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Morphisms Into Chains. Part I

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Summary. This work is the continuation of formalization of [10]. Items from 2.1 to 2.8 of Chapter 4 are proved.

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The papers [16], [7], [19], [15], [4], [17], [18], [14], [1], [20], [22], [21], [5], [6], [2], [12], [13], [23], [3], [8], [11], and [9] provide the notation and terminology for this paper.

1. Preliminaries

Let X be a set. One can verify that there exists a subset of X which is trivial.

Let X be a trivial set. Note that every subset of X is trivial.

Let L be a 1-sorted structure. One can check that there exists a subset of L which is trivial.

Let L be a relational structure. Note that there exists a subset of L which is trivial.

Let L be a non empty 1-sorted structure. One can check that there exists a subset of L which is non empty and trivial.

Let L be a non empty relational structure. Note that there exists a subset of L which is non empty and trivial.

Next we state three propositions:

(1) For every set X holds $\subseteq_X$ is reflexive in X.

(2) For every set X holds $\subseteq_X$ is transitive in X.

(3) For every set X holds $\subseteq_X$ is antisymmetric in X.

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2. Main Part

Let L be a relational structure. Observe that there exists a binary relation on L which is auxiliary(i).

Let L be a transitive relational structure. Observe that there exists a binary relation on L which is auxiliary(i) and auxiliary(ii).

Let L be an antisymmetric relational structure with l.u.b.'s. Observe that there exists a binary relation on L which is auxiliary(iii).

Let L be a non empty lower-bounded antisymmetric relational structure. Note that there exists a binary relation on L which is auxiliary(iv).

Let L be a non empty relational structure and let R be a binary relation on L. We say that R is extra-order if and only if:

(Def. 1) R is auxiliary(i), auxiliary(ii), and auxiliary(iv).

Let L be a non empty relational structure. One can verify that every binary relation on L which is extra-order is also auxiliary(i), auxiliary(ii), and auxiliary(iv) and every binary relation on L which is auxiliary(i), auxiliary(ii), and auxiliary(iv) is also extra-order.

Let L be a non empty relational structure. One can verify that every binary relation on L which is extra-order and auxiliary(iii) is also auxiliary and every binary relation on L which is auxiliary is also extra-order.

Let L be a lower-bounded antisymmetric transitive non empty relational structure. One can check that there exists a binary relation on L which is extra-order.

Let L be a lower-bounded poset with l.u.b.'s and let R be an auxiliary(ii) binary relation on L. The functor $R-$LowerMap yields a map from L into $\langle \mathrm{LOWER}\, L, \subseteq \rangle$ and is defined as follows:

(Def. 2) For every element x of the carrier of L holds $R-\mathrm{LowerMap}(x) = \downarrow_R x$.

Let L be a lower-bounded poset with l.u.b.'s and let R be an auxiliary(ii) binary relation on L. One can verify that $R-$LowerMap is monotone.

Let L be a 1-sorted structure and let R be a binary relation on the carrier of L. A subset of L is called a strict chain of R if:

(Def. 3) For all sets x, y such that $x \in$ it and $y \in$ it holds $\langle x, y \rangle \in R$ or $x = y$ or $\langle y, x \rangle \in R$.

The following proposition is true

(4) Let L be a 1-sorted structure, C be a trivial subset of L, and R be a binary relation on the carrier of L. Then C is a strict chain of R.

Let L be a non empty 1-sorted structure and let R be a binary relation on the carrier of L. One can check that there exists a strict chain of R which is non empty and trivial.

One can prove the following four propositions:

(5) Let L be an antisymmetric relational structure, R be an auxiliary(i) binary relation on L, C be a strict chain of R, and x, y be elements of the carrier of L. If $x \in C$ and $y \in C$ and $x < y$, then $\langle x, y\rangle \in R$.

(6) Let L be an antisymmetric relational structure, R be an auxiliary(i) binary relation on L, and x, y be elements of the carrier of L. If $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$, then $x = y$.

(7) Let L be a non empty lower-bounded antisymmetric relational structure, x be an element of the carrier of L, and R be an auxiliary(iv) binary relation on L. Then $\{\perp_L, x\}$ is a strict chain of R.

(8) Let L be a non empty lower-bounded antisymmetric relational structure, R be an auxiliary(iv) binary relation on L, and C be a strict chain of R. Then $C \cup \{\perp_L\}$ is a strict chain of R.

Let L be a 1-sorted structure, let R be a binary relation on the carrier of L, and let C be a strict chain of R. We say that C is maximal if and only if:

(Def. 4) For every strict chain D of R such that $C \subseteq D$ holds $C = D$.

Let L be a 1-sorted structure, let R be a binary relation on the carrier of L, and let C be a set. The functor StrictChains(R, C) is defined by:

(Def. 5) For every set x holds $x \in \mathrm{StrictChains}(R, C)$ iff x is a strict chain of R and $C \subseteq x$.

Let L be a 1-sorted structure, let R be a binary relation on the carrier of L, and let C be a strict chain of R. Note that StrictChains(R, C) is non empty.

Let R be a binary relation and let X be a set. We introduce X is inductive w.r.t. R as a synonym of X has the upper Zorn property w.r.t. R.

Next we state several propositions:

(9) Let L be a 1-sorted structure, R be a binary relation on the carrier of L, and C be a strict chain of R. Then StrictChains(R, C) is inductive w.r.t. $\subseteq_{\mathrm{StrictChains}(R,C)}$ and there exists a set D such that D is maximal in $\subseteq_{\mathrm{StrictChains}(R,C)}$ and $C \subseteq D$.

(10) Let L be a non empty transitive relational structure, C be a non empty subset of the carrier of L, and X be a subset of C. Suppose sup X exists in L and $\bigsqcup_L X \in C$. Then sup X exists in sub(C) and $\bigsqcup_L X = \bigsqcup_{\mathrm{sub}(C)} X$.

(11) Let L be a non empty poset, R be an auxiliary(i) auxiliary(ii) binary relation on L, C be a non empty strict chain of R, and X be a subset of C. If sup X exists in L and C is maximal, then sup X exists in sub(C).

(12) Let L be a non empty poset, R be an auxiliary(i) auxiliary(ii) binary relation on L, C be a non empty strict chain of R, and X be a subset of C. Suppose inf $\uparrow\bigsqcup_L X \cap C$ exists in L and sup X exists in L and C is maximal. Then $\bigsqcup_{\mathrm{sub}(C)} X = \bigsqcap_L(\uparrow\bigsqcup_L X \cap C)$ and if $\bigsqcup_L X \notin C$, then $\bigsqcup_L X < \bigsqcap_L(\uparrow\bigsqcup_L X \cap C)$.

(13) Let L be a complete non empty poset, R be an auxiliary(i) auxiliary(ii)

binary relation on L, and C be a non empty strict chain of R. If C is maximal, then $\mathrm{sub}(C)$ is complete.

(14) Let L be a non empty lower-bounded antisymmetric relational structure, R be an auxiliary(iv) binary relation on L, and C be a strict chain of R. If C is maximal, then $\bot_L \in C$.

(15) Let L be a non empty upper-bounded poset, R be an auxiliary(ii) binary relation on L, C be a strict chain of R, and m be an element of the carrier of L. Suppose C is maximal and m is a maximum of C and $\langle m, \top_L \rangle \in R$. Then $\langle \top_L, \top_L \rangle \in R$ and $m = \top_L$.

Let L be a relational structure, let C be a set, and let R be a binary relation on the carrier of L. We say that R satisfies SIC on C if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let x, z be elements of the carrier of L. Suppose $x \in C$ and $z \in C$ and $\langle x, z \rangle \in R$ and $x \neq z$. Then there exists an element y of L such that $y \in C$ and $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$ and $x \neq y$.

Let L be a relational structure, let R be a binary relation on the carrier of L, and let C be a strict chain of R. We say that C satisfies SIC if and only if:

(Def. 7) R satisfies SIC on C.

We introduce C satisfies the interpolation property and C satisfies the interpolation property as synonyms of C satisfies SIC.

The following proposition is true

(16) Let L be a relational structure, C be a set, and R be an auxiliary(i) binary relation on L. Suppose R satisfies SIC on C. Let x, z be elements of the carrier of L. Suppose $x \in C$ and $z \in C$ and $\langle x, z \rangle \in R$ and $x \neq z$. Then there exists an element y of L such that $y \in C$ and $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$ and $x < y$.

Let L be a relational structure and let R be a binary relation on the carrier of L. Note that every strict chain of R which is trivial satisfies also SIC.

Let L be a non empty relational structure and let R be a binary relation on the carrier of L. One can check that there exists a strict chain of R which is non empty and trivial.

Next we state the proposition

(17) Let L be a lower-bounded poset with l.u.b.'s, R be an auxiliary(i) auxiliary(ii) binary relation on L, and C be a strict chain of R. Suppose C is maximal and R satisfies strong interpolation property. Then R satisfies SIC on C.

Let R be a binary relation and let C, y be sets. The functor $\mathrm{SetBelow}(R, C, y)$ is defined as follows:

(Def. 8) $\mathrm{SetBelow}(R, C, y) = R^{-1}(\{y\}) \cap C$.

The following proposition is true

(18) For every binary relation R and for all sets C, x, y holds $x \in \mathrm{SetBelow}(R, C, y)$ iff $\langle x, y \rangle \in R$ and $x \in C$.

Let L be a 1-sorted structure, let R be a binary relation on the carrier of L, and let C, y be sets. Then $\mathrm{SetBelow}(R, C, y)$ is a subset of L.

Next we state three propositions:

(19) Let L be a relational structure, R be an auxiliary(i) binary relation on L, C be a set, and y be an element of the carrier of L. Then $\mathrm{SetBelow}(R, C, y) \leqslant y$.

(20) Let L be a reflexive transitive relational structure, R be an auxiliary(ii) binary relation on L, C be a subset of the carrier of L, and x, y be elements of the carrier of L. If $x \leqslant y$, then $\mathrm{SetBelow}(R, C, x) \subseteq \mathrm{SetBelow}(R, C, y)$.

(21) Let L be a relational structure, R be an auxiliary(i) binary relation on L, C be a set, and x be an element of the carrier of L. If $x \in C$ and $\langle x, x \rangle \in R$ and sup $\mathrm{SetBelow}(R, C, x)$ exists in L, then $x = \sup \mathrm{SetBelow}(R, C, x)$.

Let L be a relational structure and let C be a subset of L. We say that C is sup-closed if and only if:

(Def. 9) For every subset X of C such that sup X exists in L holds $\bigsqcup_L X = \bigsqcup_{\mathrm{sub}(C)} X$.

Next we state three propositions:

(22) Let L be a complete non empty poset, R be an extra-order binary relation on L, C be a strict chain of R satisfying SIC, and p, q be elements of the carrier of L. Suppose $p \in C$ and $q \in C$ and $p < q$. Then there exists an element y of L such that $p < y$ and $\langle y, q \rangle \in R$ and $y = \sup \mathrm{SetBelow}(R, C, y)$.

(23) Let L be a lower-bounded non empty poset, R be an extra-order binary relation on L, and C be a non empty strict chain of R. Suppose that

(i) C is sup-closed,

(ii) for every element c of the carrier of L such that $c \in C$ holds sup $\mathrm{SetBelow}(R, C, c)$ exists in L, and

(iii) R satisfies SIC on C.

Let c be an element of the carrier of L. If $c \in C$, then $c = \sup \mathrm{SetBelow}(R, C, c)$.

(24) Let L be a non empty reflexive antisymmetric relational structure, R be an auxiliary(i) binary relation on L, and C be a strict chain of R. Suppose that for every element c of the carrier of L such that $c \in C$ holds sup $\mathrm{SetBelow}(R, C, c)$ exists in L and $c = \sup \mathrm{SetBelow}(R, C, c)$. Then R satisfies SIC on C.

Let L be a non empty relational structure, let R be a binary relation on the carrier of L, and let C be a set. The functor $\mathrm{SupBelow}(R, C)$ is defined by:

(Def. 10) For every set y holds $y \in \mathrm{SupBelow}(R, C)$ iff $y = \sup \mathrm{SetBelow}(R, C, y)$.

Let L be a non empty relational structure, let R be a binary relation on the carrier of L, and let C be a set. Then SupBelow(R, C) is a subset of L.

One can prove the following propositions:

(25) Let L be a non empty reflexive transitive relational structure, R be an auxiliary(i) auxiliary(ii) binary relation on L, and C be a strict chain of R. Suppose that for every element c of L holds sup SetBelow(R, C, c) exists in L. Then SupBelow(R, C) is a strict chain of R.

(26) Let L be a non empty poset, R be an auxiliary(i) auxiliary(ii) binary relation on L, and C be a subset of the carrier of L. Suppose that for every element c of L holds sup SetBelow(R, C, c) exists in L. Then SupBelow(R, C) is sup-closed.

(27) Let L be a complete non empty poset, R be an extra-order binary relation on L, C be a strict chain of R satisfying SIC, and d be an element of the carrier of L. Suppose $d \in$ SupBelow(R, C). Then $d = \bigsqcup_L \{b; b$ ranges over elements of the carrier of L: $b \in$ SupBelow$(R, C) \ \wedge \ \langle b, d \rangle \in R\}$.

(28) Let L be a complete non empty poset, R be an extra-order binary relation on L, and C be a strict chain of R satisfying SIC. Then R satisfies SIC on SupBelow(R, C).

(29) Let L be a complete non empty poset, R be an extra-order binary relation on L, C be a strict chain of R satisfying SIC, and a, b be elements of the carrier of L. Suppose $a \in C$ and $b \in C$ and $a < b$. Then there exists an element d of L such that $d \in$ SupBelow(R, C) and $a < d$ and $\langle d, b \rangle \in R$.

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