotDegree}\, a - \mathrm{TotDegree}\, b$.

(16) For every ordinal number n and for every bag b of n holds $\mathrm{TotDegree}\, b = 0$ iff $b = \mathrm{EmptyBag}\, n$.

(17) For all natural numbers i, j, n holds $\langle(\mathrm{EmptyBag}\, n)(i), \ldots, (\mathrm{EmptyBag}\, n)(j)\rangle = \mathrm{EmptyBag}(j -' i)$.

(18) For all natural numbers i, j, n and for all bags a, b of n holds $\langle(a+b)(i), \ldots, (a+b)(j)\rangle = \langle a(i), \ldots, a(j)\rangle + \langle b(i), \ldots, b(j)\rangle$.

(19) For every set X holds $\mathrm{support}\, \mathrm{EmptyBag}\, X = \emptyset$.

(20) For every set X and for every bag b of X such that $\mathrm{support}\, b = \emptyset$ holds $b = \mathrm{EmptyBag}\, X$.

(21) For all ordinal numbers n, m and for every bag b of n such that $m \in n$ holds $b{\restriction}m$ is a bag of m.

(22) For every ordinal number n and for all bags a, b of n such that $b \mid a$ holds support $b \subseteq$ support a.

3. Some Special Orders

Let n be an ordinal number and let o be an order in Bags n. We say that o is admissible if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) o is strongly connected in Bags n,

(ii) for every bag a of n holds $\langle$ EmptyBag n, $a \rangle \in o$, and

(iii) for all bags a, b, c of n such that $\langle a, b \rangle \in o$ holds $\langle a + c, b + c \rangle \in o$.

Let n be an ordinal number. We introduce LexOrder n as a synonym of BagOrder n.

One can prove the following propositions:

(23) For every ordinal number n holds LexOrder n is admissible.

(24) For every infinite ordinal number o holds LexOrder o is non well-ordering.

Let n be an ordinal number. The functor InvLexOrder n yields an order in Bags n and is defined by the condition (Def. 6).

(Def. 6) Let p, q be bags of n. Then $\langle p, q \rangle \in$ InvLexOrder n if and only if one of the following conditions is satisfied:

(i) $p = q$, or

(ii) there exists an ordinal number i such that $i \in n$ and $p(i) < q(i)$ and for every ordinal number k such that $i \in k$ and $k \in n$ holds $p(k) = q(k)$.

The following propositions are true:

(25) For every ordinal number n holds InvLexOrder n is admissible.

(26) For every ordinal number o holds InvLexOrder o is well-ordering.

Let n be an ordinal number and let o be an order in Bags n. Let us assume that for all bags a, b, c of n such that $\langle a, b \rangle \in o$ holds $\langle a + c, b + c \rangle \in o$. The functor Graded o yields an order in Bags n and is defined by:

(Def. 7) For all bags a, b of n holds $\langle a, b \rangle \in$ Graded o iff TotDegree $a <$ TotDegree b or TotDegree $a =$ TotDegree b and $\langle a, b \rangle \in o$.

The following proposition is true

(27) Let n be an ordinal number and o be an order in Bags n. Suppose for all bags a, b, c of n such that $\langle a, b \rangle \in o$ holds $\langle a + c, b + c \rangle \in o$ and o is strongly connected in Bags n. Then Graded o is admissible.

Let n be an ordinal number. The functor GrLexOrder n yielding an order in Bags n is defined as follows:

(Def. 8) GrLexOrder $n =$ Graded LexOrder n.

The functor GrInvLexOrder n yielding an order in Bags n is defined by:

(Def. 9) GrInvLexOrder $n =$ Graded InvLexOrder n.

Next we state four propositions:

(28) For every ordinal number n holds GrLexOrder n is admissible.

(29) For every infinite ordinal number o holds GrLexOrder o is non well-ordering.

(30) For every ordinal number n holds GrInvLexOrder n is admissible.

(31) For every ordinal number o holds GrInvLexOrder o is well-ordering.

Let i, n be natural numbers, let o_1 be an order in Bags$(i+1)$, and let o_2 be an order in Bags$(n -' (i+1))$. The functor BlockOrder(i, n, o_1, o_2) yielding an order in Bags n is defined by the condition (Def. 10).

(Def. 10) Let p, q be bags of n. Then $\langle p, q \rangle \in$ BlockOrder(i, n, o_1, o_2) if and only if one of the following conditions is satisfied:

(i) $\langle p(0), \ldots, p(i+1) \rangle \neq \langle q(0), \ldots, q(i+1) \rangle$ and $\langle \langle p(0), \ldots, p(i+1) \rangle, \langle q(0), \ldots, q(i+1) \rangle \rangle \in o_1$, or

(ii) $\langle p(0), \ldots, p(i+1) \rangle = \langle q(0), \ldots, q(i+1) \rangle$ and $\langle \langle p(i+1), \ldots, p(n) \rangle, \langle q(i+1), \ldots, q(n) \rangle \rangle \in o_2$.

The following proposition is true

(32) Let i, n be natural numbers, o_1 be an order in Bags$(i+1)$, and o_2 be an order in Bags$(n -' (i+1))$. If o_1 is admissible and o_2 is admissible, then BlockOrder(i, n, o_1, o_2) is admissible.

Let n be a natural number. The functor NaivelyOrderedBags n yielding a strict relational structure is defined by the conditions (Def. 11).

(Def. 11)(i) The carrier of NaivelyOrderedBags n = Bags n, and

(ii) for all bags x, y of n holds $\langle x, y \rangle \in$ the internal relation of NaivelyOrderedBags n iff $x \mid y$.

The following propositions are true:

(33) For every natural number n holds the carrier of $\prod(n \longmapsto$ OrderedNAT$)$ = Bags n.

(34) For every natural number n holds NaivelyOrderedBags $n = \prod(n \longmapsto$ OrderedNAT$)$.

(35) Let n be a natural number and o be an order in Bags n. Suppose o is admissible. Then the internal relation of NaivelyOrderedBags $n \subseteq o$ and o is well-ordering.

4. Ordering of Finite Subsets

Let R be a connected non empty poset and let X be an element of Fin (the carrier of R). Let us assume that X is non empty. The functor PosetMin X yielding an element of R is defined as follows:

(Def. 12) PosetMin $X \in X$ and PosetMin X is minimal w.r.t. X, the internal relation of R.

The functor PosetMax X yields an element of R and is defined as follows:

(Def. 13) PosetMax $X \in X$ and PosetMax X is maximal w.r.t. X, the internal relation of R.

Let R be a connected non empty poset. The functor FinOrd-Approx R yielding a function from $\mathbb{N}$ into $2^{[\text{Fin (the carrier of } R), \text{Fin (the carrier of } R)]}$ is defined by the conditions (Def. 14).

(Def. 14)(i) dom FinOrd-Approx $R = \mathbb{N}$,

(ii) (FinOrd-Approx R)(0) $= \{\langle x, y\rangle; x$ ranges over elements of Fin (the carrier of R), y ranges over elements of Fin (the carrier of R): $x = \emptyset \lor x \neq \emptyset \land y \neq \emptyset \land$ PosetMax $x \neq$ PosetMax $y \land \langle$ PosetMax x, PosetMax $y\rangle \in$ the internal relation of $R\}$, and

(iii) for every element n of $\mathbb{N}$ holds (FinOrd-Approx R)$(n+1) = \{\langle x, y\rangle; x$ ranges over elements of Fin (the carrier of R), y ranges over elements of Fin (the carrier of R): $x \neq \emptyset \ \land \ y \neq \emptyset \ \land$ PosetMax $x =$ PosetMax $y \ \land$ $\langle x \setminus \{$PosetMax $x\}, y \setminus \{$PosetMax $y\}\rangle \in$ (FinOrd-Approx R)$(n)\}$.

One can prove the following propositions:

(36) Let R be a connected non empty poset and x, y be elements of Fin (the carrier of R). Then $\langle x, y\rangle \in \bigcup$ rng FinOrd-Approx R if and only if one of the following conditions is satisfied:

(i) $x = \emptyset$, or

(ii) $x \neq \emptyset$ and $y \neq \emptyset$ and PosetMax $x \neq$ PosetMax y and $\langle$ PosetMax x, PosetMax $y\rangle \in$ the internal relation of R, or

(iii) $x \neq \emptyset$ and $y \neq \emptyset$ and PosetMax $x =$ PosetMax y and $\langle x \setminus \{$PosetMax $x\}$, $y \setminus \{$PosetMax $y\}\rangle \in \bigcup$ rng FinOrd-Approx R.

(37) For every connected non empty poset R and for every element x of Fin (the carrier of R) such that $x \neq \emptyset$ holds $\langle x, \emptyset\rangle \notin \bigcup$ rng FinOrd-Approx R.

(38) Let R be a connected non empty poset and a be an element of Fin (the carrier of R). Then $a \setminus \{$PosetMax $a\}$ is an element of Fin (the carrier of R).

(39) For every connected non empty poset R holds $\bigcup$ rng FinOrd-Approx R is an order in Fin (the carrier of R).

Let R be a connected non empty poset. The functor FinOrd R yields an order in Fin (the carrier of R) and is defined as follows:

(Def. 15) FinOrd $R = \bigcup$ rng FinOrd-Approx R.

Let R be a connected non empty poset. The functor FinPoset R yields a poset and is defined by:

(Def. 16) FinPoset $R = \langle$Fin (the carrier of R), FinOrd $R\rangle$.

Let R be a connected non empty poset. One can check that FinPoset R is non empty.

The following proposition is true

(40) Let R be a connected non empty poset and a, b be elements of FinPoset R. Then $\langle a, b\rangle \in$ the internal relation of FinPoset R if and only if there exist elements x, y of Fin (the carrier of R) such that $a = x$ but $b = y$ but $x = \emptyset$ or $x \neq \emptyset$ and $y \neq \emptyset$ and PosetMax $x \neq$ PosetMax y and $\langle$ PosetMax x, PosetMax $y\rangle \in$ the internal relation of R or $x \neq \emptyset$ and $y \neq \emptyset$ and PosetMax x = PosetMax y and $\langle x \setminus \{$PosetMax $x\}$, $y \setminus \{$PosetMax $y\}\rangle \in$ FinOrd R.

Let R be a connected non empty poset. One can verify that FinPoset R is connected.

Let R be a connected non empty relational structure and let C be a non empty set. Let us assume that R is well founded and $C \subseteq$ the carrier of R. The functor MinElement(C, R) yields an element of R and is defined by:

(Def. 17) MinElement$(C, R) \in C$ and MinElement(C, R) is minimal w.r.t. C, the internal relation of R.

Let R be a non empty relational structure, let s be a sequence of R, and let j be a natural number. The functor SeqShift(s, j) yields a sequence of R and is defined by:

(Def. 18) For every natural number i holds $(\text{SeqShift}(s, j))(i) = s(i + j)$.

One can prove the following propositions:

(41) Let R be a non empty relational structure, s be a sequence of R, and j be a natural number. If s is descending, then SeqShift(s, j) is descending.

(42) For every connected non empty poset R such that R is well founded holds FinPoset R is well founded.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.

[4] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.

[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[6] Thomas Becker and Volker Weispfenning. *Gröbner bases: A Computational Approach to Commutative Algebra*. Springer-Verlag, New York, Berlin, 1993.

[7] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.

[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[12] Adam Grabowski. Auxiliary and approximating relations. *Formalized Mathematics*, 6(**2**):179–188, 1997.
[13] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Formalized Mathematics*, 6(**1**):117–121, 1997.
[14] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[15] Andrzej Kondracki. The Chinese Remainder Theorem. *Formalized Mathematics*, 6(**4**):573–577, 1997.
[16] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[17] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(**2**):275–278, 1992.
[18] Gilbert Lee and Piotr Rudnicki. Dickson's lemma. *Formalized Mathematics*, 10(**1**):29–37, 2002.
[19] Beata Madras. On the concept of the triangulation. *Formalized Mathematics*, 5(**3**):457–462, 1996.
[20] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Formalized Mathematics*, 5(**2**):167–172, 1996.
[21] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.
[22] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[23] Piotr Rudnicki and Andrzej Trybulec. On same equivalents of well-foundedness. *Formalized Mathematics*, 6(**3**):339–343, 1997.
[24] Piotr Rudnicki and Andrzej Trybulec. Multivariate polynomials with arbitrary number of variables. *Formalized Mathematics*, 9(**1**):95–110, 2001.
[25] Christoph Schwarzweller and Andrzej Trybulec. The evaluation of multivariate polynomials. *Formalized Mathematics*, 9(**2**):331–338, 2001.
[26] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[27] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[28] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[29] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.
[30] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[31] Wojciech A. Trybulec and Grzegorz Bancerek. Kuratowski - Zorn lemma. *Formalized Mathematics*, 1(**2**):387–393, 1990.
[32] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[33] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[34] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[35] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[36] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.

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