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# Dickson's Lemma

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**Summary.** We present a Mizar formalization of the proof of Dickson's lemma following [6], chapters 4.2 and 4.3.

MML Identifier: `DICKSON`.

The papers [19], [29], [1], [7], [13], [21], [12], [8], [9], [2], [20], [26], [27], [24], [17], [18], [30], [32], [31], [28], [23], [4], [11], [5], [14], [22], [3], [15], [16], [25], and [10] provide the terminology and notation for this paper.

## 1. Preliminaries

One can prove the following two propositions:

(1) For every function $g$ and for every set $x$ such that $\operatorname{dom} g = \{x\}$ holds $g = x \dot{\longmapsto} g(x)$.

(2) For every natural number $n$ holds $n \subseteq n+1$.

The scheme *FinSegRng2* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(i); i$ ranges over natural numbers: $\mathcal{A} < i \ \wedge \ i \leqslant \mathcal{B} \ \wedge \ \mathcal{P}[i]\}$ is finite

for all values of the parameters.

The following proposition is true

(3) For every infinite set $X$ holds there exists a function from $\mathbb{N}$ into $X$ which is one-to-one.

Let $R$ be a relational structure and let $f$ be a sequence of $R$. We say that $f$ is ascending if and only if:




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(Def. 1) For every natural number $n$ holds $f(n+1) \neq f(n)$ and $\langle f(n), f(n+1)\rangle \in$ the internal relation of $R$.

Let $R$ be a relational structure and let $f$ be a sequence of $R$. We say that $f$ is weakly ascending if and only if:

(Def. 2) For every natural number $n$ holds $\langle f(n), f(n+1)\rangle \in$ the internal relation of $R$.

The following propositions are true:

(4) Let $R$ be a non empty transitive relational structure and $f$ be a sequence of $R$. Suppose $f$ is weakly ascending. Let $i$, $j$ be natural numbers. If $i < j$, then $f(i) \leqslant f(j)$.

(5) Let $R$ be a non empty relational structure. Then $R$ is connected if and only if the internal relation of $R$ is strongly connected in the carrier of $R$.

(6) Let $R$ be a binary relation and $X$ be a set. Then $R$ is reflexive in $X$ and connected in $X$ if and only if $R$ is strongly connected in $X$.

(7) Let $L$ be a relational structure, $Y$ be a set, and $a$ be an element of $L$. Then (the internal relation of $L$)-Seg($a$) misses $Y$ and $a \in Y$ if and only if $a$ is minimal w.r.t. $Y$, the internal relation of $L$.

(8) Let $L$ be a non empty transitive antisymmetric relational structure, $a$, $x$ be elements of $L$, and $N$ be a set. Suppose $a$ is minimal w.r.t. (the internal relation of $L$)-Seg($x$) $\cap N$, the internal relation of $L$. Then $a$ is minimal w.r.t. $N$, the internal relation of $L$.

## 2. More on Ordering Relations

Let $R$ be a relational structure. We say that $R$ is quasi ordered if and only if:

(Def. 3) $R$ is reflexive and transitive.

Let $R$ be a relational structure. Let us assume that $R$ is quasi ordered. The functor EqRel($R$) yielding an equivalence relation of the carrier of $R$ is defined as follows:

(Def. 4) EqRel($R$) = (the internal relation of $R$) $\cap$ (the internal relation of $R$)$^{\smile}$.

The following proposition is true

(9) Let $R$ be a relational structure and $x$, $y$ be elements of the carrier of $R$. If $R$ is quasi ordered, then $x \in [y]_{\mathrm{EqRel}(R)}$ iff $x \leqslant y$ and $y \leqslant x$.

Let $R$ be a relational structure. The functor $\leqslant_E R$ yielding a binary relation on Classes EqRel($R$) is defined as follows:

(Def. 5) For all sets $A$, $B$ holds $\langle A, B\rangle \in {\leqslant_E} R$ iff there exist elements $a$, $b$ of $R$ such that $A = [a]_{\mathrm{EqRel}(R)}$ and $B = [b]_{\mathrm{EqRel}(R)}$ and $a \leqslant b$.

We now state two propositions:

(10) For every relational structure $R$ such that $R$ is quasi ordered holds $\leqslant_E R$ partially orders Classes EqRel($R$).

(11) Let $R$ be a non empty relational structure. If $R$ is quasi ordered and connected, then $\leqslant_E R$ linearly orders Classes EqRel($R$).

Let $R$ be a binary relation. The functor $R\backslash^{\smile}$ yields a binary relation and is defined by:

(Def. 6) $R\backslash^{\smile} = R \setminus R^{\smile}$.

Let $R$ be a binary relation. Note that $R\backslash^{\smile}$ is asymmetric.

Let $X$ be a set and let $R$ be a binary relation on $X$. Then $R\backslash^{\smile}$ is a binary relation on $X$.

Let $R$ be a relational structure. The functor $R\backslash^{\smile}$ yielding a strict relational structure is defined as follows:

(Def. 7) $R\backslash^{\smile} = \langle$the carrier of $R$, the internal relation of $R\backslash^{\smile}\rangle$.

Let $R$ be a non empty relational structure. One can check that $R\backslash^{\smile}$ is non empty.

Let $R$ be a transitive relational structure. One can check that $R\backslash^{\smile}$ is transitive.

Let $R$ be a relational structure. One can check that $R\backslash^{\smile}$ is antisymmetric.

We now state several propositions:

(12) For every non empty poset $R$ and for every element $x$ of the carrier of $R$ holds $[x]_{\mathrm{EqRel}(R)} = \{x\}$.

(13) For every binary relation $R$ holds $R = R\backslash^{\smile}$ iff $R$ is asymmetric.

(14) For every binary relation $R$ such that $R$ is transitive holds $R\backslash^{\smile}$ is transitive.

(15) Let $R$ be a binary relation and $a$, $b$ be sets. If $R$ is antisymmetric, then $\langle a, b\rangle \in R\backslash^{\smile}$ iff $\langle a, b\rangle \in R$ and $a \neq b$.

(16) For every relational structure $R$ such that $R$ is well founded holds $R\backslash^{\smile}$ is well founded.

(17) For every relational structure $R$ such that $R\backslash^{\smile}$ is well founded and $R$ is antisymmetric holds $R$ is well founded.

## 3. Foundedness Properties

The following two propositions are true:

(18) Let $L$ be a relational structure, $N$ be a set, and $x$ be an element of $L\backslash^{\smile}$. Then $x$ is minimal w.r.t. $N$, the internal relation of $L\backslash^{\smile}$ if and only if $x \in N$ and for every element $y$ of $L$ such that $y \in N$ and $\langle y, x\rangle \in$ the internal relation of $L$ holds $\langle x, y\rangle \in$ the internal relation of $L$.

(19) Let $R$, $S$ be non empty relational structures and $m$ be a map from $R$ into $S$. Suppose that
 (i) $R$ is quasi ordered,
 (ii) $S$ is antisymmetric,
 (iii) $S\backslash^{\smile}$ is well founded, and
 (iv) for all elements $a$, $b$ of $R$ holds if $a \leqslant b$, then $m(a) \leqslant m(b)$ and if $m(a) = m(b)$, then $\langle a, b\rangle \in \mathrm{EqRel}(R)$.
 Then $R\backslash^{\smile}$ is well founded.

Let $R$ be a non empty relational structure and let $N$ be a subset of the carrier of $R$. The functor $\mathrm{MinClasses}\, N$ yields a family of subsets of the carrier of $R$ and is defined by the condition (Def. 8).

(Def. 8) Let $x$ be a set. Then $x \in \mathrm{MinClasses}\, N$ if and only if there exists an element $y$ of $R\backslash^{\smile}$ such that $y$ is minimal w.r.t. $N$, the internal relation of $R\backslash^{\smile}$ and $x = [y]_{\mathrm{EqRel}(R)} \cap N$.

Next we state several propositions:

(20) Let $R$ be a non empty relational structure, $N$ be a subset of the carrier of $R$, and $x$ be a set. Suppose $R$ is quasi ordered and $x \in \mathrm{MinClasses}\, N$. Let $y$ be an element of $R\backslash^{\smile}$. If $y \in x$, then $y$ is minimal w.r.t. $N$, the internal relation of $R\backslash^{\smile}$.

(21) Let $R$ be a non empty relational structure. Then $R\backslash^{\smile}$ is well founded if and only if for every subset $N$ of the carrier of $R$ such that $N \neq \emptyset$ there exists a set $x$ such that $x \in \mathrm{MinClasses}\, N$.

(22) Let $R$ be a non empty relational structure, $N$ be a subset of the carrier of $R$, and $y$ be an element of $R\backslash^{\smile}$. If $y$ is minimal w.r.t. $N$, the internal relation of $R\backslash^{\smile}$, then $\mathrm{MinClasses}\, N$ is non empty.

(23) Let $R$ be a non empty relational structure, $N$ be a subset of the carrier of $R$, and $x$ be a set. If $R$ is quasi ordered and $x \in \mathrm{MinClasses}\, N$, then $x$ is non empty.

(24) Let $R$ be a non empty relational structure. Suppose $R$ is quasi ordered. Then $R$ is connected and $R\backslash^{\smile}$ is well founded if and only if for every non empty subset $N$ of the carrier of $R$ holds $\overline{\overline{\mathrm{MinClasses}\, N}} = 1$.

(25) Let $R$ be a non empty poset. Then the following statements are equivalent
 (i) the internal relation of $R$ well orders the carrier of $R$,
 (ii) for every non empty subset $N$ of the carrier of $R$ holds $\overline{\overline{\mathrm{MinClasses}\, N}} = 1$.

Let $R$ be a relational structure, let $N$ be a subset of the carrier of $R$, and let $B$ be a set. We say that $B$ is Dickson basis of $N$, $R$ if and only if:

(Def. 9) $B \subseteq N$ and for every element $a$ of $R$ such that $a \in N$ there exists an element $b$ of $R$ such that $b \in B$ and $b \leqslant a$.

The following two propositions are true:

(26) For every relational structure $R$ holds $\emptyset$ is Dickson basis of $\emptyset_{\text{the carrier of } R}$, $R$.

(27) Let $R$ be a non empty relational structure, $N$ be a non empty subset of the carrier of $R$, and $B$ be a set. If $B$ is Dickson basis of $N$, $R$, then $B$ is non empty.

Let $R$ be a relational structure. We say that $R$ is Dickson if and only if:

(Def. 10) For every subset $N$ of the carrier of $R$ holds there exists a set which is Dickson basis of $N$, $R$ and finite.

The following two propositions are true:

(28) For every non empty relational structure $R$ such that $R\backslash^{\smile}$ is well founded and $R$ is connected holds $R$ is Dickson.

(29) Let $R$, $S$ be relational structures. Suppose that

(i) the internal relation of $R \subseteq$ the internal relation of $S$,

(ii) $R$ is Dickson, and

(iii) the carrier of $R =$ the carrier of $S$.

Then $S$ is Dickson.

Let $f$ be a function and let $b$ be a set. Let us assume that $\operatorname{dom} f = \mathbb{N}$ and $b \in \operatorname{rng} f$. The functor $f \operatorname{mindex} b$ yielding a natural number is defined by:

(Def. 11) $f(f \operatorname{mindex} b) = b$ and for every natural number $i$ such that $f(i) = b$ holds $f \operatorname{mindex} b \leqslant i$.

Let $R$ be a non empty 1-sorted structure, let $f$ be a sequence of $R$, let $b$ be a set, and let $m$ be a natural number. Let us assume that there exists a natural number $j$ such that $m < j$ and $f(j) = b$. The functor $f \operatorname{mindex}(b, m)$ yielding a natural number is defined as follows:

(Def. 12) $f(f \operatorname{mindex}(b, m)) = b$ and $m < f \operatorname{mindex}(b, m)$ and for every natural number $i$ such that $m < i$ and $f(i) = b$ holds $f \operatorname{mindex}(b, m) \leqslant i$.

Next we state several propositions:

(30) Let $R$ be a non empty relational structure. Suppose $R$ is quasi ordered and Dickson. Let $f$ be a sequence of $R$. Then there exist natural numbers $i$, $j$ such that $i < j$ and $f(i) \leqslant f(j)$.

(31) Let $R$ be a relational structure, $N$ be a subset of the carrier of $R$, and $x$ be an element of $R\backslash^{\smile}$. Suppose $R$ is quasi ordered and $x \in N$ and (the internal relation of $R$)-Seg$(x) \cap N \subseteq [x]_{\mathrm{EqRel}(R)}$. Then $x$ is minimal w.r.t. $N$, the internal relation of $R\backslash^{\smile}$.

(32) Let $R$ be a non empty relational structure. Suppose $R$ is quasi ordered and for every sequence $f$ of $R$ there exist natural numbers $i$, $j$ such that $i < j$ and $f(i) \leqslant f(j)$. Let $N$ be a non empty subset of the carrier of $R$. Then MinClasses $N$ is finite and MinClasses $N$ is non empty.

(33) Let $R$ be a non empty relational structure. Suppose $R$ is quasi ordered and for every non empty subset $N$ of the carrier of $R$ holds MinClasses $N$ is finite and MinClasses $N$ is non empty. Then $R$ is Dickson.

(34) For every non empty relational structure $R$ such that $R$ is quasi ordered and Dickson holds $R\backslash^{\smile}$ is well founded.

(35) Let $R$ be a non empty poset and $N$ be a non empty subset of the carrier of $R$. Suppose $R$ is Dickson. Then there exists a set $B$ such that $B$ is Dickson basis of $N$, $R$ and for every set $C$ such that $C$ is Dickson basis of $N$, $R$ holds $B \subseteq C$.

Let $R$ be a non empty relational structure and let $N$ be a subset of the carrier of $R$. Let us assume that $R$ is Dickson. The functor Dickson-Bases$(N, R)$ yields a non empty family of subsets of the carrier of $R$ and is defined as follows:

(Def. 13) For every set $B$ holds $B \in$ Dickson-Bases$(N, R)$ iff $B$ is Dickson basis of $N$, $R$.

We now state several propositions:

(36) Let $R$ be a non empty relational structure and $s$ be a sequence of $R$. If $R$ is Dickson, then there exists a sequence of $R$ which is a subsequence of $s$ and weakly ascending.

(37) For every relational structure $R$ such that $R$ is empty holds $R$ is Dickson.

(38) Let $M$, $N$ be relational structures. Suppose $M$ is Dickson and $N$ is Dickson and $M$ is quasi ordered and $N$ is quasi ordered. Then $[\!: M, N :\!]$ is quasi ordered and $[\!: M, N :\!]$ is Dickson.

(39) Let $R$, $S$ be relational structures. Suppose $R$ and $S$ are isomorphic and $R$ is Dickson and quasi ordered. Then $S$ is quasi ordered and Dickson.

(40) Let $p$ be a relational structure yielding many sorted set indexed by 1 and $z$ be an element of 1. Then $p(z)$ and $\prod p$ are isomorphic.

Let $X$ be a set, let $p$ be a relational structure yielding many sorted set indexed by $X$, and let $Y$ be a subset of $X$. Note that $p{\restriction}Y$ is relational structure yielding.

Next we state three propositions:

(41) Let $n$ be a non empty natural number and $p$ be a relational structure yielding many sorted set indexed by $n$. Then $\prod p$ is non empty if and only if $p$ is nonempty.

(42) Let $n$ be a non empty natural number, $p$ be a relational structure yielding many sorted set indexed by $n+1$, $n_1$ be a subset of $n+1$, and $n_2$ be an element of $n+1$. If $n_1 = n$ and $n_2 = n$, then $[\!: \prod(p{\restriction}n_1), p(n_2) :\!]$ and $\prod p$ are isomorphic.

(43) Let $n$ be a non empty natural number and $p$ be a relational structure yielding many sorted set indexed by $n$. Suppose that for every element

$i$ of $n$ holds $p(i)$ is Dickson and $p(i)$ is quasi ordered. Then $\prod p$ is quasi ordered and $\prod p$ is Dickson.

Let $p$ be a relational structure yielding many sorted set indexed by $\emptyset$. One can check the following observations:

* $\prod p$ is non empty,
* $\prod p$ is antisymmetric,
* $\prod p$ is quasi ordered, and
* $\prod p$ is Dickson.

The binary relation NATOrd on $\mathbb{N}$ is defined by:

(Def. 14) NATOrd $= \{\langle x, y\rangle; x$ ranges over elements of $\mathbb{N}$, $y$ ranges over elements of $\mathbb{N}$: $x \leqslant y\}$.

We now state four propositions:

(44) NATOrd is reflexive in $\mathbb{N}$.

(45) NATOrd is antisymmetric in $\mathbb{N}$.

(46) NATOrd is strongly connected in $\mathbb{N}$.

(47) NATOrd is transitive in $\mathbb{N}$.

The non empty relational structure OrderedNAT is defined as follows:

(Def. 15) OrderedNAT $= \langle \mathbb{N}, \text{NATOrd}\rangle$.

One can verify the following observations:

* OrderedNAT is connected,
* OrderedNAT is Dickson,
* OrderedNAT is quasi ordered,
* OrderedNAT is antisymmetric,
* OrderedNAT is transitive, and
* OrderedNAT is well founded.

Let $n$ be a natural number. One can check the following observations:

* $\prod(n \longmapsto \text{OrderedNAT})$ is non empty,
* $\prod(n \longmapsto \text{OrderedNAT})$ is Dickson,
* $\prod(n \longmapsto \text{OrderedNAT})$ is quasi ordered, and
* $\prod(n \longmapsto \text{OrderedNAT})$ is antisymmetric.

We now state three propositions:

(48) Let $M$ be a relational structure. Suppose $M$ is Dickson and quasi ordered. Then $[\!: M, \text{OrderedNAT} :\!]$ is quasi ordered and $[\!: M, \text{OrderedNAT} :\!]$ is Dickson.

(49) Let $R$, $S$ be non empty relational structures. Suppose that

(i) $R$ is Dickson and quasi ordered,

(ii) $S$ is quasi ordered,

(iii) the internal relation of $R \subseteq$ the internal relation of $S$, and

(iv) the carrier of $R$ = the carrier of $S$.
Then $S\backslash^{\smile}$ is well founded.

(50) Let $R$ be a non empty relational structure. Suppose $R$ is quasi ordered. Then $R$ is Dickson if and only if for every non empty relational structure $S$ such that $S$ is quasi ordered and the internal relation of $R \subseteq$ the internal relation of $S$ and the carrier of $R$ = the carrier of $S$ holds $S\backslash^{\smile}$ is well founded.

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