FORMALIZED MATHEMATICS
Volume 11, Number 1, 2003
University of Białystok

Dimension of Real Unitary Space

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Summary. In this article we describe the dimension of real unitary space. Most of theorems are restricted from real linear space. In the last section, we introduce affine subset of real unitary space.

MML Identifier: RUSUB_4.

The papers [14], [13], [19], [2], [3], [4], [1], [5], [11], [18], [6], [10], [17], [16], [12], [15], [9], [8], and [7] provide the terminology and notation for this paper.

1. Finite-dimensional Real Unitary Space

One can prove the following two propositions:

(1) Let V be a real unitary space, A, B be finite subsets of V, and v be a vector of V. Suppose $v \in \mathrm{Lin}(A \cup B)$ and $v \notin \mathrm{Lin}(B)$. Then there exists a vector w of V such that $w \in A$ and $w \in \mathrm{Lin}(((A \cup B) \setminus \{w\}) \cup \{v\})$.

(2) Let V be a real unitary space and A, B be finite subsets of V. Suppose the unitary space structure of $V = \mathrm{Lin}(A)$ and B is linearly independent. Then $\overline{\overline{B}} \leqslant \overline{\overline{A}}$ and there exists a finite subset C of V such that $C \subseteq A$ and $\overline{\overline{C}} = \overline{\overline{A}} - \overline{\overline{B}}$ and the unitary space structure of $V = \mathrm{Lin}(B \cup C)$.

Let V be a real unitary space. We say that V is finite dimensional if and only if:

(Def. 1) There exists a finite subset of the carrier of V which is a basis of V.

ISSN 1426–2630

Let us mention that there exists a real unitary space which is strict and finite dimensional.

Let V be a real unitary space. Let us observe that V is finite dimensional if and only if:

(Def. 2) There exists a finite subset of V which is a basis of V.

We now state several propositions:

(3) For every real unitary space V such that V is finite dimensional holds every basis of V is finite.

(4) Let V be a real unitary space and A be a subset of V. Suppose V is finite dimensional and A is linearly independent. Then A is finite.

(5) For every real unitary space V and for all bases A, B of V such that V is finite dimensional holds $\overline{\overline{A}} = \overline{\overline{B}}$.

(6) For every real unitary space V holds $\mathbf{0}_V$ is finite dimensional.

(7) Let V be a real unitary space and W be a subspace of V. If V is finite dimensional, then W is finite dimensional.

Let V be a real unitary space. Note that there exists a subspace of V which is finite dimensional and strict.

Let V be a finite dimensional real unitary space. Observe that every subspace of V is finite dimensional.

Let V be a finite dimensional real unitary space. Observe that there exists a subspace of V which is strict.

2. Dimension of Real Unitary Space

Let V be a real unitary space. Let us assume that V is finite dimensional. The functor $\dim(V)$ yielding a natural number is defined by:

(Def. 3) For every basis I of V holds $\dim(V) = \overline{\overline{I}}$.

One can prove the following propositions:

(8) For every finite dimensional real unitary space V and for every subspace W of V holds $\dim(W) \leqslant \dim(V)$.

(9) Let V be a finite dimensional real unitary space and A be a subset of V. If A is linearly independent, then $\overline{\overline{A}} = \dim(\mathrm{Lin}(A))$.

(10) For every finite dimensional real unitary space V holds $\dim(V) = \dim(\Omega_V)$.

(11) Let V be a finite dimensional real unitary space and W be a subspace of V. Then $\dim(V) = \dim(W)$ if and only if $\Omega_V = \Omega_W$.

(12) For every finite dimensional real unitary space V holds $\dim(V) = 0$ iff $\Omega_V = \mathbf{0}_V$.

(13) Let V be a finite dimensional real unitary space. Then $\dim(V) = 1$ if and only if there exists a vector v of V such that $v \neq 0_V$ and $\Omega_V = \mathrm{Lin}(\{v\})$.

(14) Let V be a finite dimensional real unitary space. Then $\dim(V) = 2$ if and only if there exist vectors u, v of V such that $u \neq v$ and $\{u, v\}$ is linearly independent and $\Omega_V = \mathrm{Lin}(\{u, v\})$.

(15) For every finite dimensional real unitary space V and for all subspaces W_1, W_2 of V holds $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$.

(16) For every finite dimensional real unitary space V and for all subspaces W_1, W_2 of V holds $\dim(W_1 \cap W_2) \geqslant (\dim(W_1) + \dim(W_2)) - \dim(V)$.

(17) Let V be a finite dimensional real unitary space and W_1, W_2 be subspaces of V. If V is the direct sum of W_1 and W_2, then $\dim(V) = \dim(W_1) + \dim(W_2)$.

3. Fixed-dimensional Subspace Family

We now state the proposition

(18) Let V be a finite dimensional real unitary space, W be a subspace of V, and n be a natural number. Then $n \leqslant \dim(V)$ if and only if there exists a strict subspace W of V such that $\dim(W) = n$.

Let V be a finite dimensional real unitary space and let n be a natural number. The functor $\mathrm{Sub}_n(V)$ yields a set and is defined as follows:

(Def. 4) For every set x holds $x \in \mathrm{Sub}_n(V)$ iff there exists a strict subspace W of V such that $W = x$ and $\dim(W) = n$.

Next we state three propositions:

(19) Let V be a finite dimensional real unitary space and n be a natural number. If $n \leqslant \dim(V)$, then $\mathrm{Sub}_n(V)$ is non empty.

(20) For every finite dimensional real unitary space V and for every natural number n such that $\dim(V) < n$ holds $\mathrm{Sub}_n(V) = \emptyset$.

(21) Let V be a finite dimensional real unitary space, W be a subspace of V, and n be a natural number. Then $\mathrm{Sub}_n(W) \subseteq \mathrm{Sub}_n(V)$.

4. Affine Set

Let V be a non empty RLS structure and let S be a subset of V. We say that S is Affine if and only if:

(Def. 5) For all vectors x, y of V and for every real number a such that $x \in S$ and $y \in S$ holds $(1 - a) \cdot x + a \cdot y \in S$.

One can prove the following propositions:

(22) For every non empty RLS structure V holds Ω_V is Affine and $\emptyset_V$ is Affine.

(23) For every real linear space-like non empty RLS structure V and for every vector v of V holds $\{v\}$ is Affine.

Let V be a non empty RLS structure. Observe that there exists a subset of V which is non empty and Affine and there exists a subset of V which is empty and Affine.

Let V be a real linear space and let W be a subspace of V. The functor $\mathrm{Up}(W)$ yielding a non empty subset of V is defined by:

(Def. 6) $\mathrm{Up}(W) =$ the carrier of W.

Let V be a real unitary space and let W be a subspace of V. The functor $\mathrm{Up}(W)$ yielding a non empty subset of V is defined by:

(Def. 7) $\mathrm{Up}(W) =$ the carrier of W.

We now state two propositions:

(24) For every real linear space V and for every subspace W of V holds $\mathrm{Up}(W)$ is Affine and $0_V \in$ the carrier of W.

(25) Let V be a real linear space and A be a Affine subset of V. Suppose $0_V \in A$. Let x be a vector of V and a be a real number. If $x \in A$, then $a \cdot x \in A$.

Let V be a non empty RLS structure and let S be a non empty subset of V. We say that S is Subspace-like if and only if the conditions (Def. 8) are satisfied.

(Def. 8)(i) The zero of $V \in S$, and

(ii) for all elements x, y of the carrier of V and for every real number a such that $x \in S$ and $y \in S$ holds $x + y \in S$ and $a \cdot x \in S$.

One can prove the following propositions:

(26) Let V be a real linear space and A be a non empty Affine subset of V. If $0_V \in A$, then A is Subspace-like and $A =$ the carrier of $\mathrm{Lin}(A)$.

(27) For every real linear space V and for every subspace W of V holds $\mathrm{Up}(W)$ is Subspace-like.

(28) For every real linear space V and for every strict subspace W of V holds $W = \mathrm{Lin}(\mathrm{Up}(W))$.

(29) Let V be a real unitary space and A be a non empty Affine subset of V. If $0_V \in A$, then $A =$ the carrier of $\mathrm{Lin}(A)$.

(30) For every real unitary space V and for every subspace W of V holds $\mathrm{Up}(W)$ is Subspace-like.

(31) For every real unitary space V and for every strict subspace W of V holds $W = \mathrm{Lin}(\mathrm{Up}(W))$.

Let V be a non empty loop structure, let M be a subset of the carrier of V, and let v be an element of the carrier of V. The functor $v + M$ yields a subset of V and is defined as follows:

(Def. 9) $v+M=\{v+u; u$ ranges over elements of the carrier of V: $u \in M\}$.

We now state three propositions:

(32) Let V be a real linear space, W be a strict subspace of V, M be a subset of the carrier of V, and v be a vector of V. If $\mathrm{Up}(W)=M$, then $v+W=v+M$.

(33) Let V be an Abelian add-associative real linear space-like non empty RLS structure, M be a Affine subset of V, and v be a vector of V. Then $v+M$ is Affine.

(34) Let V be a real unitary space, W be a strict subspace of V, M be a subset of the carrier of V, and v be a vector of V. If $\mathrm{Up}(W)=M$, then $v+W=v+M$.

Let V be a non empty loop structure and let M, N be subsets of the carrier of V. The functor $M+N$ yields a subset of V and is defined as follows:

(Def. 10) $M+N=\{u+v; u$ ranges over elements of the carrier of V, v ranges over elements of the carrier of V: $u \in M \ \wedge\ v \in N\}$.

We now state the proposition

(35) For every Abelian non empty loop structure V and for all subsets M, N of V holds $N+M=M+N$.

Let V be an Abelian non empty loop structure and let M, N be subsets of V. Let us observe that the functor $M+N$ is commutative.

Next we state four propositions:

(36) Let V be a non empty loop structure, M be a subset of V, and v be an element of the carrier of V. Then $\{v\}+M=v+M$.

(37) Let V be an Abelian add-associative real linear space-like non empty RLS structure, M be a Affine subset of V, and v be a vector of V. Then $\{v\}+M$ is Affine.

(38) For every non empty RLS structure V and for all Affine subsets M, N of V holds $M \cap N$ is Affine.

(39) Let V be an Abelian add-associative real linear space-like non empty RLS structure and M, N be Affine subsets of V. Then $M+N$ is Affine.

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Received October 9, 2002