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Convex Hull, Set of Convex Combinations and Convex Cone

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Summary. In this article, there are two themes. One of them is the proof that convex hull of a given subset M consists of all convex combinations of M. Another is definitions of cone and convex cone and some properties of them.

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The terminology and notation used in this paper are introduced in the following articles: [8], [11], [7], [2], [12], [3], [5], [1], [4], [10], [9], and [6].

1. Equality of Convex Hull and Set of Convex Combinations

Let V be a real linear space. The functor ConvexComb(V) yielding a set is defined by:

(Def. 1) For every set L holds $L \in \mathrm{ConvexComb}(V)$ iff L is a convex combination of V.

Let V be a real linear space and let M be a non empty subset of V. The functor ConvexComb(M) yielding a set is defined as follows:

(Def. 2) For every set L holds $L \in \mathrm{ConvexComb}(M)$ iff L is a convex combination of M.

We now state several propositions:

(1) Let V be a real linear space and v be a vector of V. Then there exists a convex combination L of V such that $\sum L = v$ and for every non empty subset A of V such that $v \in A$ holds L is a convex combination of A.

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(2) Let V be a real linear space and v_1, v_2 be vectors of V. Suppose $v_1 \neq v_2$. Then there exists a convex combination L of V such that for every non empty subset A of V if $\{v_1, v_2\} \subseteq A$, then L is a convex combination of A.

(3) Let V be a real linear space and v_1, v_2, v_3 be vectors of V. Suppose $v_1 \neq v_2$ and $v_1 \neq v_3$ and $v_2 \neq v_3$. Then there exists a convex combination L of V such that for every non empty subset A of V if $\{v_1, v_2, v_3\} \subseteq A$, then L is a convex combination of A.

(4) Let V be a real linear space and M be a non empty subset of V. Then M is convex if and only if $\{\sum L; L$ ranges over convex combinations of M: $L \in \mathrm{ConvexComb}(V)\} \subseteq M$.

(5) Let V be a real linear space and M be a non empty subset of V. Then $\mathrm{conv}\, M = \{\sum L; L$ ranges over convex combinations of M: $L \in \mathrm{ConvexComb}(V)\}$.

2. Cone and Convex Cone

Let V be a non empty RLS structure and let M be a subset of V. We say that M is cone if and only if:

(Def. 3) For every real number r and for every vector v of V such that $r > 0$ and $v \in M$ holds $r \cdot v \in M$.

One can prove the following proposition

(6) For every non empty RLS structure V and for every subset M of V such that $M = \emptyset$ holds M is cone.

Let V be a non empty RLS structure. Observe that there exists a subset of V which is cone.

Let V be a non empty RLS structure. Observe that there exists a subset of V which is empty and cone.

Let V be a real linear space. Observe that there exists a subset of V which is non empty and cone.

The following propositions are true:

(7) Let V be a non empty RLS structure and M be a cone subset of V. Suppose V is real linear space-like. Then M is convex if and only if for all vectors u, v of V such that $u \in M$ and $v \in M$ holds $u + v \in M$.

(8) Let V be a real linear space and M be a subset of V. Then M is convex and cone if and only if for every linear combination L of M such that the support of $L \neq \emptyset$ and for every vector v of V such that $v \in$ the support of L holds $L(v) > 0$ holds $\sum L \in M$.

(9) For every non empty RLS structure V and for all subsets M, N of V such that M is cone and N is cone holds $M \cap N$ is cone.

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