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# The Ordering of Points on a Curve. Part IV[1]

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The notation and terminology used in this paper are introduced in the following articles: [19], [21], [22], [2], [3], [10], [20], [13], [14], [18], [6], [17], [5], [11], [1], [7], [8], [4], [9], [16], [12], and [15].

## 1. Preliminaries

For simplicity, we adopt the following rules: $n$ denotes an element of $\mathbb{N}$, $V$ denotes a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, $s$, $s_1$, $s_2$, $t$, $t_1$, $t_2$ denote points of $\mathcal{E}^n_{\mathrm{T}}$, $C$ denotes a simple closed curve, $P$ denotes a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and $a$, $p$, $p_1$, $p_2$, $q$, $q_1$, $q_2$ denote points of $\mathcal{E}^2_{\mathrm{T}}$.

Next we state several propositions:

(1) For all real numbers $a$, $b$ holds $(a-b)^{\mathbf{2}} = (b-a)^{\mathbf{2}}$.

(2) Let $S$, $T$ be non empty topological spaces, $f$ be a map from $S$ into $T$, and $A$ be a subset of $T$. If $f$ is a homeomorphism and $A$ is connected, then $f^{-1}(A)$ is connected.

(3) Let $S$, $T$ be non empty topological structures, $f$ be a map from $S$ into $T$, and $A$ be a subset of $T$. If $f$ is a homeomorphism and $A$ is compact, then $f^{-1}(A)$ is compact.

(4) $\mathrm{proj2}^{\circ}\,\mathrm{NorthHalfline}\,a$ is lower bounded.

(5) $\mathrm{proj2}^{\circ}\,\mathrm{SouthHalfline}\,a$ is upper bounded.

(6) $\mathrm{proj1}^{\circ}\,\mathrm{WestHalfline}\,a$ is upper bounded.

(7) $\mathrm{proj1}^{\circ}\,\mathrm{EastHalfline}\,a$ is lower bounded.

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Let us consider $a$. One can verify the following observations:

* $\mathrm{proj2}^\circ\,\mathrm{NorthHalfline}\,a$ is non empty,
* $\mathrm{proj2}^\circ\,\mathrm{SouthHalfline}\,a$ is non empty,
* $\mathrm{proj1}^\circ\,\mathrm{WestHalfline}\,a$ is non empty, and
* $\mathrm{proj1}^\circ\,\mathrm{EastHalfline}\,a$ is non empty.

Next we state four propositions:

(8) $\inf(\mathrm{proj2}^\circ\,\mathrm{NorthHalfline}\,a) = a_{\mathbf{2}}$.

(9) $\sup(\mathrm{proj2}^\circ\,\mathrm{SouthHalfline}\,a) = a_{\mathbf{2}}$.

(10) $\sup(\mathrm{proj1}^\circ\,\mathrm{WestHalfline}\,a) = a_{\mathbf{1}}$.

(11) $\inf(\mathrm{proj1}^\circ\,\mathrm{EastHalfline}\,a) = a_{\mathbf{1}}$.

Let us consider $a$. One can verify the following observations:

* $\mathrm{NorthHalfline}\,a$ is closed,
* $\mathrm{SouthHalfline}\,a$ is closed,
* $\mathrm{EastHalfline}\,a$ is closed, and
* $\mathrm{WestHalfline}\,a$ is closed.

One can prove the following propositions:

(12) If $a \in \mathrm{BDD}\,P$, then $\mathrm{NorthHalfline}\,a \not\subseteq \mathrm{UBD}\,P$.

(13) If $a \in \mathrm{BDD}\,P$, then $\mathrm{SouthHalfline}\,a \not\subseteq \mathrm{UBD}\,P$.

(14) If $a \in \mathrm{BDD}\,P$, then $\mathrm{EastHalfline}\,a \not\subseteq \mathrm{UBD}\,P$.

(15) If $a \in \mathrm{BDD}\,P$, then $\mathrm{WestHalfline}\,a \not\subseteq \mathrm{UBD}\,P$.

(16) Let $P$ be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and $p_1$, $p_2$, $q$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $P$ is an arc from $p_1$ to $p_2$ and $q \neq p_2$, then $p_2 \notin \mathrm{LSegment}(P, p_1, p_2, q)$.

(17) Let $P$ be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and $p_1$, $p_2$, $q$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $P$ is an arc from $p_1$ to $p_2$ and $q \neq p_1$, then $p_1 \notin \mathrm{RSegment}(P, p_1, p_2, q)$.

(18) Let $C$ be a simple closed curve, $P$ be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P$ is an arc from $p_1$ to $p_2$ and $P \subseteq C$. Then there exists a non empty subset $R$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $R$ is an arc from $p_1$ to $p_2$ and $P \cup R = C$ and $P \cap R = \{p_1, p_2\}$.

(19) Let $P$ be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and $p_1$, $p_2$, $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P$ is an arc from $p_1$ to $p_2$ and $q_1 \in P$ and $q_2 \in P$ and $q_1 \neq p_1$ and $q_1 \neq p_2$ and $q_2 \neq p_1$ and $q_2 \neq p_2$ and $q_1 \neq q_2$. Then there exists a non empty subset $Q$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $Q$ is an arc from $q_1$ to $q_2$ and $Q \subseteq P$ and $Q$ misses $\{p_1, p_2\}$.

## 2. Two Special Points on a Simple Closed Curve

Let us consider $p$, $P$. The functor $\mathrm{North\text{-}Bound}(p, P)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 1) North-Bound$(p, P) = [p_{\mathbf{1}}, \inf(\text{proj2}^{\circ}(P \cap \text{NorthHalfline}\, p))]$.

The functor South-Bound$(p, P)$ yields a point of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 2) South-Bound$(p, P) = [p_{\mathbf{1}}, \sup(\text{proj2}^{\circ}(P \cap \text{SouthHalfline}\, p))]$.

One can prove the following propositions:

(20) (North-Bound$(p, P))_{\mathbf{1}} = p_{\mathbf{1}}$ and (South-Bound$(p, P))_{\mathbf{1}} = p_{\mathbf{1}}$.

(21) (North-Bound$(p, P))_{\mathbf{2}} = \inf(\text{proj2}^{\circ}(P \cap \text{NorthHalfline}\, p))$ and (South-Bound$(p, P))_{\mathbf{2}} = \sup(\text{proj2}^{\circ}(P \cap \text{SouthHalfline}\, p))$.

(22) For every compact subset $C$ of $\mathcal{E}^2_{\rm T}$ such that $p \in \text{BDD}\, C$ holds North-Bound$(p, C) \in C$ and North-Bound$(p, C) \in$ NorthHalfline $p$ and South-Bound$(p, C) \in C$ and South-Bound$(p, C) \in$ SouthHalfline $p$.

(23) For every compact subset $C$ of $\mathcal{E}^2_{\rm T}$ such that $p \in \text{BDD}\, C$ holds (South-Bound$(p, C))_{\mathbf{2}} < p_{\mathbf{2}}$ and $p_{\mathbf{2}} <$ (North-Bound$(p, C))_{\mathbf{2}}$.

(24) For every compact subset $C$ of $\mathcal{E}^2_{\rm T}$ such that $p \in \text{BDD}\, C$ holds $\inf(\text{proj2}^{\circ}(C \cap \text{NorthHalfline}\, p)) > \sup(\text{proj2}^{\circ}(C \cap \text{SouthHalfline}\, p))$.

(25) For every compact subset $C$ of $\mathcal{E}^2_{\rm T}$ such that $p \in \text{BDD}\, C$ holds South-Bound$(p, C) \neq$ North-Bound$(p, C)$.

(26) For every subset $C$ of the carrier of $\mathcal{E}^2_{\rm T}$ holds $\mathcal{L}$(North-Bound$(p, C)$, South-Bound$(p, C))$ is vertical.

(27) For every compact subset $C$ of $\mathcal{E}^2_{\rm T}$ such that $p \in \text{BDD}\, C$ holds $\mathcal{L}$(North-Bound$(p, C)$, South-Bound$(p, C)) \cap C =$ {North-Bound$(p, C)$, South-Bound$(p, C)$}.

(28) Let $C$ be a compact subset of $\mathcal{E}^2_{\rm T}$. Suppose $p \in \text{BDD}\, C$ and $q \in \text{BDD}\, C$ and $p_{\mathbf{1}} \neq q_{\mathbf{1}}$. Then North-Bound$(p, C)$, South-Bound$(q, C)$, North-Bound$(q, C)$, South-Bound$(p, C)$ are mutually different.

## 3. An Order of Points on a Simple Closed Curve

Let us consider $n$, $V$, $s_1$, $s_2$, $t_1$, $t_2$. We say that $s_1$, $s_2$ separate $t_1$, $t_2$ on $V$ if and only if:

(Def. 3) For every subset $A$ of the carrier of $\mathcal{E}^n_{\rm T}$ such that $A$ is an arc from $s_1$ to $s_2$ and $A \subseteq V$ holds $A$ meets $\{t_1, t_2\}$.

We introduce $s_1$, $s_2$ are neighbours wrt $t_1$, $t_2$ on $V$ as an antonym of $s_1$, $s_2$ separate $t_1$, $t_2$ on $V$.

We now state a number of propositions:

(29) $t$, $t$ separate $s_1$, $s_2$ on $V$.

(30) If $s_1$, $s_2$ separate $t_1$, $t_2$ on $V$, then $s_2$, $s_1$ separate $t_1$, $t_2$ on $V$.

(31) If $s_1$, $s_2$ separate $t_1$, $t_2$ on $V$, then $s_1$, $s_2$ separate $t_2$, $t_1$ on $V$.

(32) $s$, $t_1$ separate $s$, $t_2$ on $V$.

(33) $t_1$, $s$ separate $t_2$, $s$ on $V$.

(34) $t_1$, $s$ separate $s$, $t_2$ on $V$.

(35) $s$, $t_1$ separate $t_2$, $s$ on $V$.

(36) Let $p_1$, $p_2$, $q$ be points of $\mathcal{E}^2_{\rm T}$. Suppose $q \in C$ and $p_1 \in C$ and $p_2 \in C$ and $p_1 \neq p_2$ and $p_1 \neq q$ and $p_2 \neq q$. Then $p_1$, $p_2$ are neighbours wrt $q$, $q$ on $C$.

(37) If $p_1 \neq p_2$ and $p_1 \in C$ and $p_2 \in C$, then if $p_1$, $p_2$ separate $q_1$, $q_2$ on $C$, then $q_1$, $q_2$ separate $p_1$, $p_2$ on $C$.

(38) Suppose $p_1 \in C$ and $p_2 \in C$ and $q_1 \in C$ and $p_1 \neq p_2$ and $q_1 \neq p_1$ and $q_1 \neq p_2$ and $q_2 \neq p_1$ and $q_2 \neq p_2$. Then $p_1$, $p_2$ are neighbours wrt $q_1$, $q_2$ on $C$ or $p_1$, $q_1$ are neighbours wrt $p_2$, $q_2$ on $C$.

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