FORMALIZED MATHEMATICS
Volume 11, Number 3, 2003
University of Białystok

Inner Products and Angles of Complex Numbers

Wenpai Chang
Shinshu University
Nagano

Yatsuka Nakamura
Shinshu University
Nagano

Piotr Rudnicki
University of Alberta
Edmonton

Summary. An inner product of complex numbers is defined and used to characterize the (counter-clockwise) angle between (a,0) and (0,b) in the complex plane. For complex a, b and c we then define the (counter-clockwise) angle between (a,c) and (c, b) and prove theorems about the sum of internal and external angles of a triangle.

MML Identifier: `COMPLEX2`.

The papers [9], [13], [10], [12], [14], [3], [7], [15], [5], [6], [8], [11], [2], [1], and [4] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For all real numbers a, b holds $-(a+bi) = -a + (-b)i$.

(2) For all real numbers a, b such that $b > 0$ there exists a real number r such that $r = b \cdot -\lfloor \frac{a}{b} \rfloor + a$ and $0 \leqslant r$ and $r < b$.

(3) Let a, b, c be real numbers. Suppose $a > 0$ and $b \geqslant 0$ and $c \geqslant 0$ and $b < a$ and $c < a$. Let i be an integer. If $b = c + a \cdot i$, then $b = c$.

(4) For all real numbers a, b holds $\sin(a-b) = \sin a \cdot \cos b - \cos a \cdot \sin b$ and $\cos(a-b) = \cos a \cdot \cos b + \sin a \cdot \sin b$.

(5) For every real number a holds $\sin(a-\pi) = -\sin(a)$ and $\cos(a-\pi) = -\cos(a)$.

(6) For every real number a holds $\sin(a-\pi) = -\sin a$ and $\cos(a-\pi) = -\cos a$.

ISSN 1426–2630

(7) For all real numbers a, b such that $a \in]0, \frac{\pi}{2}[$ and $b \in]0, \frac{\pi}{2}[$ holds $a < b$ iff $\sin a < \sin b$.

(8) For all real numbers a, b such that $a \in]\frac{\pi}{2}, \pi[$ and $b \in]\frac{\pi}{2}, \pi[$ holds $a < b$ iff $\sin a > \sin b$.

(9) For every real number a and for every integer i holds $\sin a = \sin(2 \cdot \pi \cdot i + a)$.

(10) For every real number a and for every integer i holds $\cos a = \cos(2 \cdot \pi \cdot i + a)$.

(11) For every real number a such that $\sin a = 0$ holds $\cos a \neq 0$.

(12) For all real numbers a, b such that $0 \leqslant a$ and $a < 2 \cdot \pi$ and $0 \leqslant b$ and $b < 2 \cdot \pi$ and $\sin a = \sin b$ and $\cos a = \cos b$ holds $a = b$.

2. More on the Argument of a Complex Number

Let us observe that $\mathbb{C}_{\mathrm{F}}$ is non empty.

Let z be an element of $\mathbb{C}$. The functor $\mathrm{Ftize}(z)$ yields an element of the carrier of $\mathbb{C}_{\mathrm{F}}$ and is defined as follows:

(Def. 1) $\mathrm{Ftize}(z) = z$.

We now state four propositions:

(13) For every element z of $\mathbb{C}$ holds $\Re(z) = \Re(\mathrm{Ftize}(z))$ and $\Im(z) = \Im(\mathrm{Ftize}(z))$.

(14) For all elements x, y of $\mathbb{C}$ holds $\mathrm{Ftize}(x + y) = \mathrm{Ftize}(x) + \mathrm{Ftize}(y)$.

(15) For every element z of $\mathbb{C}$ holds $z = 0_{\mathbb{C}}$ iff $\mathrm{Ftize}(z) = 0_{\mathbb{C}_{\mathrm{F}}}$.

(16) For every element z of $\mathbb{C}$ holds $|z| = |\mathrm{Ftize}(z)|$.

Let z be an element of $\mathbb{C}$. The functor $\mathrm{Arg}\, z$ yields a real number and is defined as follows:

(Def. 2) $\mathrm{Arg}\, z = \mathrm{Arg}\, \mathrm{Ftize}(z)$.

One can prove the following propositions:

(17) For every element z of $\mathbb{C}$ and for every element u of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $z = u$ holds $\mathrm{Arg}\, z = \mathrm{Arg}\, u$.

(18) For every element z of $\mathbb{C}$ holds $0 \leqslant \mathrm{Arg}\, z$ and $\mathrm{Arg}\, z < 2 \cdot \pi$.

(19) For every element z of $\mathbb{C}$ holds $z = |z| \cdot \cos \mathrm{Arg}\, z + (|z| \cdot \sin \mathrm{Arg}\, z)i$.

(20) $\mathrm{Arg}(0_{\mathbb{C}}) = 0$.

(21) Let z be an element of $\mathbb{C}$ and r be a real number. If $z \neq 0$ and $z = |z| \cdot \cos r + (|z| \cdot \sin r)i$ and $0 \leqslant r$ and $r < 2 \cdot \pi$, then $r = \mathrm{Arg}\, z$.

(22) For every element z of $\mathbb{C}$ such that $z \neq 0_{\mathbb{C}}$ holds if $\mathrm{Arg}\, z < \pi$, then $\mathrm{Arg}(-z) = \mathrm{Arg}\, z + \pi$ and if $\mathrm{Arg}\, z \geqslant \pi$, then $\mathrm{Arg}(-z) = \mathrm{Arg}\, z - \pi$.

(23) For every real number r such that $r \geqslant 0$ holds $\mathrm{Arg}(r + 0i) = 0$.

(24) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z = 0$ iff $z = |z| + 0i$.

(25) For every element z of $\mathbb{C}$ such that $z \neq 0_{\mathbb{C}}$ holds $\operatorname{Arg} z < \pi$ iff $\operatorname{Arg}(-z) \geqslant \pi$.

(26) For all elements x, y of $\mathbb{C}$ such that $x \neq y$ or $x-y \neq 0_{\mathbb{C}}$ holds $\operatorname{Arg}(x-y) < \pi$ iff $\operatorname{Arg}(y-x) \geqslant \pi$.

(27) For every element z of $\mathbb{C}$ holds $\operatorname{Arg} z \in]0, \pi[$ iff $\Im(z) > 0$.

(28) For every element z of $\mathbb{C}$ such that $\operatorname{Arg} z \neq 0$ holds $\operatorname{Arg} z < \pi$ iff $\sin \operatorname{Arg} z > 0$.

(29) For all elements x, y of $\mathbb{C}$ such that $\operatorname{Arg} x < \pi$ and $\operatorname{Arg} y < \pi$ holds $\operatorname{Arg}(x+y) < \pi$.

(30) For every real number x such that $x > 0$ holds $\operatorname{Arg}(0+xi) = \frac{\pi}{2}$.

(31) For every element z of $\mathbb{C}$ holds $\operatorname{Arg} z \in]0, \frac{\pi}{2}[$ iff $\Re(z) > 0$ and $\Im(z) > 0$.

(32) For every element z of $\mathbb{C}$ holds $\operatorname{Arg} z \in]\frac{\pi}{2}, \pi[$ iff $\Re(z) < 0$ and $\Im(z) > 0$.

(33) For every element z of $\mathbb{C}$ such that $\Im(z) > 0$ holds $\sin \operatorname{Arg} z > 0$.

(34) For every element z of $\mathbb{C}$ holds $\operatorname{Arg} z = 0$ iff $\Re(z) \geqslant 0$ and $\Im(z) = 0$.

(35) For every element z of $\mathbb{C}$ holds $\operatorname{Arg} z = \pi$ iff $\Re(z) < 0$ and $\Im(z) = 0$.

(36) For every element z of $\mathbb{C}$ holds $\Im(z) = 0$ iff $\operatorname{Arg} z = 0$ or $\operatorname{Arg} z = \pi$.

(37) For every element z of $\mathbb{C}$ such that $\operatorname{Arg} z \leqslant \pi$ holds $\Im(z) \geqslant 0$.

(38) For every element z of $\mathbb{C}$ such that $z \neq 0$ holds $\cos \operatorname{Arg}(-z) = -\cos \operatorname{Arg} z$ and $\sin \operatorname{Arg}(-z) = -\sin \operatorname{Arg} z$.

(39) For every element a of $\mathbb{C}$ such that $a \neq 0$ holds $\cos \operatorname{Arg} a = \frac{\Re(a)}{|a|}$ and $\sin \operatorname{Arg} a = \frac{\Im(a)}{|a|}$.

(40) For every element a of $\mathbb{C}$ and for every real number r such that $r > 0$ holds $\operatorname{Arg}(a \cdot (r+0i)) = \operatorname{Arg} a$.

(41) For every element a of $\mathbb{C}$ and for every real number r such that $r < 0$ holds $\operatorname{Arg}(a \cdot (r+0i)) = \operatorname{Arg}(-a)$.

3. Inner Product

Let x, y be elements of $\mathbb{C}$. The functor $(x|y)$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 3) $(x|y) = x \cdot \overline{y}$.

In the sequel a, b, c, d, x, y, z are elements of $\mathbb{C}$.

The following propositions are true:

(42) $(x|y) = (\Re(x) \cdot \Re(y) + \Im(x) \cdot \Im(y)) + (-\Re(x) \cdot \Im(y) + \Im(x) \cdot \Re(y))i$.

(43) $(z|z) = (\Re(z) \cdot \Re(z) + \Im(z) \cdot \Im(z)) + 0i$ and $(z|z) = (\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}) + 0i$.

(44) $(z|z) = |z|^{\mathbf{2}} + 0i$ and $|z|^{\mathbf{2}} = \Re((z|z))$.

(45) $|(x|y)| = |x| \cdot |y|$.

(46) If $(x|x) = 0$, then $x = 0$.

(47) $(y|x) = \overline{(x|y)}$.

(48) $((x+y)|z) = (x|z) + (y|z)$.

(49) $(x|(y+z)) = (x|y) + (x|z)$.

(50) $((a \cdot x)|y) = a \cdot (x|y)$.

(51) $(x|(a \cdot y)) = \overline{a} \cdot (x|y)$.

(52) $((a \cdot x)|y) = (x|(\overline{a} \cdot y))$.

(53) $((a \cdot x + b \cdot y)|z) = a \cdot (x|z) + b \cdot (y|z)$.

(54) $(x|(a \cdot y + b \cdot z)) = \overline{a} \cdot (x|y) + \overline{b} \cdot (x|z)$.

(55) $((-x)|y) = (x|-y)$.

(56) $((-x)|y) = -(x|y)$.

(57) $-(x|y) = (x|-y)$.

(58) $((-x)|-y) = (x|y)$.

(59) $((x-y)|z) = (x|z) - (y|z)$.

(60) $(x|(y-z)) = (x|y) - (x|z)$.

(61) $(0_{\mathbb{C}}|x) = 0_{\mathbb{C}}$ and $(x|0_{\mathbb{C}}) = 0_{\mathbb{C}}$.

(62) $((x+y)|(x+y)) = (x|x) + (x|y) + (y|x) + (y|y)$.

(63) $((x-y)|(x-y)) = ((x|x) - (x|y) - (y|x)) + (y|y)$.

(64) $\Re((x|y)) = 0$ iff $\Im((x|y)) = |x| \cdot |y|$ or $\Im((x|y)) = -|x| \cdot |y|$.

4. Rotation

Let a be an element of $\mathbb{C}$ and let r be a real number. The functor $a \circlearrowleft r$ yielding an element of $\mathbb{C}$ is defined as follows:

(Def. 4) $a \circlearrowleft r = |a| \cdot \cos(r + \operatorname{Arg} a) + (|a| \cdot \sin(r + \operatorname{Arg} a))i$.

In the sequel r denotes a real number.

We now state a number of propositions:

(65) $a \circlearrowleft 0 = a$.

(66) $a \circlearrowleft r = 0_{\mathbb{C}}$ iff $a = 0_{\mathbb{C}}$.

(67) $|a \circlearrowleft r| = |a|$.

(68) If $a \neq 0_{\mathbb{C}}$, then there exists an integer i such that $\operatorname{Arg}(a \circlearrowleft r) = 2 \cdot \pi \cdot i + (r + \operatorname{Arg} a)$.

(69) $a \circlearrowleft -\operatorname{Arg} a = |a| + 0i$.

(70) $\Re(a \circlearrowleft r) = \Re(a) \cdot \cos r - \Im(a) \cdot \sin r$ and $\Im(a \circlearrowleft r) = \Re(a) \cdot \sin r + \Im(a) \cdot \cos r$.

(71) $a + b \circlearrowleft r = (a \circlearrowleft r) + (b \circlearrowleft r)$.

(72) $-a \circlearrowleft r = -(a \circlearrowleft r)$.

(73) $a - b \circlearrowleft r = (a \circlearrowleft r) - (b \circlearrowleft r)$.

(74) $a \circlearrowleft \pi = -a$.

5. Angles

Let a, b be elements of $\mathbb{C}$. The functor $\measuredangle(a,b)$ yielding a real number is defined by:

(Def. 5) $\measuredangle(a,b) = \begin{cases} \operatorname{Arg}(b \circlearrowleft -\operatorname{Arg} a), & \text{if } \operatorname{Arg} a = 0 \text{ or } b \neq 0, \\ 2 \cdot \pi - \operatorname{Arg} a, & \text{otherwise.} \end{cases}$

Next we state several propositions:

(75) If $r \geqslant 0$, then $\measuredangle(r + 0i, a) = \operatorname{Arg} a$.

(76) If $\operatorname{Arg} a = \operatorname{Arg} b$ and $a \neq 0$ and $b \neq 0$, then $\operatorname{Arg}(a \circlearrowleft r) = \operatorname{Arg}(b \circlearrowleft r)$.

(77) If $r > 0$, then $\measuredangle(a,b) = \measuredangle(a \cdot (r + 0i), b \cdot (r + 0i))$.

(78) If $a \neq 0$ and $b \neq 0$ and $\operatorname{Arg} a = \operatorname{Arg} b$, then $\operatorname{Arg}(-a) = \operatorname{Arg}(-b)$.

(79) If $a \neq 0$ and $b \neq 0$, then $\measuredangle(a,b) = \measuredangle(a \circlearrowleft r, b \circlearrowleft r)$.

(80) If $r < 0$ and $a \neq 0$ and $b \neq 0$, then $\measuredangle(a,b) = \measuredangle(a \cdot (r + 0i), b \cdot (r + 0i))$.

(81) If $a \neq 0$ and $b \neq 0$, then $\measuredangle(a,b) = \measuredangle(-a,-b)$.

(82) If $b \neq 0$ and $\measuredangle(a,b) = 0$, then $\measuredangle(a,-b) = \pi$.

(83) If $a \neq 0$ and $b \neq 0$, then $\cos \measuredangle(a,b) = \frac{\Re((a|b))}{|a| \cdot |b|}$ and $\sin \measuredangle(a,b) = -\frac{\Im((a|b))}{|a| \cdot |b|}$.

Let x, y, z be elements of $\mathbb{C}$. The functor $\measuredangle(x,y,z)$ yielding a real number is defined as follows:

(Def. 6) $\measuredangle(x,y,z) = \begin{cases} \operatorname{Arg}(z-y) - \operatorname{Arg}(x-y), & \text{if } \operatorname{Arg}(z-y) - \operatorname{Arg}(x-y) \geqslant 0, \\ 2 \cdot \pi + (\operatorname{Arg}(z-y) - \operatorname{Arg}(x-y)), & \text{otherwise.} \end{cases}$

One can prove the following propositions:

(84) $0 \leqslant \measuredangle(x,y,z)$ and $\measuredangle(x,y,z) < 2 \cdot \pi$.

(85) $\measuredangle(x,y,z) = \measuredangle(x-y, 0_{\mathbb{C}}, z-y)$.

(86) $\measuredangle(a,b,c) = \measuredangle(a+d, b+d, c+d)$.

(87) $\measuredangle(a,b) = \measuredangle(a, 0_{\mathbb{C}}, b)$.

(88) If $\measuredangle(x,y,z) = 0$, then $\operatorname{Arg}(x-y) = \operatorname{Arg}(z-y)$ and $\measuredangle(z,y,x) = 0$.

(89) If $a \neq 0_{\mathbb{C}}$ and $b \neq 0_{\mathbb{C}}$, then $\Re((a|b)) = 0$ iff $\measuredangle(a, 0_{\mathbb{C}}, b) = \frac{\pi}{2}$ or $\measuredangle(a, 0_{\mathbb{C}}, b) = \frac{3}{2} \cdot \pi$.

(90) If $a \neq 0_{\mathbb{C}}$ and $b \neq 0_{\mathbb{C}}$, then $\Im((a|b)) = |a| \cdot |b|$ or $\Im((a|b)) = -|a| \cdot |b|$ iff $\measuredangle(a, 0_{\mathbb{C}}, b) = \frac{\pi}{2}$ or $\measuredangle(a, 0_{\mathbb{C}}, b) = \frac{3}{2} \cdot \pi$.

(91) If $x \neq y$ and if $z \neq y$ and if $\measuredangle(x,y,z) = \frac{\pi}{2}$ or $\measuredangle(x,y,z) = \frac{3}{2} \cdot \pi$, then $|x-y|^{\mathbf{2}} + |z-y|^{\mathbf{2}} = |x-z|^{\mathbf{2}}$.

(92) If $a \neq b$ and $b \neq c$, then $\measuredangle(a,b,c) = \measuredangle(a \circlearrowleft r, b \circlearrowleft r, c \circlearrowleft r)$.

(93) $\measuredangle(a,b,a) = 0$.

(94) $\measuredangle(a,b,c) \neq 0$ iff $\measuredangle(a,b,c) + \measuredangle(c,b,a) = 2 \cdot \pi$.

(95) If $\measuredangle(a,b,c) \neq 0$, then $\measuredangle(c,b,a) \neq 0$.

(96) If $\measuredangle(a,b,c) = \pi$, then $\measuredangle(c,b,a) = \pi$.

(97) If $a \neq b$ and $a \neq c$ and $b \neq c$, then $\measuredangle(a, b, c) \neq 0$ or $\measuredangle(b, c, a) \neq 0$ or $\measuredangle(c, a, b) \neq 0$.

(98) If $a \neq b$ and $b \neq c$ and $0 < \measuredangle(a, b, c)$ and $\measuredangle(a, b, c) < \pi$, then $\measuredangle(a, b, c) + \measuredangle(b, c, a) + \measuredangle(c, a, b) = \pi$ and $0 < \measuredangle(b, c, a)$ and $0 < \measuredangle(c, a, b)$.

(99) If $a \neq b$ and $b \neq c$ and $\measuredangle(a, b, c) > \pi$, then $\measuredangle(a, b, c) + \measuredangle(b, c, a) + \measuredangle(c, a, b) = 5 \cdot \pi$ and $\measuredangle(b, c, a) > \pi$ and $\measuredangle(c, a, b) > \pi$.

(100) If $a \neq b$ and $b \neq c$ and $\measuredangle(a, b, c) = \pi$, then $\measuredangle(b, c, a) = 0$ and $\measuredangle(c, a, b) = 0$.

(101) If $a \neq b$ and $a \neq c$ and $b \neq c$ and $\measuredangle(a, b, c) = 0$, then $\measuredangle(b, c, a) = 0$ and $\measuredangle(c, a, b) = \pi$ or $\measuredangle(b, c, a) = \pi$ and $\measuredangle(c, a, b) = 0$.

(102) $\measuredangle(a, b, c) + \measuredangle(b, c, a) + \measuredangle(c, a, b) = \pi$ or $\measuredangle(a, b, c) + \measuredangle(b, c, a) + \measuredangle(c, a, b) = 5 \cdot \pi$ iff $a \neq b$ and $a \neq c$ and $b \neq c$.

References

[1] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[3] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[4] Anna Justyna Milewska. The field of complex numbers. *Formalized Mathematics*, 9(**2**):265–269, 2001.
[5] Anna Justyna Milewska. The Hahn Banach theorem in the vector space over the field of complex numbers. *Formalized Mathematics*, 9(**2**):363–371, 2001.
[6] Robert Milewski. Trigonometric form of complex numbers. *Formalized Mathematics*, 9(**3**):455–460, 2001.
[7] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[8] Andrzej Trybulec. Introduction to arithmetics. *To appear in Formalized Mathematics.*
[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[10] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[11] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[12] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[13] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[15] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

Received May 29, 2003