FORMALIZED MATHEMATICS
Volume 11, Number 1, 2003
University of Białystok

Linear Combinations in Real Unitary Space

Noboru Endou
Gifu National College of Technology

Takashi Mitsuishi
Miyagi University

Yasunari Shidama
Shinshu University
Nagano

Summary. In this article, we mainly discuss linear combination of vectors in Real Unitary Space and dimension of the space. As the result, we obtain some theorems that are similar to those in Real Linear Space.

MML Identifier: RUSUB_3.

The articles [11], [5], [16], [2], [17], [1], [3], [4], [15], [10], [6], [14], [13], [9], [12], [8], and [7] provide the terminology and notation for this paper.

1. Definition and Fundamental Properties of Linear Combination

Let V be a real unitary space and let A be a subset of the carrier of V. The functor $\mathrm{Lin}(A)$ yielding a strict subspace of V is defined by:

(Def. 1) The carrier of $\mathrm{Lin}(A) = \{\sum l : l$ ranges over linear combinations of $A\}$.

We now state a number of propositions:

(1) Let V be a real unitary space, A be a subset of the carrier of V, and x be a set. Then $x \in \mathrm{Lin}(A)$ if and only if there exists a linear combination l of A such that $x = \sum l$.

(2) Let V be a real unitary space, A be a subset of the carrier of V, and x be a set. If $x \in A$, then $x \in \mathrm{Lin}(A)$.

(3) For every real unitary space V holds $\mathrm{Lin}(\emptyset_{\text{the carrier of } V}) = \mathbf{0}_V$.

(4) For every real unitary space V and for every subset A of the carrier of V such that $\mathrm{Lin}(A) = \mathbf{0}_V$ holds $A = \emptyset$ or $A = \{0_V\}$.

ISSN 1426–2630

(5) Let V be a real unitary space, W be a strict subspace of V, and A be a subset of the carrier of V. If $A =$ the carrier of W, then $\mathrm{Lin}(A) = W$.

(6) Let V be a strict real unitary space and A be a subset of the carrier of V. If $A =$ the carrier of V, then $\mathrm{Lin}(A) = V$.

(7) Let V be a real unitary space and A, B be subsets of the carrier of V. If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a subspace of $\mathrm{Lin}(B)$.

(8) Let V be a strict real unitary space and A, B be subsets of the carrier of V. If $\mathrm{Lin}(A) = V$ and $A \subseteq B$, then $\mathrm{Lin}(B) = V$.

(9) For every real unitary space V and for all subsets A, B of the carrier of V holds $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(10) For every real unitary space V and for all subsets A, B of the carrier of V holds $\mathrm{Lin}(A \cap B)$ is a subspace of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(11) Let V be a real unitary space and A be a subset of the carrier of V. Suppose A is linearly independent. Then there exists a subset B of the carrier of V such that $A \subseteq B$ and B is linearly independent and $\mathrm{Lin}(B) =$ the unitary space structure of V.

(12) Let V be a real unitary space and A be a subset of the carrier of V. Suppose $\mathrm{Lin}(A) = V$. Then there exists a subset B of the carrier of V such that $B \subseteq A$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

2. Definition of the Basis of Real Unitary Space

Let V be a real unitary space. A subset of the carrier of V is said to be a basis of V if:

(Def. 2) It is linearly independent and Lin(it) = the unitary space structure of V.

One can prove the following three propositions:

(13) Let V be a strict real unitary space and A be a subset of the carrier of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

(14) Let V be a real unitary space and A be a subset of the carrier of V. If $\mathrm{Lin}(A) = V$, then there exists a basis I of V such that $I \subseteq A$.

(15) Let V be a real unitary space and A be a subset of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

3. Some Theorems of Lin, Sum, Carrier

We now state a number of propositions:

(16) Let V be a real unitary space, L be a linear combination of V, A be a subset of V, and F be a finite sequence of elements of the carrier of V. Suppose $\operatorname{rng} F \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum(L\,F) = \sum K$.

(17) Let V be a real unitary space, L be a linear combination of V, and A be a subset of V. Suppose the support of $L \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum L = \sum K$.

(18) Let V be a real unitary space, W be a subspace of V, and L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Let K be a linear combination of W. Suppose $K = L\restriction$the carrier of W. Then the support of $L =$ the support of K and $\sum L = \sum K$.

(19) Let V be a real unitary space, W be a subspace of V, and K be a linear combination of W. Then there exists a linear combination L of V such that the support of $K =$ the support of L and $\sum K = \sum L$.

(20) Let V be a real unitary space, W be a subspace of V, and L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Then there exists a linear combination K of W such that the support of $K =$ the support of L and $\sum K = \sum L$.

(21) For every real unitary space V and for every basis I of V and for every vector v of V holds $v \in \operatorname{Lin}(I)$.

(22) Let V be a real unitary space, W be a subspace of V, and A be a subset of W. Suppose A is linearly independent. Then there exists a subset B of V such that B is linearly independent and $B = A$.

(23) Let V be a real unitary space, W be a subspace of V, and A be a subset of V. Suppose A is linearly independent and $A \subseteq$ the carrier of W. Then there exists a subset B of W such that B is linearly independent and $B = A$.

(24) Let V be a real unitary space, W be a subspace of V, and A be a basis of W. Then there exists a basis B of V such that $A \subseteq B$.

(25) Let V be a real unitary space and A be a subset of V. Suppose A is linearly independent. Let v be a vector of V. If $v \in A$, then for every subset B of V such that $B = A \setminus \{v\}$ holds $v \notin \operatorname{Lin}(B)$.

(26) Let V be a real unitary space, I be a basis of V, and A be a non empty subset of V. Suppose A misses I. Let B be a subset of V. If $B = I \cup A$, then B is linearly dependent.

(27) Let V be a real unitary space, W be a subspace of V, and A be a subset of V. If $A \subseteq$ the carrier of W, then $\operatorname{Lin}(A)$ is a subspace of W.

(28) Let V be a real unitary space, W be a subspace of V, A be a subset of V, and B be a subset of W. If $A = B$, then $\mathrm{Lin}(A) = \mathrm{Lin}(B)$.

4. Subspaces of Real Unitary Space Generated by One, Two, or Three Vectors

We now state a number of propositions:

(29) Let V be a real unitary space, v be a vector of V, and x be a set. Then $x \in \mathrm{Lin}(\{v\})$ if and only if there exists a real number a such that $x = a \cdot v$.

(30) For every real unitary space V and for every vector v of V holds $v \in \mathrm{Lin}(\{v\})$.

(31) Let V be a real unitary space, v, w be vectors of V, and x be a set. Then $x \in v + \mathrm{Lin}(\{w\})$ if and only if there exists a real number a such that $x = v + a \cdot w$.

(32) Let V be a real unitary space, w_1, w_2 be vectors of V, and x be a set. Then $x \in \mathrm{Lin}(\{w_1, w_2\})$ if and only if there exist real numbers a, b such that $x = a \cdot w_1 + b \cdot w_2$.

(33) For every real unitary space V and for all vectors w_1, w_2 of V holds $w_1 \in \mathrm{Lin}(\{w_1, w_2\})$ and $w_2 \in \mathrm{Lin}(\{w_1, w_2\})$.

(34) Let V be a real unitary space, v, w_1, w_2 be vectors of V, and x be a set. Then $x \in v + \mathrm{Lin}(\{w_1, w_2\})$ if and only if there exist real numbers a, b such that $x = v + a \cdot w_1 + b \cdot w_2$.

(35) Let V be a real unitary space, v_1, v_2, v_3 be vectors of V, and x be a set. Then $x \in \mathrm{Lin}(\{v_1, v_2, v_3\})$ if and only if there exist real numbers a, b, c such that $x = a \cdot v_1 + b \cdot v_2 + c \cdot v_3$.

(36) For every real unitary space V and for all vectors w_1, w_2, w_3 of V holds $w_1 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$ and $w_2 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$ and $w_3 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$.

(37) Let V be a real unitary space, v, w_1, w_2, w_3 be vectors of V, and x be a set. Then $x \in v + \mathrm{Lin}(\{w_1, w_2, w_3\})$ if and only if there exist real numbers a, b, c such that $x = v + a \cdot w_1 + b \cdot w_2 + c \cdot w_3$.

(38) For every real unitary space V and for all vectors v, w of V such that $v \in \mathrm{Lin}(\{w\})$ and $v \neq 0_V$ holds $\mathrm{Lin}(\{v\}) = \mathrm{Lin}(\{w\})$.

(39) Let V be a real unitary space and v_1, v_2, w_1, w_2 be vectors of V. Suppose $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent and $v_1 \in \mathrm{Lin}(\{w_1, w_2\})$ and $v_2 \in \mathrm{Lin}(\{w_1, w_2\})$. Then $\mathrm{Lin}(\{w_1, w_2\}) = \mathrm{Lin}(\{v_1, v_2\})$ and $\{w_1, w_2\}$ is linearly independent and $w_1 \neq w_2$.

5. Auxiliary Theorems

We now state several propositions:

(40) For every real unitary space V and for every set x holds $x \in \mathbf{0}_V$ iff $x = 0_V$.

(41) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a subspace of W_3, then $W_1 \cap W_2$ is a subspace of W_3.

(42) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. Suppose W_1 is a subspace of W_2 and a subspace of W_3. Then W_1 is a subspace of $W_2 \cap W_3$.

(43) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. Suppose W_1 is a subspace of W_3 and W_2 is a subspace of W_3. Then W_1+W_2 is a subspace of W_3.

(44) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a subspace of W_2, then W_1 is a subspace of $W_2 + W_3$.

(45) Let V be a real unitary space, W_1, W_2 be subspaces of V, and v be a vector of V. If W_1 is a subspace of W_2, then $v + W_1 \subseteq v + W_2$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[6] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[7] Noboru Endou, Takashi Mitsuishi, and Yasunari Shidama. Operations on subspaces in real unitary space. *Formalized Mathematics*, 11(**1**):9–16, 2003.
[8] Noboru Endou, Takashi Mitsuishi, and Yasunari Shidama. Subspaces and cosets of subspace of real unitary space. *Formalized Mathematics*, 11(**1**):1–7, 2003.
[9] Jan Popiołek. Introduction to Banach and Hilbert spaces - part I. *Formalized Mathematics*, 2(**4**):511–516, 1991.
[10] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[12] Wojciech A. Trybulec. Basis of real linear space. *Formalized Mathematics*, 1(**5**):847–850, 1990.
[13] Wojciech A. Trybulec. Linear combinations in real linear space. *Formalized Mathematics*, 1(**3**):581–588, 1990.
[14] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.

[15] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[17] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received October 9, 2002