FORMALIZED MATHEMATICS
Volume 11, Number 3, 2003
University of Białystok

General Fashoda Meet Theorem for Unit Circle and Square

Yatsuka Nakamura
Shinshu University
Nagano

Summary. Here we will prove Fashoda meet theorem for the unit circle and for a square, when 4 points on the boundary are ordered cyclically. Also, the concepts of general rectangle and general circle are defined.

MML Identifier: `JGRAPH_6`.

The articles [8], [22], [26], [3], [4], [25], [1], [9], [2], [6], [13], [23], [19], [18], [16], [17], [11], [24], [7], [14], [15], [21], [20], [10], [5], and [12] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(2)[1] For all real numbers a, b, r such that $0 \leqslant r$ and $r \leqslant 1$ and $a \leqslant b$ holds $a \leqslant (1-r) \cdot a + r \cdot b$ and $(1-r) \cdot a + r \cdot b \leqslant b$.

(3) For all real numbers a, b such that $a \geqslant 0$ and $b > 0$ or $a > 0$ and $b \geqslant 0$ holds $a + b > 0$.

(4) For all real numbers a, b such that $-1 \leqslant a$ and $a \leqslant 1$ and $-1 \leqslant b$ and $b \leqslant 1$ holds $a^{\mathbf{2}} \cdot b^{\mathbf{2}} \leqslant 1$.

(5) For all real numbers a, b such that $a \geqslant 0$ and $b \geqslant 0$ holds $a \cdot \sqrt{b} = \sqrt{a^{\mathbf{2}} \cdot b}$.

(6) For all real numbers a, b such that $-1 \leqslant a$ and $a \leqslant 1$ and $-1 \leqslant b$ and $b \leqslant 1$ holds $(-b) \cdot \sqrt{1 + a^{\mathbf{2}}} \leqslant \sqrt{1 + b^{\mathbf{2}}}$ and $-\sqrt{1 + b^{\mathbf{2}}} \leqslant b \cdot \sqrt{1 + a^{\mathbf{2}}}$.

[1]The proposition (1) has been removed.

ISSN 1426–2630

(7) For all real numbers a, b such that $-1 \leqslant a$ and $a \leqslant 1$ and $-1 \leqslant b$ and $b \leqslant 1$ holds $b \cdot \sqrt{1 + a^{\mathbf{2}}} \leqslant \sqrt{1 + b^{\mathbf{2}}}$.

(8) For all real numbers a, b such that $a \geqslant b$ holds $a \cdot \sqrt{1 + b^{\mathbf{2}}} \geqslant b \cdot \sqrt{1 + a^{\mathbf{2}}}$.

(9) Let a, c, d be real numbers and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $c \leqslant d$ and $p \in \mathcal{L}([a, c], [a, d])$, then $p_{\mathbf{1}} = a$ and $c \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant d$.

(10) For all real numbers a, c, d and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $c < d$ and $p_{\mathbf{1}} = a$ and $c \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant d$ holds $p \in \mathcal{L}([a, c], [a, d])$.

(11) Let a, b, d be real numbers and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $a \leqslant b$ and $p \in \mathcal{L}([a, d], [b, d])$, then $p_{\mathbf{2}} = d$ and $a \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant b$.

(12) For all real numbers a, b and for every subset B of $\mathbb{I}$ such that $B = [a, b]$ holds B is closed.

(13) Let X be a topological structure, Y, Z be non empty topological structures, f be a map from X into Y, and g be a map from X into Z. Then $\mathrm{dom}\, f = \mathrm{dom}\, g$ and $\mathrm{dom}\, f =$ the carrier of X and $\mathrm{dom}\, f = \Omega_X$.

(14) Let X be a non empty topological space and B be a non empty subset of X. Then there exists a map f from $X \restriction B$ into X such that for every point p of $X \restriction B$ holds $f(p) = p$ and f is continuous.

(15) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = r_1 - a$ and g is continuous.

(16) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = a - r_1$ and g is continuous.

(17) Let X be a non empty topological space, n be a natural number, p be a point of $\mathcal{E}^n_{\mathrm{T}}$, and f be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f is continuous. Then there exists a map g from X into $\mathcal{E}^n_{\mathrm{T}}$ such that for every point r of X holds $g(r) = f(r) \cdot p$ and g is continuous.

(18) $\mathrm{SqCirc}([-1, 0]) = [-1, 0]$.

(19) For every compact non empty subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ holds $\mathrm{SqCirc}([-1, 0]) = \text{W-min}\, P$.

(20) Let X be a non empty topological space, n be a natural number, and g_1, g_2 be maps from X into $\mathcal{E}^n_{\mathrm{T}}$. Suppose g_1 is continuous and g_2 is continuous. Then there exists a map g from X into $\mathcal{E}^n_{\mathrm{T}}$ such that for every point r of X holds $g(r) = g_1(r) + g_2(r)$ and g is continuous.

(21) Let X be a non empty topological space, n be a natural number, p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$, and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathcal{E}^n_{\mathrm{T}}$ such that for every point r of X holds $g(r) = f_1(r) \cdot p_1 + f_2(r) \cdot p_2$ and

g is continuous.

(22) For every function f and for every set A such that f is one-to-one and $A \subseteq \operatorname{dom} f$ holds $(f^{-1})^{\circ} f^{\circ} A = A$.

2. General Fashoda Theorem for Unit Circle

In the sequel p, p_1, p_2, p_3, q, q_1, q_2 are points of $\mathcal{E}^2_{\mathrm{T}}$.

One can prove the following propositions:

(23) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leqslant 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leqslant (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geqslant -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geqslant (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leqslant -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geqslant (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geqslant -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leqslant (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leqslant -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_1$ and $f(I) \in K_2$ and $g(O) \in K_3$ and $g(I) \in K_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(24) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leqslant 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leqslant (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geqslant -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geqslant (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leqslant -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geqslant (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geqslant -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leqslant (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leqslant -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_1$ and $f(I) \in K_2$ and $g(O) \in K_4$ and $g(I) \in K_3$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(25) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p_8; p_8$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_8| \leqslant 1\}$ and $f(0) = p_3$ and $f(1) = p_1$ and $g(0) = p_2$ and $g(1) = p_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(26) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p_8; p_8$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$:

$|p_8| \leqslant 1\}$ and $f(0) = p_3$ and $f(1) = p_1$ and $g(0) = p_4$ and $g(1) = p_2$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(27) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and p_1, p_2, p_3, p_4 are in this order on P. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p_8; p_8$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_8| \leqslant 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

3. General Rectangles and Circles

Let a, b, c, d be real numbers. The functor $\mathrm{Rectangle}(a, b, c, d)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the condition (Def. 1).

(Def. 1) $\mathrm{Rectangle}(a, b, c, d) = \{p : p_{\mathbf{1}} = a \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = d \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b \vee p_{\mathbf{1}} = b \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = c \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b\}$.

The following proposition is true

(28) Let a, b, c, d be real numbers and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $a \leqslant b$ and $c \leqslant d$ and $p \in \mathrm{Rectangle}(a, b, c, d)$, then $a \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant b$ and $c \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant d$.

Let a, b, c, d be real numbers. The functor $\mathrm{InsideOfRectangle}(a, b, c, d)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 2) $\mathrm{InsideOfRectangle}(a, b, c, d) = \{p : a < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < b \wedge c < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < d\}$.

Let a, b, c, d be real numbers. The functor $\mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 3) $\mathrm{ClosedInsideOfRectangle}(a, b, c, d) = \{p : a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d\}$.

Let a, b, c, d be real numbers. The functor $\mathrm{OutsideOfRectangle}(a, b, c, d)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 4) $\mathrm{OutsideOfRectangle}(a, b, c, d) = \{p : a \nleqslant p_{\mathbf{1}} \vee p_{\mathbf{1}} \nleqslant b \vee c \nleqslant p_{\mathbf{2}} \vee p_{\mathbf{2}} \nleqslant d\}$.

Let a, b, c, d be real numbers. The functor $\mathrm{ClosedOutsideOfRectangle}(a, b, c, d)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 5) $\mathrm{ClosedOutsideOfRectangle}(a, b, c, d) = \{p : a \not< p_{\mathbf{1}} \vee p_{\mathbf{1}} \not< b \vee c \not< p_{\mathbf{2}} \vee p_{\mathbf{2}} \not< d\}$.

Next we state four propositions:

(29) Let a, b, r be real numbers and K_5, C_1 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $r \geqslant 0$ and $K_5 = \{q : |q| = 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_2 - [a, b]| = r\}$. Then $(\mathrm{AffineMap}(r, a, r, b))^{\circ} K_5 = C_1$.

(30) Let P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose there exists a map from $\mathcal{E}^2_{\mathrm{T}}\restriction P$ into $\mathcal{E}^2_{\mathrm{T}}\restriction Q$ which is a homeomorphism and P is a simple closed curve. Then Q is a simple closed curve.

(31) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that P satisfies conditions of simple closed curve holds P is compact.

(32) Let a, b, r be real numbers and C_1 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $r > 0$ and $C_1 = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p - [a, b]| = r\}$. Then C_1 is a simple closed curve.

Let a, b, r be real numbers. Let us assume that $r > 0$. The functor $\mathrm{Circle}(a, b, r)$ yielding a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 6) $\mathrm{Circle}(a, b, r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p - [a, b]| = r\}$.

Let a, b, r be real numbers. The functor $\mathrm{InsideOfCircle}(a, b, r)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 7) $\mathrm{InsideOfCircle}(a, b, r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p - [a, b]| < r\}$.

Let a, b, r be real numbers. The functor $\mathrm{ClosedInsideOfCircle}(a, b, r)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 8) $\mathrm{ClosedInsideOfCircle}(a, b, r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p - [a, b]| \leqslant r\}$.

Let a, b, r be real numbers. The functor $\mathrm{OutsideOfCircle}(a, b, r)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 9) $\mathrm{OutsideOfCircle}(a, b, r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p - [a, b]| > r\}$.

Let a, b, r be real numbers. The functor $\mathrm{ClosedOutsideOfCircle}(a, b, r)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 10) $\mathrm{ClosedOutsideOfCircle}(a, b, r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p - [a, b]| \geqslant r\}$.

One can prove the following propositions:

(33) Let r be a real number. Then $\mathrm{InsideOfCircle}(0, 0, r) = \{p : |p| < r\}$ and if $r > 0$, then $\mathrm{Circle}(0, 0, r) = \{p_2 : |p_2| = r\}$ and $\mathrm{OutsideOfCircle}(0, 0, r) = \{p_3 : |p_3| > r\}$ and $\mathrm{ClosedInsideOfCircle}(0, 0, r) = \{q : |q| \leqslant r\}$ and $\mathrm{ClosedOutsideOfCircle}(0, 0, r) = \{q_2 : |q_2| \geqslant r\}$.

(34) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_5 = \{p : -1 < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < 1 \wedge -1 < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_2| < 1\}$. Then $\mathrm{SqCirc}^\circ K_5 = C_1$.

(35) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_5 = \{p : -1 \nless p_{\mathbf{1}} \vee p_{\mathbf{1}} \nless 1 \vee -1 \nless p_{\mathbf{2}} \vee p_{\mathbf{2}} \nless 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_2| > 1\}$. Then $\mathrm{SqCirc}^\circ K_5 = C_1$.

(36) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_5 = \{p : -1 \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant 1 \wedge -1 \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_2| \leqslant 1\}$. Then $\mathrm{SqCirc}^\circ K_5 = C_1$.

(37) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $K_5 = \{p : -1 \not< p_{\mathbf{1}} \ \vee \ p_{\mathbf{1}} \not< 1 \ \vee \ -1 \not< p_{\mathbf{2}} \ \vee \ p_{\mathbf{2}} \not< 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_2| \geqslant 1\}$. Then $\mathrm{SqCirc}^\circ K_5 = C_1$.

(38) Let P_0, P_1, P_2, P_{11}, K_0, K_6, K_7, K_{11} be subsets of $\mathcal{E}^2_{\rm T}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $P = \mathrm{Circle}(0,0,1)$ and $P_0 = \mathrm{InsideOfCircle}(0,0,1)$ and $P_1 = \mathrm{OutsideOfCircle}(0,0,1)$ and $P_2 = \mathrm{ClosedInsideOfCircle}(0,0,1)$ and $P_{11} = \mathrm{ClosedOutsideOfCircle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $K_0 = \mathrm{InsideOfRectangle}(-1,1,-1,1)$ and $K_6 = \mathrm{OutsideOfRectangle}(-1,1,-1,1)$ and $K_7 = \mathrm{ClosedInsideOfRectangle}(-1,1,-1,1)$ and $K_{11} = \mathrm{ClosedOutsideOfRectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$. Then $f^\circ K = P$ and $(f^{-1})^\circ P = K$ and $f^\circ K_0 = P_0$ and $(f^{-1})^\circ P_0 = K_0$ and $f^\circ K_6 = P_1$ and $(f^{-1})^\circ P_1 = K_6$ and $f^\circ K_7 = P_2$ and $f^\circ K_{11} = P_{11}$ and $(f^{-1})^\circ P_2 = K_7$ and $(f^{-1})^\circ P_{11} = K_{11}$.

4. Order of Points on Rectangle

The following propositions are true:

(39) Let a, b, c, d be real numbers. Suppose $a \leqslant b$ and $c \leqslant d$. Then

(i) $\mathcal{L}([a,c],[a,d]) = \{p_1 : (p_1)_{\mathbf{1}} = a \ \wedge \ (p_1)_{\mathbf{2}} \leqslant d \ \wedge \ (p_1)_{\mathbf{2}} \geqslant c\}$,

(ii) $\mathcal{L}([a,d],[b,d]) = \{p_2 : (p_2)_{\mathbf{1}} \leqslant b \ \wedge \ (p_2)_{\mathbf{1}} \geqslant a \ \wedge \ (p_2)_{\mathbf{2}} = d\}$,

(iii) $\mathcal{L}([a,c],[b,c]) = \{q_1 : (q_1)_{\mathbf{1}} \leqslant b \ \wedge \ (q_1)_{\mathbf{1}} \geqslant a \ \wedge \ (q_1)_{\mathbf{2}} = c\}$, and

(iv) $\mathcal{L}([b,c],[b,d]) = \{q_2 : (q_2)_{\mathbf{1}} = b \ \wedge \ (q_2)_{\mathbf{2}} \leqslant d \ \wedge \ (q_2)_{\mathbf{2}} \geqslant c\}$.

(40) Let a, b, c, d be real numbers. Suppose $a \leqslant b$ and $c \leqslant d$. Then $\{p : p_{\mathbf{1}} = a \ \wedge \ c \leqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{2}} \leqslant d \ \vee \ p_{\mathbf{2}} = d \ \wedge \ a \leqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{1}} \leqslant b \ \vee \ p_{\mathbf{1}} = b \ \wedge \ c \leqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{2}} \leqslant d \ \vee \ p_{\mathbf{2}} = c \ \wedge \ a \leqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{1}} \leqslant b\} = \mathcal{L}([a,c],[a,d]) \cup \mathcal{L}([a,d],[b,d]) \cup (\mathcal{L}([a,c],[b,c]) \cup \mathcal{L}([b,c],[b,d]))$.

(41) For all real numbers a, b, c, d such that $a \leqslant b$ and $c \leqslant d$ holds $\mathcal{L}([a,c],[a,d]) \cap \mathcal{L}([a,c],[b,c]) = \{[a,c]\}$.

(42) For all real numbers a, b, c, d such that $a \leqslant b$ and $c \leqslant d$ holds $\mathcal{L}([a,c],[b,c]) \cap \mathcal{L}([b,c],[b,d]) = \{[b,c]\}$.

(43) For all real numbers a, b, c, d such that $a \leqslant b$ and $c \leqslant d$ holds $\mathcal{L}([a,d],[b,d]) \cap \mathcal{L}([b,c],[b,d]) = \{[b,d]\}$.

(44) For all real numbers a, b, c, d such that $a \leqslant b$ and $c \leqslant d$ holds $\mathcal{L}([a,c],[a,d]) \cap \mathcal{L}([a,d],[b,d]) = \{[a,d]\}$.

(45) $\{q : -1 = q_{\mathbf{1}} \ \wedge \ -1 \leqslant q_{\mathbf{2}} \ \wedge \ q_{\mathbf{2}} \leqslant 1 \ \vee \ q_{\mathbf{1}} = 1 \ \wedge \ -1 \leqslant q_{\mathbf{2}} \ \wedge \ q_{\mathbf{2}} \leqslant 1 \ \vee \ -1 = q_{\mathbf{2}} \ \wedge \ -1 \leqslant q_{\mathbf{1}} \ \wedge \ q_{\mathbf{1}} \leqslant 1 \ \vee \ 1 = q_{\mathbf{2}} \ \wedge \ -1 \leqslant q_{\mathbf{1}} \ \wedge \ q_{\mathbf{1}} \leqslant 1\} = \{p : p_{\mathbf{1}} = -1 \ \wedge \ -1 \leqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{2}} \leqslant 1 \ \vee \ p_{\mathbf{2}} = 1 \ \wedge \ -1 \leqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{1}} \leqslant 1 \ \vee \ p_{\mathbf{1}} = 1 \ \wedge \ -1 \leqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{2}} \leqslant 1 \ \vee \ p_{\mathbf{2}} = -1 \ \wedge \ -1 \leqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{1}} \leqslant 1\}$.

(46) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then W-bound $K = a$.

(47) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then N-bound $K = d$.

(48) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then E-bound $K = b$.

(49) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then S-bound $K = c$.

(50) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then NW-corner $K = [a, d]$.

(51) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then NE-corner $K = [b, d]$.

(52) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then SW-corner $K = [a, c]$.

(53) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then SE-corner $K = [b, c]$.

(54) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then W-most $K = \mathcal{L}([a,c],[a,d])$.

(55) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then E-most $K = \mathcal{L}([b,c],[b,d])$.

(56) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then W-min $K = [a, c]$ and E-max $K = [b, d]$.

(57) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$. Then $\mathcal{L}([a,c],[a,d]) \cup \mathcal{L}([a,d],[b,d])$ is an arc from W-min K to E-max K and $\mathcal{L}([a,c],[b,c]) \cup \mathcal{L}([b,c],[b,d])$ is an arc from E-max K to W-min K.

(58) Let P, P_1, P_3 be subsets of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, f_1, f_2 be finite sequences of elements of $\mathcal{E}^2_{\rm T}$, and p_0, p_1, p_5, p_{10} be points of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $P = \{p : p_{\mathbf{1}} = a \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = d \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b \vee p_{\mathbf{1}} = b \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = c \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b\}$ and $p_0 = [a, c]$ and $p_1 = [b, d]$ and $p_5 = [a,$

$d]$ and $p_{10} = [b, c]$ and $f_1 = \langle p_0, p_5, p_1\rangle$ and $f_2 = \langle p_0, p_{10}, p_1\rangle$. Then f_1 is a special sequence and $\widetilde{\mathcal{L}}(f_1) = \mathcal{L}(p_0, p_5) \cup \mathcal{L}(p_5, p_1)$ and f_2 is a special sequence and $\widetilde{\mathcal{L}}(f_2) = \mathcal{L}(p_0, p_{10}) \cup \mathcal{L}(p_{10}, p_1)$ and $P = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$ and $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{p_0, p_1\}$ and $(f_1)_1 = p_0$ and $(f_1)_{\mathrm{len}\, f_1} = p_1$ and $(f_2)_1 = p_0$ and $(f_2)_{\mathrm{len}\, f_2} = p_1$.

(59) Let P, P_1, P_3 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, f_1, f_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $P = \{p : p_{\mathbf{1}} = a \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = d \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b \vee p_{\mathbf{1}} = b \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = c \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b\}$ and $p_1 = [a, c]$ and $p_2 = [b, d]$ and $f_1 = \langle [a, c], [a, d], [b, d]\rangle$ and $f_2 = \langle [a, c], [b, c], [b, d]\rangle$ and $P_1 = \widetilde{\mathcal{L}}(f_1)$ and $P_3 = \widetilde{\mathcal{L}}(f_2)$. Then P_1 is an arc from p_1 to p_2 and P_3 is an arc from p_1 to p_2 and P_1 is non empty and P_3 is non empty and $P = P_1 \cup P_3$ and $P_1 \cap P_3 = \{p_1, p_2\}$.

(60) For all real numbers a, b, c, d such that $a < b$ and $c < d$ holds $\mathrm{Rectangle}(a, b, c, d)$ is a simple closed curve.

(61) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a, b, c, d)$ and $a < b$ and $c < d$, then $\mathrm{UpperArc}\, K = \mathcal{L}([a, c], [a, d]) \cup \mathcal{L}([a, d], [b, d])$.

(62) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a, b, c, d)$ and $a < b$ and $c < d$, then $\mathrm{LowerArc}\, K = \mathcal{L}([a, c], [b, c]) \cup \mathcal{L}([b, c], [b, d])$.

(63) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a, b, c, d)$ and $a < b$ and $c < d$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction \mathrm{UpperArc}\, K$ such that
f is a homeomorphism and $f(0) = \text{W-min}\, K$ and $f(1) = \text{E-max}\, K$ and $\mathrm{rng}\, f = \mathrm{UpperArc}\, K$ and for every real number r such that $r \in [0, \frac{1}{2}]$ holds $f(r) = (1 - 2 \cdot r) \cdot [a, c] + 2 \cdot r \cdot [a, d]$ and for every real number r such that $r \in [\frac{1}{2}, 1]$ holds $f(r) = (1 - (2 \cdot r - 1)) \cdot [a, d] + (2 \cdot r - 1) \cdot [b, d]$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([a, c], [a, d])$ holds $0 \leqslant \frac{\frac{p_{\mathbf{2}} - c}{d - c}}{2}$ and $\frac{\frac{p_{\mathbf{2}} - c}{d - c}}{2} \leqslant 1$ and $f(\frac{\frac{p_{\mathbf{2}} - c}{d - c}}{2}) = p$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([a, d], [b, d])$ holds $0 \leqslant \frac{\frac{p_{\mathbf{1}} - a}{b - a}}{2} + \frac{1}{2}$ and $\frac{\frac{p_{\mathbf{1}} - a}{b - a}}{2} + \frac{1}{2} \leqslant 1$ and $f(\frac{\frac{p_{\mathbf{1}} - a}{b - a}}{2} + \frac{1}{2}) = p$.

(64) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a, b, c, d)$ and $a < b$ and $c < d$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction \mathrm{LowerArc}\, K$ such that
f is a homeomorphism and $f(0) = \text{E-max}\, K$ and $f(1) = \text{W-min}\, K$ and $\mathrm{rng}\, f = \mathrm{LowerArc}\, K$ and for every real number r such that $r \in [0, \frac{1}{2}]$ holds $f(r) = (1 - 2 \cdot r) \cdot [b, d] + 2 \cdot r \cdot [b, c]$ and for every real number r such that $r \in [\frac{1}{2}, 1]$ holds $f(r) = (1 - (2 \cdot r - 1)) \cdot [b, c] + (2 \cdot r - 1) \cdot [a, c]$ and for every

point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([b,d],[b,c])$ holds $0 \leqslant \frac{\frac{p_{\mathbf{2}}-d}{c-d}}{2}$ and $\frac{\frac{p_{\mathbf{2}}-d}{c-d}}{2} \leqslant 1$ and $f(\frac{\frac{p_{\mathbf{2}}-d}{c-d}}{2}) = p$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([b,c],[a,c])$ holds $0 \leqslant \frac{\frac{p_{\mathbf{1}}-b}{a-b}}{2} + \frac{1}{2}$ and $\frac{\frac{p_{\mathbf{1}}-b}{a-b}}{2} + \frac{1}{2} \leqslant 1$ and $f(\frac{\frac{p_{\mathbf{1}}-b}{a-b}}{2} + \frac{1}{2}) = p$.

(65) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,c],[a,d])$ and $p_2 \in \mathcal{L}([a,c],[a,d])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if $(p_1)_{\mathbf{2}} \leqslant (p_2)_{\mathbf{2}}$.

(66) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,d],[b,d])$ and $p_2 \in \mathcal{L}([a,d],[b,d])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if $(p_1)_{\mathbf{1}} \leqslant (p_2)_{\mathbf{1}}$.

(67) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([b,c],[b,d])$ and $p_2 \in \mathcal{L}([b,c],[b,d])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if $(p_1)_{\mathbf{2}} \geqslant (p_2)_{\mathbf{2}}$.

(68) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,c],[b,c])$ and $p_2 \in \mathcal{L}([a,c],[b,c])$. Then $\mathrm{LE}(p_1,p_2,K)$ and $p_1 \neq \text{W-min}\,K$ if and only if $(p_1)_{\mathbf{1}} \geqslant (p_2)_{\mathbf{1}}$ and $p_2 \neq \text{W-min}\,K$.

(69) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,c],[a,d])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if one of the following conditions is satisfied:

(i) $p_2 \in \mathcal{L}([a,c],[a,d])$ and $(p_1)_{\mathbf{2}} \leqslant (p_2)_{\mathbf{2}}$, or

(ii) $p_2 \in \mathcal{L}([a,d],[b,d])$, or

(iii) $p_2 \in \mathcal{L}([b,d],[b,c])$, or

(iv) $p_2 \in \mathcal{L}([b,c],[a,c])$ and $p_2 \neq \text{W-min}\,K$.

(70) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,d],[b,d])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if one of the following conditions is satisfied:

(i) $p_2 \in \mathcal{L}([a,d],[b,d])$ and $(p_1)_{\mathbf{1}} \leqslant (p_2)_{\mathbf{1}}$, or

(ii) $p_2 \in \mathcal{L}([b,d],[b,c])$, or

(iii) $p_2 \in \mathcal{L}([b,c],[a,c])$ and $p_2 \neq \text{W-min}\,K$.

(71) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([b,d],[b,c])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if one of the following conditions is satisfied:

(i) $p_2 \in \mathcal{L}([b,d],[b,c])$ and $(p_1)_{\mathbf{2}} \geqslant (p_2)_{\mathbf{2}}$, or

(ii) $p_2 \in \mathcal{L}([b,c],[a,c])$ and $p_2 \neq$ W-min K.

(72) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $K = \text{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([b,c],[a,c])$ and $p_1 \neq$ W-min K. Then $\text{LE}(p_1,p_2,K)$ if and only if the following conditions are satisfied:

(i) $p_2 \in \mathcal{L}([b,c],[a,c])$,

(ii) $(p_1)_{\mathbf{1}} \geqslant (p_2)_{\mathbf{1}}$, and

(iii) $p_2 \neq$ W-min K.

(73) Let x be a set and a, b, c, d be real numbers. Suppose $x \in \text{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$. Then $x \in \mathcal{L}([a,c],[a,d])$ or $x \in \mathcal{L}([a,d],[b,d])$ or $x \in \mathcal{L}([b,d],[b,c])$ or $x \in \mathcal{L}([b,c],[a,c])$.

5. General Fashoda Theorem for Square

The following propositions are true:

(74) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$. Suppose $K = \text{Rectangle}(-1,1,-1,1)$ and $\text{LE}(p_1,p_2,K)$ and $p_1 \in \mathcal{L}([-1,-1],[-1,1])$. Then $p_2 \in \mathcal{L}([-1,-1],[-1,1])$ and $(p_2)_{\mathbf{2}} \geqslant (p_1)_{\mathbf{2}}$ or $p_2 \in \mathcal{L}([-1,1],[1,1])$ or $p_2 \in \mathcal{L}([1,1],[1,-1])$ or $p_2 \in \mathcal{L}([1,-1],[-1,-1])$ and $p_2 \neq [-1,-1]$.

(75) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose $P = \text{Circle}(0,0,1)$ and $K = \text{Rectangle}(-1,1,-1,1)$ and $f = \text{SqCirc}$ and $p_1 \in \mathcal{L}([-1,-1],[-1,1])$ and $(p_1)_{\mathbf{2}} \geqslant 0$ and $\text{LE}(p_1,p_2,K)$. Then $\text{LE}(f(p_1),f(p_2),P)$.

(76) Let p_1, p_2, p_3 be points of $\mathcal{E}^2_{\rm T}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose $P = \text{Circle}(0,0,1)$ and $K = \text{Rectangle}(-1,1,-1,1)$ and $f = \text{SqCirc}$ and $p_1 \in \mathcal{L}([-1,-1],[-1,1])$ and $(p_1)_{\mathbf{2}} \geqslant 0$ and $\text{LE}(p_1,p_2,K)$ and $\text{LE}(p_2,p_3,K)$. Then $\text{LE}(f(p_2),f(p_3),P)$.

(77) Let p be a point of $\mathcal{E}^2_{\rm T}$ and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. If $f = \text{SqCirc}$ and $p_{\mathbf{1}} = -1$ and $p_{\mathbf{2}} < 0$, then $f(p)_{\mathbf{1}} < 0$ and $f(p)_{\mathbf{2}} < 0$.

(78) Let p be a point of $\mathcal{E}^2_{\rm T}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. If $P = \text{Circle}(0,0,1)$ and $K = \text{Rectangle}(-1,1,-1,1)$ and $f = \text{SqCirc}$, then $f(p)_{\mathbf{1}} \geqslant 0$ iff $p_{\mathbf{1}} \geqslant 0$.

(79) Let p be a point of $\mathcal{E}^2_{\rm T}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. If $P = \text{Circle}(0,0,1)$ and $K = \text{Rectangle}(-1,1,-1,1)$ and $f = \text{SqCirc}$, then $f(p)_{\mathbf{2}} \geqslant 0$ iff $p_{\mathbf{2}} \geqslant 0$.

(80) Let p, q be points of $\mathcal{E}^2_{\rm T}$ and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. If $f = \text{SqCirc}$ and $p \in \mathcal{L}([-1,-1],[-1,1])$ and $q \in \mathcal{L}([1,-1],[-1,-1])$, then $f(p)_{\mathbf{1}} \leqslant f(q)_{\mathbf{1}}$.

(81) Let p, q be points of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f = \mathrm{SqCirc}$ and $p \in \mathcal{L}([-1,-1],[-1,1])$ and $q \in \mathcal{L}([-1,-1],[-1,1])$ and $p_{\mathbf{2}} \geqslant q_{\mathbf{2}}$ and $p_{\mathbf{2}} < 0$. Then $f(p)_{\mathbf{2}} \geqslant f(q)_{\mathbf{2}}$.

(82) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$. Suppose $\mathrm{LE}(p_1,p_2,K)$ and $\mathrm{LE}(p_2,p_3,K)$ and $\mathrm{LE}(p_3,p_4,K)$. Then $f(p_1)$, $f(p_2)$, $f(p_3)$, $f(p_4)$ are in this order on P.

(83) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and not $\mathrm{LE}(p_1,p_2,P)$, then $\mathrm{LE}(p_2,p_1,P)$.

(84) Let p_1, p_2, p_3 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_3 \in P$. Then $\mathrm{LE}(p_1,p_2,P)$ and $\mathrm{LE}(p_2,p_3,P)$ or $\mathrm{LE}(p_1,p_3,P)$ and $\mathrm{LE}(p_3,p_2,P)$ or $\mathrm{LE}(p_2,p_1,P)$ and $\mathrm{LE}(p_1,p_3,P)$ or $\mathrm{LE}(p_2,p_3,P)$ and $\mathrm{LE}(p_3,p_1,P)$ or $\mathrm{LE}(p_3,p_1,P)$ and $\mathrm{LE}(p_1,p_2,P)$ or $\mathrm{LE}(p_3,p_2,P)$ and $\mathrm{LE}(p_2,p_1,P)$.

(85) Let p_1, p_2, p_3 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_3 \in P$ and $\mathrm{LE}(p_2,p_3,P)$. Then $\mathrm{LE}(p_1,p_2,P)$ or $\mathrm{LE}(p_2,p_1,P)$ and $\mathrm{LE}(p_1,p_3,P)$ or $\mathrm{LE}(p_3,p_1,P)$.

(86) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_3 \in P$ and $p_4 \in P$ and $\mathrm{LE}(p_2,p_3,P)$ and $\mathrm{LE}(p_3,p_4,P)$. Then $\mathrm{LE}(p_1,p_2,P)$ or $\mathrm{LE}(p_2,p_1,P)$ and $\mathrm{LE}(p_1,p_3,P)$ or $\mathrm{LE}(p_3,p_1,P)$ and $\mathrm{LE}(p_1,p_4,P)$ or $\mathrm{LE}(p_4,p_1,P)$.

(87) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$ and $\mathrm{LE}(f(p_1),f(p_2),P)$ and $\mathrm{LE}(f(p_2),f(p_3),P)$ and $\mathrm{LE}(f(p_3),f(p_4),P)$. Then p_1, p_2, p_3, p_4 are in this order on K.

(88) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$. Then p_1, p_2, p_3, p_4 are in this order on K if and only if $f(p_1)$, $f(p_2)$, $f(p_3)$, $f(p_4)$ are in this order on P.

(89) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, K be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and K_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(-1,1,-1,1)$ and p_1, p_2, p_3, p_4 are in this order on K. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $K_0 = \mathrm{ClosedInsideOfRectangle}(-1,1,-1,1)$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and $\mathrm{rng}\, f \subseteq K_0$ and $\mathrm{rng}\, g \subseteq K_0$.

Then rng f meets rng g.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[2] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.
[6] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[7] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[8] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[9] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces - fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[10] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.
[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Simple closed curves. *Formalized Mathematics*, 2(**5**):663–664, 1991.
[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[14] Artur Korniłowicz. The ordering of points on a curve. Part III. *Formalized Mathematics*, 10(**3**):169–171, 2002.
[15] Yatsuka Nakamura. On Outside Fashoda Meet Theorem. *Formalized Mathematics*, 9(**4**):697–704, 2001.
[16] Yatsuka Nakamura. On the simple closed curve property of the circle and the Fashoda Meet Theorem. *Formalized Mathematics*, 9(**4**):801–808, 2001.
[17] Yatsuka Nakamura and Andrzej Trybulec. A decomposition of a simple closed curves and the order of their points. *Formalized Mathematics*, 6(**4**):563–572, 1997.
[18] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[19] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[20] Agnieszka Sakowicz, Jarosław Gryko, and Adam Grabowski. Sequences in $\mathcal{E}^N_{\mathrm{T}}$. *Formalized Mathematics*, 5(**1**):93–96, 1996.
[21] Andrzej Trybulec. Introduction to arithmetics. *To appear in Formalized Mathematics.*
[22] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[23] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[24] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[26] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received February 25, 2003