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The Inner Product of Finite Sequences and of Points of n-dimensional Topological Space

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Summary. First, we define the inner product to finite sequences of real value. Next, we extend it to points of n-dimensional topological space $\mathcal{E}^n_{\mathrm{T}}$. At the end, orthogonality is introduced to this space.

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The notation and terminology used in this paper are introduced in the following articles: [11], [3], [9], [7], [1], [2], [6], [8], [4], [5], and [10].

1. Preliminaries

For simplicity, we use the following convention: i, n denote natural numbers, x, y, a denote real numbers, v denotes an element of $\mathbb{R}^n$, and p, p_1, p_2, p_3, q, q_1, q_2 denote points of $\mathcal{E}^n_{\mathrm{T}}$.

We now state several propositions:

(1) For every i such that $i \in \operatorname{Seg} n$ holds $(v \bullet \langle\underbrace{0,\ldots,0}_{n}\rangle)(i) = 0$.

(2) $v \bullet \langle\underbrace{0,\ldots,0}_{n}\rangle = \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(3) For every finite sequence x of elements of $\mathbb{R}$ holds $(-1) \cdot x = -x$.

(4) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $x - y = x + -y$.

(5) For every finite sequence x of elements of $\mathbb{R}$ holds $\operatorname{len}(-x) = \operatorname{len} x$.

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(6) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\operatorname{len}(x_1 + x_2) = \operatorname{len} x_1$.

(7) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\operatorname{len}(x_1 - x_2) = \operatorname{len} x_1$.

(8) For every real number a and for every finite sequence x of elements of $\mathbb{R}$ holds $\operatorname{len}(a \cdot x) = \operatorname{len} x$.

(9) For all finite sequences x, y, z of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $(x + y) \bullet z = x \bullet z + y \bullet z$.

2. Inner Product of Finite Sequences

Let x_1, x_2 be finite sequences of elements of $\mathbb{R}$. The functor $|(x_1, x_2)|$ yielding a real number is defined as follows:

(Def. 1) $|(x_1, x_2)| = \sum(x_1 \bullet x_2)$.

Let us observe that the functor $|(x_1, x_2)|$ is commutative.

We now state a number of propositions:

(10) Let y_1, y_2 be finite sequences of elements of $\mathbb{R}$ and x_1, x_2 be elements of $\mathcal{R}^n$. If $x_1 = y_1$ and $x_2 = y_2$, then $|(y_1, y_2)| = \frac{1}{4} \cdot (|x_1 + x_2|^{\mathbf{2}} - |x_1 - x_2|^{\mathbf{2}})$.

(11) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x, x)| \geqslant 0$.

(12) For every finite sequence x of elements of $\mathbb{R}$ holds $|x|^{\mathbf{2}} = |(x, x)|$.

(13) For every finite sequence x of elements of $\mathbb{R}$ holds $|x| = \sqrt{|(x, x)|}$.

(14) For every finite sequence x of elements of $\mathbb{R}$ holds $0 \leqslant |x|$.

(15) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x, x)| = 0$ iff $x = \underbrace{\langle 0, \ldots, 0 \rangle}_{\operatorname{len} x}$.

(16) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x, x)| = 0$ iff $|x| = 0$.

(17) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x, \underbrace{\langle 0, \ldots, 0 \rangle}_{\operatorname{len} x})| = 0$.

(18) For every finite sequence x of elements of $\mathbb{R}$ holds $|(\underbrace{\langle 0, \ldots, 0 \rangle}_{\operatorname{len} x}, x)| = 0$.

(19) For all finite sequences x, y, z of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $|(x + y, z)| = |(x, z)| + |(y, z)|$.

(20) For all finite sequences x, y of elements of $\mathbb{R}$ and for every real number a such that $\operatorname{len} x = \operatorname{len} y$ holds $|(a \cdot x, y)| = a \cdot |(x, y)|$.

(21) For all finite sequences x, y of elements of $\mathbb{R}$ and for every real number a such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x, a \cdot y)| = a \cdot |(x, y)|$.

(22) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(-x_1, x_2)| = -|(x_1, x_2)|$.

(23) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(x_1, -x_2)| = -|(x_1, x_2)|$.

(24) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(-x_1, -x_2)| = |(x_1, x_2)|$.

(25) For all finite sequences x_1, x_2, x_3 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} x_3$ holds $|(x_1 - x_2, x_3)| = |(x_1, x_3)| - |(x_2, x_3)|$.

(26) Let x, y be real numbers and x_1, x_2, x_3 be finite sequences of elements of $\mathbb{R}$. If $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} x_3$, then $|(x \cdot x_1 + y \cdot x_2, x_3)| = x \cdot |(x_1, x_3)| + y \cdot |(x_2, x_3)|$.

(27) For all finite sequences x, y_1, y_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$ holds $|(x, y_1 + y_2)| = |(x, y_1)| + |(x, y_2)|$.

(28) For all finite sequences x, y_1, y_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$ holds $|(x, y_1 - y_2)| = |(x, y_1)| - |(x, y_2)|$.

(29) Let x_1, x_2, y_1, y_2 be finite sequences of elements of $\mathbb{R}$. If $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $|(x_1+x_2, y_1+y_2)| = |(x_1, y_1)| + |(x_1, y_2)| + |(x_2, y_1)| + |(x_2, y_2)|$.

(30) Let x_1, x_2, y_1, y_2 be finite sequences of elements of $\mathbb{R}$. If $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $|(x_1 - x_2, y_1 - y_2)| = (|(x_1, y_1)| - |(x_1, y_2)| - |(x_2, y_1)|) + |(x_2, y_2)|$.

(31) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x + y, x + y)| = |(x, x)| + 2 \cdot |(x, y)| + |(y, y)|$.

(32) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x - y, x - y)| = (|(x, x)| - 2 \cdot |(x, y)|) + |(y, y)|$.

(33) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x + y|^{\mathbf{2}} = |x|^{\mathbf{2}} + 2 \cdot |(y, x)| + |y|^{\mathbf{2}}$.

(34) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x - y|^{\mathbf{2}} = (|x|^{\mathbf{2}} - 2 \cdot |(y, x)|) + |y|^{\mathbf{2}}$.

(35) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x + y|^{\mathbf{2}} + |x - y|^{\mathbf{2}} = 2 \cdot (|x|^{\mathbf{2}} + |y|^{\mathbf{2}})$.

(36) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x + y|^{\mathbf{2}} - |x - y|^{\mathbf{2}} = 4 \cdot |(x, y)|$.

(37) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $||(x, y)|| \leqslant |x| \cdot |y|$.

(38) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x + y| \leqslant |x| + |y|$.

3. Inner Product of Points of $\mathcal{E}^n_{\mathrm{T}}$

Let us consider n and let p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $|(p, q)|$ yielding a real number is defined as follows:

(Def. 2) There exist finite sequences f, g of elements of $\mathbb{R}$ such that $f = p$ and $g = q$ and $|(p, q)| = |(f, g)|$.

Let us observe that the functor $|(p, q)|$ is commutative.

We now state a number of propositions:

(39) For every natural number n and for all points p_1, p_2 of $\mathcal{E}^n_{\mathrm{T}}$ holds $|(p_1, p_2)| = \frac{1}{4} \cdot (|p_1 + p_2|^{\mathbf{2}} - |p_1 - p_2|^{\mathbf{2}})$.

(40) $|(p_1 + p_2, p_3)| = |(p_1, p_3)| + |(p_2, p_3)|$.

(41) For every real number x holds $|(x \cdot p_1, p_2)| = x \cdot |(p_1, p_2)|$.

(42) For every real number x holds $|(p_1, x \cdot p_2)| = x \cdot |(p_1, p_2)|$.

(43) $|(-p_1, p_2)| = -|(p_1, p_2)|$.

(44) $|(p_1, -p_2)| = -|(p_1, p_2)|$.

(45) $|(-p_1, -p_2)| = |(p_1, p_2)|$.

(46) $|(p_1 - p_2, p_3)| = |(p_1, p_3)| - |(p_2, p_3)|$.

(47) $|(x \cdot p_1 + y \cdot p_2, p_3)| = x \cdot |(p_1, p_3)| + y \cdot |(p_2, p_3)|$.

(48) $|(p, q_1 + q_2)| = |(p, q_1)| + |(p, q_2)|$.

(49) $|(p, q_1 - q_2)| = |(p, q_1)| - |(p, q_2)|$.

(50) $|(p_1 + p_2, q_1 + q_2)| = |(p_1, q_1)| + |(p_1, q_2)| + |(p_2, q_1)| + |(p_2, q_2)|$.

(51) $|(p_1 - p_2, q_1 - q_2)| = (|(p_1, q_1)| - |(p_1, q_2)| - |(p_2, q_1)|) + |(p_2, q_2)|$.

(52) $|(p + q, p + q)| = |(p, p)| + 2 \cdot |(p, q)| + |(q, q)|$.

(53) $|(p - q, p - q)| = (|(p, p)| - 2 \cdot |(p, q)|) + |(q, q)|$.

(54) $|(p, 0_{\mathcal{E}^n_{\mathrm{T}}})| = 0$.

(55) $|(0_{\mathcal{E}^n_{\mathrm{T}}}, p)| = 0$.

(56) $|(0_{\mathcal{E}^n_{\mathrm{T}}}, 0_{\mathcal{E}^n_{\mathrm{T}}})| = 0$.

(57) $|(p, p)| \geqslant 0$.

(58) $|(p, p)| = |p|^{\mathbf{2}}$.

(59) $|p| = \sqrt{|(p, p)|}$.

(60) $0 \leqslant |p|$.

(61) $|0_{\mathcal{E}^n_{\mathrm{T}}}| = 0$.

(62) $|(p, p)| = 0$ iff $|p| = 0$.

(63) $|(p, p)| = 0$ iff $p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(64) $|p| = 0$ iff $p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(65) $p \neq 0_{\mathcal{E}^n_{\mathrm{T}}}$ iff $|(p, p)| > 0$.

(66) $p \neq 0_{\mathcal{E}^n_{\mathrm{T}}}$ iff $|p| > 0$.

(67) $|p + q|^{\mathbf{2}} = |p|^{\mathbf{2}} + 2 \cdot |(q, p)| + |q|^{\mathbf{2}}$.

(68) $|p - q|^{\mathbf{2}} = (|p|^{\mathbf{2}} - 2 \cdot |(q, p)|) + |q|^{\mathbf{2}}$.

(69) $|p + q|^{\mathbf{2}} + |p - q|^{\mathbf{2}} = 2 \cdot (|p|^{\mathbf{2}} + |q|^{\mathbf{2}})$.

(70) $|p + q|^{\mathbf{2}} - |p - q|^{\mathbf{2}} = 4 \cdot |(p, q)|$.

(71) $|(p,q)| = \frac{1}{4} \cdot (|p+q|^{\mathbf{2}} - |p-q|^{\mathbf{2}})$.

(72) $|(p,q)| \leqslant |(p,p)| + |(q,q)|$.

(73) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ holds $||(p,q)|| \leqslant |p| \cdot |q|$.

(74) $|p+q| \leqslant |p| + |q|$.

Let us consider n, p, q. We say that p, q are orthogonal if and only if:

(Def. 3) $|(p,q)| = 0$.

Let us note that the predicate p, q are orthogonal is symmetric.

The following propositions are true:

(75) p, $0_{\mathcal{E}^n_{\mathrm{T}}}$ are orthogonal.

(76) $0_{\mathcal{E}^n_{\mathrm{T}}}$, p are orthogonal.

(77) p, p are orthogonal iff $p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(78) If p, q are orthogonal, then $a \cdot p$, q are orthogonal.

(79) If p, q are orthogonal, then p, $a \cdot q$ are orthogonal.

(80) If for every q holds p, q are orthogonal, then $p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

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