FORMALIZED MATHEMATICS
Volume 11, Number 1, 2003
University of Białystok

Hermitan Functionals. Canonical Construction of Scalar Product in Quotient Vector Space[1]

Jarosław Kotowicz
University of Białystok

Summary. In the article we present antilinear functionals, sesquilinear and hermitan forms. We prove Schwarz and Minkowski inequalities, and Parallelogram Law for non-negative hermitan form. The proof of Schwarz inequality is based on [14]. The incorrect proof of this fact can be found in [11]. The construction of scalar product in quotient vector space from non-negative hermitan functions is the main result of the article.

MML Identifier: `HERMITAN`.

The notation and terminology used in this paper have been introduced in the following articles: [16], [5], [20], [6], [15], [3], [1], [19], [10], [21], [4], [17], [2], [7], [18], [12], [13], [9], and [8].

1. Auxiliary Facts about Complex Numbers

The following propositions are true:

(1) For every element a of $\mathbb{C}$ such that $a = \overline{a}$ holds $\Im(a) = 0$.

(2) For every element a of $\mathbb{C}$ such that $a \neq 0_{\mathbb{C}}$ holds $|\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i| = 1$ and $\Re((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i) \cdot a) = |a|$ and $\Im((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i) \cdot a) = 0$.

(3) For every element a of $\mathbb{C}$ there exists an element b of $\mathbb{C}$ such that $|b| = 1$ and $\Re(b \cdot a) = |a|$ and $\Im(b \cdot a) = 0$.

(4) For every element a of $\mathbb{C}$ holds $a \cdot \overline{a} = |a|^{\mathbf{2}} + 0i$.

[1]This work has been partially supported by TRIAL-SOLUTION grant IST-2001-35447 and SPUB-M grant of KBN.

ISSN 1426–2630

(5) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $a = \overline{a}$ holds $\Im(a) = 0$.

(6) $\overline{i_{\mathbb{C}_{\mathrm{F}}}} = (i)^{-1}$.

(7) $i_{\mathbb{C}_{\mathrm{F}}} \cdot \overline{i_{\mathbb{C}_{\mathrm{F}}}} = \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}$.

(8) Let a be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. Suppose $a \neq 0_{\mathbb{C}_{\mathrm{F}}}$. Then $|\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|} i_{\mathbb{C}_{\mathrm{F}}}| = 1$ and $\Re((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|} i_{\mathbb{C}_{\mathrm{F}}}) \cdot a) = |a|$ and $\Im((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|} i_{\mathbb{C}_{\mathrm{F}}}) \cdot a) = 0$.

(9) Let a be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. Then there exists an element b of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $|b| = 1$ and $\Re(b \cdot a) = |a|$ and $\Im(b \cdot a) = 0$.

(10) For all elements a, b of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(a - b) = \Re(a) - \Re(b)$ and $\Im(a - b) = \Im(a) - \Im(b)$.

(11) For all elements a, b of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(a) = 0$ holds $\Re(a \cdot b) = \Re(a) \cdot \Re(b)$ and $\Im(a \cdot b) = \Re(a) \cdot \Im(b)$.

(12) For all elements a, b of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(a) = 0$ and $\Im(b) = 0$ holds $\Im(a \cdot b) = 0$.

(13) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(a) = \Re(\overline{a})$.

(14) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(a) = 0$ holds $a = \overline{a}$.

(15) For all real numbers r, s holds $(r + 0 i_{\mathbb{C}_{\mathrm{F}}}) \cdot (s + 0 i_{\mathbb{C}_{\mathrm{F}}}) = r \cdot s + 0 i_{\mathbb{C}_{\mathrm{F}}}$.

(16) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $a \cdot \overline{a} = |a|^{\mathbf{2}} + 0 i_{\mathbb{C}_{\mathrm{F}}}$.

(17) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $0 \leqslant \Re(a)$ and $\Im(a) = 0$ holds $|a| = \Re(a)$.

(18) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(a) + \Re(\overline{a}) = 2 \cdot \Re(a)$.

2. Antilinear Functionals in Complex Vector Spaces

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f be a functional in V. We say that f is complex-homogeneous if and only if:

(Def. 1) For every vector v of V and for every scalar a of V holds $f(a \cdot v) = \overline{a} \cdot f(v)$.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that 0Functional V is complex-homogeneous.

Let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can verify that every functional in V which is complex-homogeneous is also 0-preserving.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can check that there exists a functional in V which is additive, complex-homogeneous, and 0-preserving.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. An antilinear functional of V is an additive complex-homogeneous functional in V.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f, g be complex-homogeneous functionals in V. Observe that $f + g$ is complex-homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f be a complex-homogeneous functional in V. One can verify that $-f$ is complex-homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$, let a be a scalar of V, and let f be a complex-homogeneous functional in V. One can verify that $a \cdot f$ is complex-homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f, g be complex-homogeneous functionals in V. One can check that $f - g$ is complex-homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f be a functional in V. The functor $\overline{f}$ yields a functional in V and is defined by:

(Def. 2) For every vector v of V holds $\overline{f}(v) = \overline{f(v)}$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f be an additive functional in V. Note that $\overline{f}$ is additive.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f be a homogeneous functional in V. Note that $\overline{f}$ is complex-homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f be a complex-homogeneous functional in V. Note that $\overline{f}$ is homogeneous.

Let V be a non trivial vector space over $\mathbb{C}_{\rm F}$ and let f be a non constant functional in V. One can check that $\overline{f}$ is non constant.

Let V be a non trivial vector space over $\mathbb{C}_{\rm F}$. One can check that there exists a functional in V which is additive, complex-homogeneous, non constant, and non trivial.

The following propositions are true:

(19) For every non empty vector space structure V over $\mathbb{C}_{\rm F}$ and for every functional f in V holds $\overline{\overline{f}} = f$.

(20) For every non empty vector space structure V over $\mathbb{C}_{\rm F}$ holds $\overline{\text{0Functional}\,V} = \text{0Functional}\,V$.

(21) For every non empty vector space structure V over $\mathbb{C}_{\rm F}$ and for all functionals f, g in V holds $\overline{f+g} = \overline{f} + \overline{g}$.

(22) For every non empty vector space structure V over $\mathbb{C}_{\rm F}$ and for every functional f in V holds $\overline{-f} = -\overline{f}$.

(23) Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$, f be a functional in V, and a be a scalar of V. Then $\overline{a \cdot f} = \overline{a} \cdot \overline{f}$.

(24) For every non empty vector space structure V over $\mathbb{C}_{\rm F}$ and for all functionals f, g in V holds $\overline{f-g} = \overline{f} - \overline{g}$.

(25) Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$, f be a functional in V, and v be a vector of V. Then $f(v) = 0_{\mathbb{C}_{\rm F}}$ if and only if $\overline{f}(v) = 0_{\mathbb{C}_{\rm F}}$.

(26) For every non empty vector space structure V over $\mathbb{C}_{\rm F}$ and for every functional f in V holds $\ker f = \ker \overline{f}$.

(27) Let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and f be an antilinear functional of V. Then $\ker f$ is linearly closed.

(28) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, W be a subspace of V, and f be an antilinear functional of V. If the carrier of $W \subseteq \ker \overline{f}$, then ${}^{f}/_{W}$ is complex-homogeneous.

Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$ and let f be an antilinear functional of V. The functor $\mathrm{QcFunctional}\, f$ yields an antilinear functional of ${}^{V}/_{\mathrm{Ker}\,\overline{f}}$ and is defined as follows:

(Def. 3) $\mathrm{QcFunctional}\, f = {}^{f}/_{\mathrm{Ker}\,\overline{f}}$.

We now state the proposition

(29) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, f be an antilinear functional of V, A be a vector of ${}^{V}/_{\mathrm{Ker}\,\overline{f}}$, and v be a vector of V. If $A = v + \mathrm{Ker}\,\overline{f}$, then $(\mathrm{QcFunctional}\, f)(A) = f(v)$.

Let V be a non trivial vector space over $\mathbb{C}_{\mathrm{F}}$ and let f be a non constant antilinear functional of V. One can check that $\mathrm{QcFunctional}\, f$ is non constant.

Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$ and let f be an antilinear functional of V. Observe that $\mathrm{QcFunctional}\, f$ is non degenerated.

3. Sesquilinear Forms in Complex Vector Spaces

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$ and let f be a form of V, W. We say that f is complex-homogeneous wrt. second argument if and only if:

(Def. 4) For every vector v of V holds $f(v, \cdot)$ is complex-homogeneous.

We now state the proposition

(30) Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$, v be a vector of V, w be a vector of W, a be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$, and f be a form of V, W. Suppose f is complex-homogeneous wrt. second argument. Then $f(\langle v, a \cdot w\rangle) = \overline{a} \cdot f(\langle v, w\rangle)$.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f be a form of V, V. We say that f is hermitan if and only if:

(Def. 5) For all vectors v, u of V holds $f(\langle v, u\rangle) = \overline{f(\langle u, v\rangle)}$.

We say that f is diagonal real valued if and only if:

(Def. 6) For every vector v of V holds $\Im(f(\langle v, v\rangle)) = 0$.

We say that f is diagonal plus-real valued if and only if:

(Def. 7) For every vector v of V holds $0 \leqslant \Re(f(\langle v, v\rangle))$.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$. Observe that $\mathrm{NulForm}(V, W)$ is complex-homogeneous wrt. second argument.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that NulForm(V, V) is hermitan and NulForm(V, V) is diagonal plus-real valued.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that every form of V, V which is hermitan is also diagonal real valued.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can check that there exists a form of V, V which is diagonal plus-real valued, hermitan, diagonal real valued, additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, and complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$. One can check that there exists a form of V, W which is additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, and complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$. A sesquilinear form of V, W is an additive wrt. first argument homogeneous wrt. first argument additive wrt. second argument complex-homogeneous wrt. second argument form of V, W.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can check that every form of V, V which is hermitan and additive wrt. second argument is also additive wrt. first argument.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that every form of V, V which is hermitan and additive wrt. first argument is also additive wrt. second argument.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that every form of V, V which is hermitan and homogeneous wrt. first argument is also complex-homogeneous wrt. second argument.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Note that every form of V, V which is hermitan and complex-homogeneous wrt. second argument is also homogeneous wrt. first argument.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. A hermitan form of V is a hermitan additive wrt. first argument homogeneous wrt. first argument form of V, V.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$, let f be a functional in V, and let g be a complex-homogeneous functional in W. Note that $f \otimes g$ is complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$, let f be a complex-homogeneous wrt. second argument form of V, W, and let v be a vector of V. One can verify that $f(v, \cdot)$ is complex-homogeneous.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$ and let f, g be complex-homogeneous wrt. second argument forms of V, W. One can verify that $f + g$ is complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$, let f be a complex-homogeneous wrt. second argument form of V, W, and let a be a scalar of V. Observe that $a \cdot f$ is complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let f be a complex-homogeneous wrt. second argument form of V, W. One can check that $-f$ is complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let f, g be complex-homogeneous wrt. second argument forms of V, W. Observe that $f - g$ is complex-homogeneous wrt. second argument.

Let V, W be non trivial vector spaces over $\mathbb{C}_{\rm F}$. Observe that there exists a form of V, W which is additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, complex-homogeneous wrt. second argument, non constant, and non trivial.

Let V, W be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let f be a form of V, W. The functor $\overline{f}$ yielding a form of V, W is defined by:

(Def. 8) For every vector v of V and for every vector w of W holds $\overline{f}(\langle v, w\rangle) = \overline{f(\langle v, w\rangle)}$.

Let V, W be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let f be an additive wrt. second argument form of V, W. Note that $\overline{f}$ is additive wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let f be an additive wrt. first argument form of V, W. Note that $\overline{f}$ is additive wrt. first argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let f be a homogeneous wrt. second argument form of V, W. One can check that $\overline{f}$ is complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let f be a complex-homogeneous wrt. second argument form of V, W. Note that $\overline{f}$ is homogeneous wrt. second argument.

Let V, W be non trivial vector spaces over $\mathbb{C}_{\rm F}$ and let f be a non constant form of V, W. One can verify that $\overline{f}$ is non constant.

The following proposition is true

(31) Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$, f be a functional in V, and v be a vector of V. Then $f \otimes \overline{f}(\langle v, v\rangle) = |f(v)|^{\mathbf{2}} + 0i_{\mathbb{C}_{\rm F}}$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f be a functional in V. One can verify that $f \otimes \overline{f}$ is diagonal plus-real valued, hermitan, and diagonal real valued.

Let V be a non trivial vector space over $\mathbb{C}_{\rm F}$. Note that there exists a form of V, V which is diagonal plus-real valued, hermitan, diagonal real valued, additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, complex-homogeneous wrt. second argument, non constant, and non trivial.

We now state a number of propositions:

(32) For all non empty vector space structures V, W over $\mathbb{C}_{\rm F}$ and for every form f of V, W holds $\overline{\overline{f}} = f$.

(33) For all non empty vector space structures V, W over $\mathbb{C}_{\rm F}$ holds $\overline{\mathrm{NulForm}(V,W)} = \mathrm{NulForm}(V,W)$.

(34) For all non empty vector space structures V, W over $\mathbb{C}_{\rm F}$ and for all forms f, g of V, W holds $\overline{f+g} = \overline{f} + \overline{g}$.

(35) For all non empty vector space structures V, W over $\mathbb{C}_{\rm F}$ and for every form f of V, W holds $\overline{-f} = -\overline{f}$.

(36) Let V, W be non empty vector space structures over $\mathbb{C}_{\rm F}$, f be a form of V, W, and a be an element of $\mathbb{C}_{\rm F}$. Then $\overline{a \cdot f} = \overline{a} \cdot \overline{f}$.

(37) For all non empty vector space structures V, W over $\mathbb{C}_{\rm F}$ and for all forms f, g of V, W holds $\overline{f-g} = \overline{f} - \overline{g}$.

(38) Let V, W be vector spaces over $\mathbb{C}_{\rm F}$, v be a vector of V, w, t be vectors of W, and f be an additive wrt. second argument complex-homogeneous wrt. second argument form of V, W. Then $f(\langle v, w-t\rangle) = f(\langle v, w\rangle) - f(\langle v, t\rangle)$.

(39) Let V, W be vector spaces over $\mathbb{C}_{\rm F}$, v, u be vectors of V, w, t be vectors of W, and f be a sesquilinear form of V, W. Then $f(\langle v-u, w-t\rangle) = f(\langle v, w\rangle) - f(\langle v, t\rangle) - (f(\langle u, w\rangle) - f(\langle u, t\rangle))$.

(40) Let V, W be add-associative right zeroed right complementable vector space-like non empty vector space structures over $\mathbb{C}_{\rm F}$, v, u be vectors of V, w, t be vectors of W, a, b be elements of the carrier of $\mathbb{C}_{\rm F}$, and f be a sesquilinear form of V, W. Then $f(\langle v + a\cdot u, w + b\cdot t\rangle) = f(\langle v, w\rangle) + \overline{b}\cdot f(\langle v, t\rangle) + (a\cdot f(\langle u, w\rangle) + a\cdot(\overline{b}\cdot f(\langle u, t\rangle)))$.

(41) Let V, W be vector spaces over $\mathbb{C}_{\rm F}$, v, u be vectors of V, w, t be vectors of W, a, b be elements of the carrier of $\mathbb{C}_{\rm F}$, and f be a sesquilinear form of V, W. Then $f(\langle v-a\cdot u, w-b\cdot t\rangle) = f(\langle v, w\rangle) - \overline{b}\cdot f(\langle v, t\rangle) - (a\cdot f(\langle u, w\rangle) - a\cdot(\overline{b}\cdot f(\langle u, t\rangle)))$.

(42) Let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\rm F}$, f be a complex-homogeneous wrt. second argument form of V, V, and v be a vector of V. Then $f(\langle v, 0_V\rangle) = 0_{\mathbb{C}_{\rm F}}$.

(43) Let V be a vector space over $\mathbb{C}_{\rm F}$, v, w be vectors of V, and f be a hermitan form of V. Then $f(\langle v, w\rangle)+f(\langle v, w\rangle)+f(\langle v, w\rangle)+f(\langle v, w\rangle) = ((f(\langle v+w, v+w\rangle) - f(\langle v-w, v-w\rangle)) + i_{\mathbb{C}_{\rm F}}\cdot f(\langle v+i_{\mathbb{C}_{\rm F}}\cdot w, v+i_{\mathbb{C}_{\rm F}}\cdot w\rangle)) - i_{\mathbb{C}_{\rm F}}\cdot f(\langle v-i_{\mathbb{C}_{\rm F}}\cdot w, v-i_{\mathbb{C}_{\rm F}}\cdot w\rangle)$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$, let f be a form of V, V, and let v be a vector of V. The functor $||v||_f^2$ yields a real number and is defined as follows:

(Def. 9) $||v||_f^2 = \Re(f(\langle v, v\rangle))$.

The following propositions are true:

(44) Let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\rm F}$, f be a diagonal plus-real valued diagonal real valued form of V, V, and v be a vector of V. Then $|f(\langle v, v\rangle)| = \Re(f(\langle v, v\rangle))$ and $||v||_f^2 = |f(\langle v, v\rangle)|$.

(45) Let V be a vector space over $\mathbb{C}_{\rm F}$, v, w be vectors of V, f be a sesquilinear form of V, V, r be a real number, and a be an element of the carrier of $\mathbb{C}_{\rm F}$. Suppose $|a| = 1$ and $\Re(a \cdot f(\langle w, v\rangle)) = |f(\langle w, v\rangle)|$ and $\Im(a \cdot f(\langle w, v\rangle)) = 0$. Then $f(\langle v - (r + 0i_{\mathbb{C}_{\rm F}}) \cdot a \cdot w, v - (r + 0i_{\mathbb{C}_{\rm F}}) \cdot a \cdot w\rangle) = (f(\langle v, v\rangle) - (r+0i_{\mathbb{C}_{\rm F}}) \cdot (a \cdot f(\langle w, v\rangle)) - (r+0i_{\mathbb{C}_{\rm F}}) \cdot (\overline{a} \cdot f(\langle v, w\rangle))) + (r^{\mathbf{2}} + 0i_{\mathbb{C}_{\rm F}}) \cdot f(\langle w, w\rangle)$.

(46) Let V be a vector space over $\mathbb{C}_{\rm F}$, v, w be vectors of V, f be a diagonal plus-real valued hermitan form of V, r be a real number, and a be an element of the carrier of $\mathbb{C}_{\rm F}$. Suppose $|a| = 1$ and $\Re(a \cdot f(\langle w, v\rangle)) = |f(\langle w, v\rangle)|$ and $\Im(a \cdot f(\langle w, v\rangle)) = 0$. Then $\Re(f(\langle v - (r + 0i_{\mathbb{C}_{\rm F}}) \cdot a \cdot w, v - (r + 0i_{\mathbb{C}_{\rm F}}) \cdot a \cdot w\rangle)) = (||v||_f^2 - 2 \cdot |f(\langle w, v\rangle)| \cdot r) + ||w||_f^2 \cdot r^{\mathbf{2}}$ and $0 \leqslant (||v||_f^2 - 2 \cdot |f(\langle w, v\rangle)| \cdot r) + ||w||_f^2 \cdot r^{\mathbf{2}}$.

(47) Let V be a vector space over $\mathbb{C}_{\rm F}$, v, w be vectors of V, and f be a diagonal plus-real valued hermitan form of V. If $||w||_f^2 = 0$, then $|f(\langle w, v\rangle)| = 0$.

(48) Let V be a vector space over $\mathbb{C}_{\rm F}$, v, w be vectors of V, and f be a diagonal plus-real valued hermitan form of V. Then $|f(\langle v, w\rangle)|^{\mathbf{2}} \leqslant ||v||_f^2 \cdot ||w||_f^2$.

(49) Let V be a vector space over $\mathbb{C}_{\rm F}$, f be a diagonal plus-real valued hermitan form of V, and v, w be vectors of V. Then $|f(\langle v, w\rangle)|^{\mathbf{2}} \leqslant |f(\langle v, v\rangle)| \cdot |f(\langle w, w\rangle)|$.

(50) Let V be a vector space over $\mathbb{C}_{\rm F}$, f be a diagonal plus-real valued hermitan form of V, and v, w be vectors of V. Then $||v + w||_f^2 \leqslant (\sqrt{||v||_f^2} + \sqrt{||w||_f^2})^{\mathbf{2}}$.

(51) Let V be a vector space over $\mathbb{C}_{\rm F}$, f be a diagonal plus-real valued hermitan form of V, and v, w be vectors of V. Then $|f(\langle v + w, v + w\rangle)| \leqslant (\sqrt{|f(\langle v, v\rangle)|} + \sqrt{|f(\langle w, w\rangle)|})^{\mathbf{2}}$.

(52) Let V be a vector space over $\mathbb{C}_{\rm F}$, f be a hermitan form of V, and v, w be elements of the carrier of V. Then $||v+w||_f^2 + ||v-w||_f^2 = 2 \cdot ||v||_f^2 + 2 \cdot ||w||_f^2$.

(53) Let V be a vector space over $\mathbb{C}_{\rm F}$, f be a diagonal plus-real valued hermitan form of V, and v, w be elements of the carrier of V. Then $|f(\langle v + w, v + w\rangle)| + |f(\langle v - w, v - w\rangle)| = 2 \cdot |f(\langle v, v\rangle)| + 2 \cdot |f(\langle w, w\rangle)|$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f be a form of V, V. The functor $||\cdot||_f$ yields a RFunctional of V and is defined as follows:

(Def. 10) For every element v of the carrier of V holds $(||\cdot||_f)(v) = \sqrt{||v||_f^2}$.

Let V be a vector space over $\mathbb{C}_{\rm F}$ and let f be a diagonal plus-real valued hermitan form of V. Then $||\cdot||_f$ is a Semi-Norm of V.

4. Kernel of Hermitan Forms and Hermitan Forms in Quotient Vector Spaces

Let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f be a complex-homogeneous wrt. second argument form of V, V. Note that diagker f is non empty.

We now state several propositions:

(54) Let V be a vector space over $\mathbb{C}_{\rm F}$ and f be a diagonal plus-real valued hermitan form of V. Then diagker f is linearly closed.

(55) For every vector space V over $\mathbb{C}_{\rm F}$ and for every diagonal plus-real valued hermitan form f of V holds diagker f = leftker f.

(56) For every vector space V over $\mathbb{C}_{\rm F}$ and for every diagonal plus-real valued hermitan form f of V holds diagker f = rightker f.

(57) For every non empty vector space structure V over $\mathbb{C}_{\rm F}$ and for every form f of V, V holds diagker f = diagker $\overline{f}$.

(58) For all non empty vector space structures V, W over $\mathbb{C}_{\rm F}$ and for every form f of V, W holds leftker f = leftker $\overline{f}$ and rightker f = rightker $\overline{f}$.

(59) For every vector space V over $\mathbb{C}_{\rm F}$ and for every diagonal plus-real valued hermitan form f of V holds LKer f = RKer $\overline{f}$.

(60) Let V be a vector space over $\mathbb{C}_{\rm F}$, f be a diagonal plus-real valued diagonal real valued form of V, V, and v be a vector of V. If $\Re(f(\langle v, v\rangle)) = 0$, then $f(\langle v, v\rangle) = 0_{\mathbb{C}_{\rm F}}$.

(61) Let V be a vector space over $\mathbb{C}_{\rm F}$, f be a diagonal plus-real valued hermitan form of V, and v be a vector of V. Suppose $\Re(f(\langle v, v\rangle)) = 0$ and f is non degenerated on left and non degenerated on right. Then $v = 0_V$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$, let W be a vector space over $\mathbb{C}_{\rm F}$, and let f be an additive wrt. second argument complex-homogeneous wrt. second argument form of V, W. The functor RQForm$^*(f)$ yielding an additive wrt. second argument complex-homogeneous wrt. second argument form of V, ${}^{W}/_{\mathrm{RKer}\,\overline{f}}$ is defined as follows:

(Def. 11) $\mathrm{RQForm}^*(f) = \overline{\mathrm{RQForm}(\overline{f})}$.

We now state the proposition

(62) Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$, W be a vector space over $\mathbb{C}_{\rm F}$, f be an additive wrt. second argument complex-

homogeneous wrt. second argument form of V, W, v be a vector of V, and w be a vector of W. Then $(\mathrm{RQForm}^*(f))(\langle v, w + \mathrm{RKer}\,\overline{f}\rangle) = f(\langle v, w\rangle)$.

Let V, W be vector spaces over $\mathbb{C}_{\mathrm{F}}$ and let f be a sesquilinear form of V, W. Note that $\mathrm{LQForm}(f)$ is additive wrt. second argument and complex-homogeneous wrt. second argument and $\mathrm{RQForm}^*(f)$ is additive wrt. first argument and homogeneous wrt. first argument.

Let V, W be vector spaces over $\mathbb{C}_{\mathrm{F}}$ and let f be a sesquilinear form of V, W. The functor $\mathrm{QForm}^*\, f$ yields a sesquilinear form of ${}^{V}/_{\mathrm{LKer}\,f}$, ${}^{W}/_{\mathrm{RKer}\,\overline{f}}$ and is defined by the condition (Def. 12).

(Def. 12) Let A be a vector of ${}^{V}/_{\mathrm{LKer}\,f}$, B be a vector of ${}^{W}/_{\mathrm{RKer}\,\overline{f}}$, v be a vector of V, and w be a vector of W. If $A = v + \mathrm{LKer}\,f$ and $B = w + \mathrm{RKer}\,\overline{f}$, then $(\mathrm{QForm}^*\, f)(\langle A, B\rangle) = f(\langle v, w\rangle)$.

Let V, W be non trivial vector spaces over $\mathbb{C}_{\mathrm{F}}$ and let f be a non constant sesquilinear form of V, W. Observe that $\mathrm{QForm}^*\, f$ is non constant.

Let V be a right zeroed non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, let W be a vector space over $\mathbb{C}_{\mathrm{F}}$, and let f be an additive wrt. second argument complex-homogeneous wrt. second argument form of V, W. One can verify that $\mathrm{RQForm}^*(f)$ is non degenerated on right.

One can prove the following propositions:

(63) Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, W be a vector space over $\mathbb{C}_{\mathrm{F}}$, and f be an additive wrt. second argument complex-homogeneous wrt. second argument form of V, W. Then $\mathrm{leftker}\, f = \mathrm{leftker}(\mathrm{RQForm}^*(f))$.

(64) For all vector spaces V, W over $\mathbb{C}_{\mathrm{F}}$ and for every sesquilinear form f of V, W holds $\mathrm{RKer}\,\overline{f} = \mathrm{RKer}\,\overline{\mathrm{LQForm}(f)}$.

(65) For all vector spaces V, W over $\mathbb{C}_{\mathrm{F}}$ and for every sesquilinear form f of V, W holds $\mathrm{LKer}\, f = \mathrm{LKer}(\mathrm{RQForm}^*(f))$.

(66) For all vector spaces V, W over $\mathbb{C}_{\mathrm{F}}$ and for every sesquilinear form f of V, W holds $\mathrm{QForm}^*\, f = \mathrm{RQForm}^*(\mathrm{LQForm}(f))$ and $\mathrm{QForm}^*\, f = \mathrm{LQForm}(\mathrm{RQForm}^*(f))$.

(67) Let V, W be vector spaces over $\mathbb{C}_{\mathrm{F}}$ and f be a sesquilinear form of V, W. Then $\mathrm{leftker}(\mathrm{QForm}^*\, f) = \mathrm{leftker}(\mathrm{RQForm}^*(\mathrm{LQForm}(f)))$ and $\mathrm{rightker}(\mathrm{QForm}^*\, f) = \mathrm{rightker}(\mathrm{RQForm}^*(\mathrm{LQForm}(f)))$ and $\mathrm{leftker}(\mathrm{QForm}^*\, f) = \mathrm{leftker}(\mathrm{LQForm}(\mathrm{RQForm}^*(f)))$ and $\mathrm{rightker}(\mathrm{QForm}^*\, f) = \mathrm{rightker}(\mathrm{LQForm}(\mathrm{RQForm}^*(f)))$.

Let V, W be vector spaces over $\mathbb{C}_{\mathrm{F}}$ and let f be a sesquilinear form of V, W. Note that $\mathrm{RQForm}^*(\mathrm{LQForm}(f))$ is non degenerated on left and non degenerated on right and $\mathrm{LQForm}(\mathrm{RQForm}^*(f))$ is non degenerated on left and non degenerated on right.

Let V, W be vector spaces over $\mathbb{C}_{\rm F}$ and let f be a sesquilinear form of V, W. Note that $\mathrm{QForm}^* f$ is non degenerated on left and non degenerated on right.

5. Scalar Product in Quotient Vector Space Generated by Non-Negative Hermitan Form

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f be a form of V, V. We say that f is positive diagonal valued if and only if:

(Def. 13) For every vector v of V such that $v \neq 0_V$ holds $0 < \Re(f(\langle v, v \rangle))$.

Let V be a right zeroed non empty vector space structure over $\mathbb{C}_{\rm F}$. Note that every form of V, V which is positive diagonal valued and additive wrt. first argument is also diagonal plus-real valued.

Let V be a right zeroed non empty vector space structure over $\mathbb{C}_{\rm F}$. One can verify that every form of V, V which is positive diagonal valued and additive wrt. second argument is also diagonal plus-real valued.

Let V be a vector space over $\mathbb{C}_{\rm F}$ and let f be a diagonal plus-real valued hermitan form of V. The functor $\langle\cdot|\cdot\rangle_f$ yields a diagonal plus-real valued hermitan form of $^{V}/_{\mathrm{LKer}\, f}$ and is defined as follows:

(Def. 14) $\langle\cdot|\cdot\rangle_f = \mathrm{QForm}^* f$.

Next we state three propositions:

(68) Let V be a vector space over $\mathbb{C}_{\rm F}$, f be a diagonal plus-real valued hermitan form of V, A, B be vectors of $^{V}/_{\mathrm{LKer}\, f}$, and v, w be vectors of V. If $A = v + \mathrm{LKer}\, f$ and $B = w + \mathrm{LKer}\, f$, then $(\langle\cdot|\cdot\rangle_f)(\langle A, B\rangle) = f(\langle v, w\rangle)$.

(69) For every vector space V over $\mathbb{C}_{\rm F}$ and for every diagonal plus-real valued hermitan form f of V holds $\mathrm{leftker}(\langle\cdot|\cdot\rangle_f) = \mathrm{leftker}(\mathrm{QForm}^* f)$.

(70) For every vector space V over $\mathbb{C}_{\rm F}$ and for every diagonal plus-real valued hermitan form f of V holds $\mathrm{rightker}(\langle\cdot|\cdot\rangle_f) = \mathrm{rightker}(\mathrm{QForm}^* f)$.

Let V be a vector space over $\mathbb{C}_{\rm F}$ and let f be a diagonal plus-real valued hermitan form of V. Observe that $\langle\cdot|\cdot\rangle_f$ is non degenerated on left, non degenerated on right, and positive diagonal valued.

Let V be a non trivial vector space over $\mathbb{C}_{\rm F}$ and let f be a diagonal plus-real valued non constant hermitan form of V. Note that $\langle\cdot|\cdot\rangle_f$ is non constant.

References

[1] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.
[2] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[7] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[8] Jarosław Kotowicz. Bilinear functionals in vector spaces. *Formalized Mathematics*, 11(**1**):69–86, 2003.
[9] Jarosław Kotowicz. Quotient vector spaces and functionals. *Formalized Mathematics*, 11(**1**):59–68, 2003.
[10] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[11] Krzysztof Maurin. *Analiza, II*, volume 70 of *Biblioteka Matematyczna*. PWN - Warszawa, 1991.
[12] Anna Justyna Milewska. The field of complex numbers. *Formalized Mathematics*, 9(**2**):265–269, 2001.
[13] Anna Justyna Milewska. The Hahn Banach theorem in the vector space over the field of complex numbers. *Formalized Mathematics*, 9(**2**):363–371, 2001.
[14] Walter Rudin. *Real and Complex Analysis*. Mc Graw-Hill, Inc., 1974.
[15] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[18] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Formalized Mathematics*, 1(**5**):865–870, 1990.
[19] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[21] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received November 12, 2002