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Upper and Lower Sequence on the Cage, Upper and Lower Arcs[1]

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The articles [25], [30], [2], [4], [3], [29], [5], [14], [27], [20], [24], [13], [1], [23], [10], [11], [8], [28], [16], [12], [21], [26], [7], [18], [19], [6], [22], [9], [15], and [17] provide the notation and terminology for this paper.

In this paper n is a natural number.

The following propositions are true:

(1) Let G be a Go-board and i_1, i_2, j_1, j_2 be natural numbers. Suppose $1 \leqslant j_1$ and $j_1 \leqslant \operatorname{width} G$ and $1 \leqslant j_2$ and $j_2 \leqslant \operatorname{width} G$ and $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 \leqslant \operatorname{len} G$. Then $(G \circ (i_1, j_1))_{\mathbf{1}} < (G \circ (i_2, j_2))_{\mathbf{1}}$.

(2) Let G be a Go-board and i_1, i_2, j_1, j_2 be natural numbers. Suppose $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} G$ and $1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} G$ and $1 \leqslant j_1$ and $j_1 < j_2$ and $j_2 \leqslant \operatorname{width} G$. Then $(G \circ (i_1, j_1))_{\mathbf{2}} < (G \circ (i_2, j_2))_{\mathbf{2}}$.

Let f be a non empty finite sequence and let g be a finite sequence. One can verify that $f \frown g$ is non empty.

The following propositions are true:

(3) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Then $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n) -: \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n) :- \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = \{\text{N-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))\}$.

(4) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{UpperSeq}(C,n) = ((\mathrm{Cage}(C,n))^{\text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}_{\circlearrowleft}) :- \text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

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(5) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$ and W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(6) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds W-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$ and W-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(7) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds N-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$ and N-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(8) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds N-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$ and N-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(9) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds E-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$ and E-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(10) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds E-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$ and E-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(11) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds E-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$ and E-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(12) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds S-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$ and S-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(13) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds S-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$ and S-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(14) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$ and W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(15) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and N-min $Y \in X$ holds N-min $X =$ N-min Y.

(16) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and N-max $Y \in X$ holds N-max $X =$ N-max Y.

(17) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and E-min $Y \in X$ holds E-min $X =$ E-min Y.

(18) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and E-max $Y \in X$ holds E-max $X =$ E-max Y.

(19) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and S-min $Y \in X$ holds S-min $X =$ S-min Y.

(20) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and S-max $Y \in X$ holds S-max $X =$ S-max Y.

(21) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and W-min $Y \in X$ holds W-min $X =$ W-min Y.

(22) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and W-max $Y \in X$ holds W-max $X =$ W-max Y.

(23) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that N-bound $X <$ N-bound Y holds N-bound $X \cup Y =$ N-bound Y.

(24) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that E-bound $X <$ E-bound Y holds E-bound $X \cup Y =$ E-bound Y.

(25) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that S-bound $X <$ S-bound Y holds S-bound $X \cup Y =$ S-bound X.

(26) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that W-bound $X <$ W-bound Y holds W-bound $X \cup Y =$ W-bound X.

(27) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that N-bound $X <$ N-bound Y holds N-min $X \cup Y =$ N-min Y.

(28) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that N-bound $X <$ N-bound Y holds N-max $X \cup Y =$ N-max Y.

(29) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that E-bound $X <$ E-bound Y holds E-min $X \cup Y =$ E-min Y.

(30) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that E-bound $X <$ E-bound Y holds E-max $X \cup Y =$ E-max Y.

(31) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that S-bound $X <$ S-bound Y holds S-min $X \cup Y =$ S-min X.

(32) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that S-bound $X <$ S-bound Y holds S-max $X \cup Y =$ S-max X.

(33) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that W-bound $X <$ W-bound Y holds W-min $X \cup Y =$ W-min X.

(34) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that W-bound $X <$ W-bound Y holds W-max $X \cup Y =$ W-max X.

(35) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$, then $(\downharpoonleft p, f)_{{\rm len}\downharpoonleft p, f} = f_{{\rm len}\, f}$.

(36) Let f be a non constant standard special circular sequence, p, q be points of $\mathcal{E}^2_{\rm T}$, and g be a connected subset of $\mathcal{E}^2_{\rm T}$. If $p \in$ RightComp(f) and $q \in$ LeftComp(f) and $p \in g$ and $q \in g$, then g meets $\widetilde{\mathcal{L}}(f)$.

One can verify that there exists special sequence finite sequence of elements of $\mathcal{E}^2_{\rm T}$ which is non constant, standard, and s.c.c..

Next we state a number of propositions:

(37) For every S-sequence f in $\mathbb{R}^2$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \operatorname{rng} f$ holds $\downharpoonright p, f = \operatorname{mid}(f, p \leftarrow\!\!\!\!\!\!\!\!\!\!\!\! f, \operatorname{len} f)$.

(38) Let M be a Go-board and f be a S-sequence in $\mathbb{R}^2$. Suppose f is a sequence which elements belong to M. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \operatorname{rng} f$, then $\downharpoonleft f, p$ is a sequence which elements belong to M.

(39) Let M be a Go-board and f be a S-sequence in $\mathbb{R}^2$. Suppose f is a sequence which elements belong to M. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \operatorname{rng} f$, then $\downharpoonright p, f$ is a sequence which elements belong to M.

(40) Let G be a Go-board and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to G. Let i, j be natural numbers. If $1 \leqslant i$ and $i \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j \leqslant \operatorname{width} G$, then if $G \circ (i, j) \in \widetilde{\mathcal{L}}(f)$, then $G \circ (i, j) \in \operatorname{rng} f$.

(41) Let f be a S-sequence in $\mathbb{R}^2$ and g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) g is unfolded, s.n.c., and one-to-one,

(ii) $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{f_1\}$,

(iii) $f_1 = g_{\operatorname{len} g}$,

(iv) for every natural number i such that $1 \leqslant i$ and $i + 2 \leqslant \operatorname{len} f$ holds $\mathcal{L}(f, i) \cap \mathcal{L}(f_{\operatorname{len} f}, g_1) = \emptyset$, and

(v) for every natural number i such that $2 \leqslant i$ and $i + 1 \leqslant \operatorname{len} g$ holds $\mathcal{L}(g, i) \cap \mathcal{L}(f_{\operatorname{len} f}, g_1) = \emptyset$.

Then $f \frown g$ is s.c.c..

(42) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Then there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $\text{W-min}\, C \in \operatorname{cell}(\operatorname{Gauge}(C, n), 1, i)$ and $\text{W-min}\, C \neq \operatorname{Gauge}(C, n) \circ (2, i)$.

(43) For every S-sequence f in $\mathbb{R}^2$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \widetilde{\mathcal{L}}(f)$ and $f(\operatorname{len} f) \in \widetilde{\mathcal{L}}(\downharpoonleft f, p)$ holds $f(\operatorname{len} f) = p$.

(44) For every non empty finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $\downharpoonleft f, p \neq \emptyset$.

(45) For every S-sequence f in $\mathbb{R}^2$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $(\downharpoonleft f, p)_{\operatorname{len} \downharpoonleft f, p} = p$.

(46) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$ and $p_{\mathbf{1}} = \text{E-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$, then $p = \text{E-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(47) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$ and $p_{\mathbf{1}} = \text{W-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$, then $p = \text{W-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(48) Let G be a Go-board, f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, and k be a natural number. Suppose $1 \leqslant k$ and $k < \operatorname{len} f$ and $f \frown g$ is a sequence

which elements belong to G. Then $\mathrm{left\_cell}(f \frown g, k, G) = \mathrm{left\_cell}(f, k, G)$ and $\mathrm{right\_cell}(f \frown g, k, G) = \mathrm{right\_cell}(f, k, G)$.

(49) Let D be a set, f, g be finite sequences of elements of D, and i be a natural number. If $i \leqslant \mathrm{len}\, f$, then $(f \frown\!\!\frown g) \restriction i = f \restriction i$.

(50) For every set D and for all finite sequences f, g of elements of D holds $(f \frown\!\!\frown g) \restriction \mathrm{len}\, f = f$.

(51) Let G be a Go-board, f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, and k be a natural number. Suppose $1 \leqslant k$ and $k < \mathrm{len}\, f$ and $f \frown\!\!\frown g$ is a sequence which elements belong to G. Then $\mathrm{left\_cell}(f \frown\!\!\frown g, k, G) = \mathrm{left\_cell}(f, k, G)$ and $\mathrm{right\_cell}(f \frown\!\!\frown g, k, G) = \mathrm{right\_cell}(f, k, G)$.

(52) Let G be a Go-board, f be a S-sequence in $\mathbb{R}^2$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and k be a natural number. Suppose $1 \leqslant k$ and $k < p \leftarrow f$ and f is a sequence which elements belong to G and $p \in \mathrm{rng}\, f$. Then $\mathrm{left\_cell}(\downharpoonleft f, p, k, G) = \mathrm{left\_cell}(f, k, G)$ and $\mathrm{right\_cell}(\downharpoonleft f, p, k, G) = \mathrm{right\_cell}(f, k, G)$.

(53) Let G be a Go-board, f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and k be a natural number. Suppose $1 \leqslant k$ and $k < p \leftarrow f$ and f is a sequence which elements belong to G. Then $\mathrm{left\_cell}(f -: p, k, G) = \mathrm{left\_cell}(f, k, G)$ and $\mathrm{right\_cell}(f -: p, k, G) = \mathrm{right\_cell}(f, k, G)$.

(54) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) f is unfolded, s.n.c., and one-to-one,

(ii) g is unfolded, s.n.c., and one-to-one,

(iii) $f_{\mathrm{len}\, f} = g_1$, and

(iv) $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g_1\}$.

Then $f \frown\!\!\frown g$ is s.n.c..

(55) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is one-to-one and g is one-to-one and $\mathrm{rng}\, f \cap \mathrm{rng}\, g \subseteq \{g_1\}$. Then $f \frown\!\!\frown g$ is one-to-one.

(56) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p \in \mathrm{rng}\, f$ and $p \neq f(1)$, then $\mathrm{Index}(p, f) + 1 = p \leftarrow f$.

(57) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 < i$ and $i < \mathrm{len}\, \mathrm{Gauge}(C, n)$ and $1 \leqslant j$ and $k \leqslant \mathrm{width}\, \mathrm{Gauge}(C, n)$ and $\mathrm{Gauge}(C, n) \circ (i, k) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n))$ and $\mathrm{Gauge}(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n))$. Then $j \neq k$.

(58) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \mathrm{len}\, \mathrm{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\, \mathrm{Gauge}(C, n)$ and $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i, j), \mathrm{Gauge}(C, n) \circ (i, k)) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n)) = \{\mathrm{Gauge}(C, n) \circ (i, k)\}$ and $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i, j), \mathrm{Gauge}(C, n) \circ (i, k)) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n)) = \{\mathrm{Gauge}(C, n) \circ (i, j)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i, j), \mathrm{Gauge}(C, n) \circ (i, k))$ meets $\mathrm{LowerArc}\, C$.

(59) Let C be a simple closed curve and i, j, k be natural numbers.

Suppose $1 < i$ and $i < \operatorname{len}\operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n)) = \{\operatorname{Gauge}(C,n) \circ (i,k)\}$ and $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n)) = \{\operatorname{Gauge}(C,n) \circ (i,j)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k))$ meets $\operatorname{UpperArc} C$.

(60) Let C be a simple closed curve and i, j, k be natural numbers. Suppose that $1 < i$ and $i < \operatorname{len}\operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $n > 0$ and $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k)) \cap \operatorname{UpperArc}\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) = \{\operatorname{Gauge}(C,n) \circ (i,k)\}$ and $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k)) \cap \operatorname{LowerArc}\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) = \{\operatorname{Gauge}(C,n) \circ (i,j)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k))$ meets $\operatorname{LowerArc} C$.

(61) Let C be a simple closed curve and i, j, k be natural numbers. Suppose that $1 < i$ and $i < \operatorname{len}\operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $n > 0$ and $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k)) \cap \operatorname{UpperArc}\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) = \{\operatorname{Gauge}(C,n) \circ (i,k)\}$ and $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k)) \cap \operatorname{LowerArc}\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) = \{\operatorname{Gauge}(C,n) \circ (i,j)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k))$ meets $\operatorname{UpperArc} C$.

(62) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and j be a natural number. Suppose $\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center}\operatorname{Gauge}(C,n+1), j) \in \operatorname{UpperArc}\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n+1))$ and $1 \leqslant j$ and $j \leqslant \operatorname{width}\operatorname{Gauge}(C,n+1)$. Then $\mathcal{L}(\operatorname{Gauge}(C,1) \circ (\operatorname{Center}\operatorname{Gauge}(C,1),1), \operatorname{Gauge}(C,n+1) \circ (\operatorname{Center}\operatorname{Gauge}(C,n+1),j))$ meets $\operatorname{LowerArc}\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n+1))$.

(63) Let C be a simple closed curve and j, k be natural numbers. Suppose that

(i) $1 \leqslant j$,

(ii) $j \leqslant k$,

(iii) $k \leqslant \operatorname{width}\operatorname{Gauge}(C,n+1)$,

(iv) $\mathcal{L}(\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center}\operatorname{Gauge}(C,n+1),j), \operatorname{Gauge}(C,n+1) \circ (\operatorname{Center}\operatorname{Gauge}(C,n+1),k)) \cap \operatorname{UpperArc}\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n+1)) = \{\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center}\operatorname{Gauge}(C,n+1),k)\}$, and

(v) $\mathcal{L}(\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center}\operatorname{Gauge}(C,n+1),j), \operatorname{Gauge}(C,n+1) \circ (\operatorname{Center}\operatorname{Gauge}(C,n+1),k)) \cap \operatorname{LowerArc}\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n+1)) = \{\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center}\operatorname{Gauge}(C,n+1),j)\}$.

Then $\mathcal{L}(\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center}\operatorname{Gauge}(C,n+1),j), \operatorname{Gauge}(C,n+1) \circ (\operatorname{Center}\operatorname{Gauge}(C,n+1),k))$ meets $\operatorname{LowerArc} C$.

(64) Let C be a simple closed curve and j, k be natural numbers. Suppose that

(i) $1 \leqslant j$,

(ii) $j \leqslant k$,

(iii) $k \leqslant \operatorname{width} \operatorname{Gauge}(C, n+1)$,

(iv) $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), j), \operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), k)) \cap \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1)) = \{\operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), k)\}$, and

(v) $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), j), \operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), k)) \cap \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1)) = \{\operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), j)\}$.

Then $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), j), \operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), k))$ meets $\operatorname{UpperArc} C$.

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