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Complex Valued Functions Space

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Summary. This article is an extension of [9] to complex valued functions.

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The articles [14], [5], [16], [10], [17], [3], [4], [1], [12], [11], [15], [2], [8], [13], [9], [7], and [6] provide the notation and terminology for this paper.

1. Operation of Complex Functions

We adopt the following convention: x_1, x_2, z are sets, A is a non empty set, and f, g, h are elements of $\mathbb{C}^A$.

Let us consider A. The functor $+_{\mathbb{C}^A}$ yielding a binary operation on $\mathbb{C}^A$ is defined by:

(Def. 1) For all elements f, g of $\mathbb{C}^A$ holds $+_{\mathbb{C}^A}(f, g) = (+_{\mathbb{C}})^{\circ}(f, g)$.

Let us consider A. The functor $\cdot_{\mathbb{C}^A}$ yielding a binary operation on $\mathbb{C}^A$ is defined as follows:

(Def. 2) For all elements f, g of $\mathbb{C}^A$ holds $\cdot_{\mathbb{C}^A}(f, g) = (\cdot_{\mathbb{C}})^{\circ}(f, g)$.

Let us consider A. The functor $\cdot^{\mathbb{C}}_{\mathbb{C}^A}$ yielding a function from $[:\mathbb{C}, \mathbb{C}^A:]$ into $\mathbb{C}^A$ is defined by:

(Def. 3) For every complex number z and for every element f of $\mathbb{C}^A$ and for every element x of A holds $\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle z, f\rangle)(x) = z \cdot f(x)$.

Let us consider A. The functor $\mathbf{0}_{\mathbb{C}^A}$ yielding an element of $\mathbb{C}^A$ is defined by:

(Def. 4) $\mathbf{0}_{\mathbb{C}^A} = A \longmapsto 0_{\mathbb{C}}$.

Let us consider A. The functor $\mathbf{1}_{\mathbb{C}^A}$ yields an element of $\mathbb{C}^A$ and is defined by:

(Def. 5) $\mathbf{1}_{\mathbb{C}^A} = A \longmapsto 1_{\mathbb{C}}$.

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One can prove the following propositions:

(1) $h = +_{\mathbb{C}^A}(f, g)$ iff for every element x of A holds $h(x) = f(x) + g(x)$.

(2) $h = \cdot_{\mathbb{C}^A}(f, g)$ iff for every element x of A holds $h(x) = f(x) \cdot g(x)$.

(3) For every element x of A holds $\mathbf{1}_{\mathbb{C}^A}(x) = 1_{\mathbb{C}}$.

(4) For every element x of A holds $\mathbf{0}_{\mathbb{C}^A}(x) = 0_{\mathbb{C}}$.

(5) $\mathbf{0}_{\mathbb{C}^A} \neq \mathbf{1}_{\mathbb{C}^A}$.

In the sequel a, b denote complex numbers.

The following proposition is true

(6) $h = \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle)$ iff for every element x of A holds $h(x) = a \cdot f(x)$.

In the sequel u, v, w are vectors of $\langle \mathbb{C}^A, \mathbf{0}_{\mathbb{C}^A}, +_{\mathbb{C}^A}, \cdot^{\mathbb{C}}_{\mathbb{C}^A}\rangle$.

One can prove the following propositions:

(7) $+_{\mathbb{C}^A}(f, g) = +_{\mathbb{C}^A}(g, f)$.

(8) $+_{\mathbb{C}^A}(f, +_{\mathbb{C}^A}(g, h)) = +_{\mathbb{C}^A}(+_{\mathbb{C}^A}(f, g), h)$.

(9) $\cdot_{\mathbb{C}^A}(f, g) = \cdot_{\mathbb{C}^A}(g, f)$.

(10) $\cdot_{\mathbb{C}^A}(f, \cdot_{\mathbb{C}^A}(g, h)) = \cdot_{\mathbb{C}^A}(\cdot_{\mathbb{C}^A}(f, g), h)$.

(11) $\cdot_{\mathbb{C}^A}(\mathbf{1}_{\mathbb{C}^A}, f) = f$.

(12) $+_{\mathbb{C}^A}(\mathbf{0}_{\mathbb{C}^A}, f) = f$.

(13) $+_{\mathbb{C}^A}(f, \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle -1_{\mathbb{C}}, f\rangle)) = \mathbf{0}_{\mathbb{C}^A}$.

(14) $\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle 1_{\mathbb{C}}, f\rangle) = f$.

(15) $\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, f\rangle)\rangle) = \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a \cdot b, f\rangle)$.

(16) $+_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, f\rangle)) = \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a + b, f\rangle)$.

(17) $\cdot_{\mathbb{C}^A}(f, +_{\mathbb{C}^A}(g, h)) = +_{\mathbb{C}^A}(\cdot_{\mathbb{C}^A}(f, g), \cdot_{\mathbb{C}^A}(f, h))$.

(18) $\cdot_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), g) = \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, \cdot_{\mathbb{C}^A}(f, g)\rangle)$.

2. Complex Linear Space of Complex Valued Functions

One can prove the following propositions:

(19) There exist f, g such that

(i) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1_{\mathbb{C}}$ and if $z \neq x_1$, then $f(z) = 0_{\mathbb{C}}$, and

(ii) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0_{\mathbb{C}}$ and if $z \neq x_1$, then $g(z) = 1_{\mathbb{C}}$.

(20) Suppose that

(i) $x_1 \in A$,

(ii) $x_2 \in A$,

(iii) $x_1 \neq x_2$,

(iv) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1_{\mathbb{C}}$ and if $z \neq x_1$, then $f(z) = 0_{\mathbb{C}}$, and

(v) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0_{\mathbb{C}}$ and if $z \neq x_1$, then $g(z) = 1_{\mathbb{C}}$.
Let given a, b. If $+_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle)) = \mathbf{0}_{\mathbb{C}^A}$, then $a = 0_{\mathbb{C}}$ and $b = 0_{\mathbb{C}}$.

(21) If $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$, then there exist f, g such that for all a, b such that $+_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle)) = \mathbf{0}_{\mathbb{C}^A}$ holds $a = 0_{\mathbb{C}}$ and $b = 0_{\mathbb{C}}$.

(22) Suppose that
(i) $A = \{x_1, x_2\}$,
(ii) $x_1 \neq x_2$,
(iii) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1_{\mathbb{C}}$ and if $z \neq x_1$, then $f(z) = 0_{\mathbb{C}}$, and
(iv) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0_{\mathbb{C}}$ and if $z \neq x_1$, then $g(z) = 1_{\mathbb{C}}$.
Let given h. Then there exist a, b such that $h = +_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle))$.

(23) If $A = \{x_1, x_2\}$ and $x_1 \neq x_2$, then there exist f, g such that for every h there exist a, b such that $h = +_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle))$.

(24) Suppose $A = \{x_1, x_2\}$ and $x_1 \neq x_2$. Then there exist f, g such that for all a, b such that $+_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle)) = \mathbf{0}_{\mathbb{C}^A}$ holds $a = 0_{\mathbb{C}}$ and $b = 0_{\mathbb{C}}$ and for every h there exist a, b such that $h = +_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle))$.

(25) $\langle \mathbb{C}^A, \mathbf{0}_{\mathbb{C}^A}, +_{\mathbb{C}^A}, \cdot^{\mathbb{C}}_{\mathbb{C}^A}\rangle$ is a complex linear space.

Let us consider A. The functor ComplexVectSpace(A) yields a strict complex linear space and is defined by:

(Def. 6) ComplexVectSpace(A) $= \langle \mathbb{C}^A, \mathbf{0}_{\mathbb{C}^A}, +_{\mathbb{C}^A}, \cdot^{\mathbb{C}}_{\mathbb{C}^A}\rangle$.

We now state the proposition

(26) There exists a strict complex linear space V and there exist vectors u, v of V such that for all a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0_{\mathbb{C}}$ and $b = 0_{\mathbb{C}}$ and for every vector w of V there exist a, b such that $w = a \cdot u + b \cdot v$.

Let us consider A. The functor CRing(A) yielding a strict double loop structure is defined by:

(Def. 7) CRing(A) $= \langle \mathbb{C}^A, +_{\mathbb{C}^A}, \cdot_{\mathbb{C}^A}, \mathbf{1}_{\mathbb{C}^A}, \mathbf{0}_{\mathbb{C}^A}\rangle$.

Let us consider A. Observe that CRing(A) is non empty.

We now state two propositions:

(27) Let x, y, z be elements of CRing(A). Then $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x+0_{\mathrm{CRing}(A)} = x$ and there exists an element t of CRing(A) such that $x + t = 0_{\mathrm{CRing}(A)}$ and $x \cdot y = y \cdot x$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{CRing}(A)} = x$ and $\mathbf{1}_{\mathrm{CRing}(A)} \cdot x = x$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$.

(28) CRing(A) is a commutative ring.

We introduce complex algebra structures which are extensions of double loop structure and CLS structure and are systems

⟨ a carrier, a multiplication, an addition, an external multiplication, a unity, a zero ⟩,

where the carrier is a set, the multiplication and the addition are binary operations on the carrier, the external multiplication is a function from $[\!: \mathbb{C}$, the carrier $:\!]$ into the carrier, and the unity and the zero are elements of the carrier.

Let us mention that there exists a complex algebra structure which is non empty.

Let us consider A. The functor CAlgebra(A) yielding a strict complex algebra structure is defined as follows:

(Def. 8) $\mathrm{CAlgebra}(A) = \langle \mathbb{C}^A, \cdot_{\mathbb{C}^A}, +_{\mathbb{C}^A}, \cdot^{\mathbb{C}}_{\mathbb{C}^A}, \mathbf{1}_{\mathbb{C}^A}, \mathbf{0}_{\mathbb{C}^A} \rangle$.

Let us consider A. Observe that CAlgebra(A) is non empty.

Next we state the proposition

(29) Let x, y, z be elements of CAlgebra(A) and given a, b. Then $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x + 0_{\mathrm{CAlgebra}(A)} = x$ and there exists an element t of CAlgebra(A) such that $x + t = 0_{\mathrm{CAlgebra}(A)}$ and $x \cdot y = y \cdot x$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{CAlgebra}(A)} = x$ and $x \cdot (y+z) = x \cdot y + x \cdot z$ and $a \cdot (x \cdot y) = (a \cdot x) \cdot y$ and $a \cdot (x+y) = a \cdot x + a \cdot y$ and $(a+b) \cdot x = a \cdot x + b \cdot x$ and $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.

Let I_1 be a non empty complex algebra structure. We say that I_1 is complex algebra-like if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let x, y, z be elements of I_1 and given a, b. Then $x \cdot \mathbf{1}_{(I_1)} = x$ and $x \cdot (y+z) = x \cdot y + x \cdot z$ and $a \cdot (x \cdot y) = (a \cdot x) \cdot y$ and $a \cdot (x+y) = a \cdot x + a \cdot y$ and $(a+b) \cdot x = a \cdot x + b \cdot x$ and $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.

Let us note that there exists a non empty complex algebra structure which is strict, Abelian, add-associative, right zeroed, right complementable, commutative, associative, and complex algebra-like.

A complex algebra is an Abelian add-associative right zeroed right complementable commutative associative complex algebra-like non empty complex algebra structure.

One can prove the following proposition

(30) CAlgebra(A) is a complex algebra.

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Banach Algebra of Bounded Complex Linear Operators

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Summary. This article is an extension of [16].

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The terminology and notation used here are introduced in the following articles: [18], [8], [20], [5], [7], [6], [3], [1], [17], [13], [19], [14], [2], [4], [15], [10], [11], [9], and [12].

One can prove the following propositions:

(1) Let X, Y, Z be complex linear spaces, f be a linear operator from X into Y, and g be a linear operator from Y into Z. Then $g \cdot f$ is a linear operator from X into Z.

(2) Let X, Y, Z be complex normed spaces, f be a bounded linear operator from X into Y, and g be a bounded linear operator from Y into Z. Then

(i) $g \cdot f$ is a bounded linear operator from X into Z, and

(ii) for every vector x of X holds $\|(g \cdot f)(x)\| \leqslant (\mathrm{BdLinOpsNorm}(Y, Z))(g) \cdot (\mathrm{BdLinOpsNorm}(X, Y))(f) \cdot \|x\|$ and $(\mathrm{BdLinOpsNorm}(X, Z))(g \cdot f) \leqslant (\mathrm{BdLinOpsNorm}(Y, Z))(g) \cdot (\mathrm{BdLinOpsNorm}(X, Y))(f)$.

Let X be a complex normed space and let f, g be bounded linear operators from X into X. Then $g \cdot f$ is a bounded linear operator from X into X.

Let X be a complex normed space and let f, g be elements of $\mathrm{BdLinOps}(X, X)$. The functor $f + g$ yields an element of $\mathrm{BdLinOps}(X, X)$ and is defined by:

(Def. 1) $f + g = (\mathrm{Add}_{-}(\mathrm{BdLinOps}(X, X), \mathrm{CVSpLinOps}(X, X)))(f, g)$.

Let X be a complex normed space and let f, g be elements of $\mathrm{BdLinOps}(X, X)$. The functor $g \cdot f$ yields an element of $\mathrm{BdLinOps}(X, X)$ and is defined as follows:

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(Def. 2) $g \cdot f = \text{modetrans}(g, X, X) \cdot \text{modetrans}(f, X, X)$.

Let X be a complex normed space, let f be an element of BdLinOps(X, X), and let z be a complex number. The functor $z \cdot f$ yields an element of BdLinOps(X, X) and is defined by:

(Def. 3) $z \cdot f = (\text{Mult}_{\_}(\text{BdLinOps}(X, X), \text{CVSpLinOps}(X, X)))(z, f)$.

Let X be a complex normed space. The functor FuncMult(X) yields a binary operation on BdLinOps(X, X) and is defined as follows:

(Def. 4) For all elements f, g of BdLinOps(X, X) holds $(\text{FuncMult}(X))(f, g) = f \cdot g$.

The following proposition is true

(3) For every complex normed space X holds $\text{id}_{\text{the carrier of } X}$ is a bounded linear operator from X into X.

Let X be a complex normed space. The functor FuncUnit(X) yielding an element of BdLinOps(X, X) is defined by:

(Def. 5) $\text{FuncUnit}(X) = \text{id}_{\text{the carrier of } X}$.

The following propositions are true:

(4) Let X be a complex normed space and f, g, h be bounded linear operators from X into X. Then $h = f \cdot g$ if and only if for every vector x of X holds $h(x) = f(g(x))$.

(5) For every complex normed space X and for all bounded linear operators f, g, h from X into X holds $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

(6) Let X be a complex normed space and f be a bounded linear operator from X into X. Then $f \cdot \text{id}_{\text{the carrier of } X} = f$ and $\text{id}_{\text{the carrier of } X} \cdot f = f$.

(7) For every complex normed space X and for all elements f, g, h of BdLinOps(X, X) holds $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

(8) For every complex normed space X and for every element f of BdLinOps(X, X) holds $f \cdot \text{FuncUnit}(X) = f$ and $\text{FuncUnit}(X) \cdot f = f$.

(9) For every complex normed space X and for all elements f, g, h of BdLinOps(X, X) holds $f \cdot (g + h) = f \cdot g + f \cdot h$.

(10) For every complex normed space X and for all elements f, g, h of BdLinOps(X, X) holds $(g + h) \cdot f = g \cdot f + h \cdot f$.

(11) Let X be a complex normed space, f, g be elements of BdLinOps(X, X), and a, b be complex numbers. Then $(a \cdot b) \cdot (f \cdot g) = a \cdot f \cdot (b \cdot g)$.

(12) Let X be a complex normed space, f, g be elements of BdLinOps(X, X), and a be a complex number. Then $a \cdot (f \cdot g) = (a \cdot f) \cdot g$.

Let X be a complex normed space.

The functor RingOfBoundedLinearOperators(X) yields a double loop structure and is defined by:

(Def. 6) RingOfBoundedLinearOperators$(X) = \langle$BdLinOps(X, X),

$\mathrm{Add\_}(\mathrm{BdLinOps}(X,X),\mathrm{CVSpLinOps}(X,X)),\mathrm{FuncMult}(X),\mathrm{FuncUnit}(X),$ $\mathrm{Zero\_}(\mathrm{BdLinOps}(X,X),\mathrm{CVSpLinOps}(X,X))\rangle$.

Let X be a complex normed space.

Note that RingOfBoundedLinearOperators(X) is non empty and strict.

Next we state two propositions:

(13) Let X be a complex normed space and x, y, z be elements of RingOfBoundedLinearOperators(X). Then $x+y=y+x$ and $(x+y)+z=x+(y+z)$ and $x+0_{\mathrm{RingOfBoundedLinearOperators}(X)}=x$ and there exists an element t of RingOfBoundedLinearOperators(X) such that $x+t=0_{\mathrm{RingOfBoundedLinearOperators}(X)}$ and $(x\cdot y)\cdot z=x\cdot(y\cdot z)$ and $x\cdot\mathbf{1}_{\mathrm{RingOfBoundedLinearOperators}(X)}=x$ and $\mathbf{1}_{\mathrm{RingOfBoundedLinearOperators}(X)}\cdot x=x$ and $x\cdot(y+z)=x\cdot y+x\cdot z$ and $(y+z)\cdot x=y\cdot x+z\cdot x$.

(14) For every complex normed space X holds RingOfBoundedLinearOperators(X) is a ring.

Let X be a complex normed space.

Observe that RingOfBoundedLinearOperators(X) is Abelian, add-associative, right zeroed, right complementable, associative, left unital, right unital, and distributive.

Let X be a complex normed space. The functor CAlgBdLinOps(X) yields a complex algebra structure and is defined by:

(Def. 7) $\mathrm{CAlgBdLinOps}(X)=\langle\mathrm{BdLinOps}(X,X),\mathrm{FuncMult}(X),\mathrm{Add\_}(\mathrm{BdLinOps}(X,X),\mathrm{CVSpLinOps}(X,X)),\mathrm{Mult\_}(\mathrm{BdLinOps}(X,X),\mathrm{CVSpLinOps}(X,X)),$ $\mathrm{FuncUnit}(X),\mathrm{Zero\_}(\mathrm{BdLinOps}(X,X),\mathrm{CVSpLinOps}(X,X))\rangle$.

Let X be a complex normed space. Note that CAlgBdLinOps(X) is non empty and strict.

The following proposition is true

(15) Let X be a complex normed space, x, y, z be elements of CAlgBdLinOps(X), and a, b be complex numbers. Then $x+y=y+x$ and $(x+y)+z=x+(y+z)$ and $x+0_{\mathrm{CAlgBdLinOps}(X)}=x$ and there exists an element t of CAlgBdLinOps(X) such that $x+t=0_{\mathrm{CAlgBdLinOps}(X)}$ and $(x\cdot y)\cdot z=x\cdot(y\cdot z)$ and $x\cdot\mathbf{1}_{\mathrm{CAlgBdLinOps}(X)}=x$ and $\mathbf{1}_{\mathrm{CAlgBdLinOps}(X)}\cdot x=x$ and $x\cdot(y+z)=x\cdot y+x\cdot z$ and $(y+z)\cdot x=y\cdot x+z\cdot x$ and $a\cdot(x\cdot y)=(a\cdot x)\cdot y$ and $a\cdot(x+y)=a\cdot x+a\cdot y$ and $(a+b)\cdot x=a\cdot x+b\cdot x$ and $(a\cdot b)\cdot x=a\cdot(b\cdot x)$ and $(a\cdot b)\cdot(x\cdot y)=a\cdot x\cdot(b\cdot y)$.

A complex BL algebra is an Abelian add-associative right zeroed right complementable associative complex algebra-like non empty complex algebra structure.

We now state the proposition

(16) For every complex normed space X holds CAlgBdLinOps(X) is a complex BL algebra.

Let us note that Complex-l1-Space is complete.

Let us mention that Complex-l1-Space is non trivial.

Let us note that there exists a complex Banach space which is non trivial.

The following two propositions are true:

(17) For every non trivial complex normed space X there exists a vector w of X such that $\|w\| = 1$.

(18) For every non trivial complex normed space X holds $(\mathrm{BdLinOpsNorm}(X, X))(\mathrm{id}_{\text{the carrier of } X}) = 1$.

We introduce normed complex algebra structures which are extensions of complex algebra structure and complex normed space structure and are systems

⟨ a carrier, a multiplication, an addition, an external multiplication, a unity, a zero, a norm ⟩,

where the carrier is a set, the multiplication and the addition are binary operations on the carrier, the external multiplication is a function from $[\!:\mathbb{C}$, the carrier$:\!]$ into the carrier, the unity and the zero are elements of the carrier, and the norm is a function from the carrier into $\mathbb{R}$.

One can check that there exists a normed complex algebra structure which is non empty.

Let X be a complex normed space. The functor CNAlgBdLinOps(X) yields a normed complex algebra structure and is defined by:

(Def. 8) $\mathrm{CNAlgBdLinOps}(X) = \langle \mathrm{BdLinOps}(X, X), \mathrm{FuncMult}(X), \mathrm{Add}\_(\mathrm{BdLinOps}(X, X), \mathrm{CVSpLinOps}(X, X)), \mathrm{Mult}\_(\mathrm{BdLinOps}(X, X), \mathrm{CVSpLinOps}(X, X)), \mathrm{FuncUnit}(X), \mathrm{Zero}\_(\mathrm{BdLinOps}(X, X), \mathrm{CVSpLinOps}(X, X)), \mathrm{BdLinOpsNorm}(X, X)\rangle$.

Let X be a complex normed space. Note that CNAlgBdLinOps(X) is non empty and strict.

The following propositions are true:

(19) Let X be a complex normed space, x, y, z be elements of CNAlgBdLinOps(X), and a, b be complex numbers. Then $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x+0_{\mathrm{CNAlgBdLinOps}(X)} = x$ and there exists an element t of CNAlgBdLinOps(X) such that $x+t = 0_{\mathrm{CNAlgBdLinOps}(X)}$ and $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ and $x\cdot \mathbf{1}_{\mathrm{CNAlgBdLinOps}(X)} = x$ and $\mathbf{1}_{\mathrm{CNAlgBdLinOps}(X)}\cdot x = x$ and $x\cdot(y+z) = x\cdot y+x\cdot z$ and $(y+z)\cdot x = y\cdot x+z\cdot x$ and $a\cdot(x\cdot y) = (a\cdot x)\cdot y$ and $(a\cdot b)\cdot(x\cdot y) = a\cdot x\cdot(b\cdot y)$ and $a\cdot(x+y) = a\cdot x+a\cdot y$ and $(a+b)\cdot x = a\cdot x+b\cdot x$ and $(a\cdot b)\cdot x = a\cdot(b\cdot x)$ and $1_{\mathbb{C}}\cdot x = x$.

(20) Let X be a complex normed space. Then CNAlgBdLinOps(X) is complex normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, complex algebra-like, and complex linear space-like.

Let us observe that there exists a non empty normed complex algebra structure which is complex normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, complex algebra-like, complex linear space-like, and strict.

A normed complex algebra is a complex normed space-like Abelian add-associative right zeroed right complementable associative complex algebra-like complex linear space-like non empty normed complex algebra structure.

Let X be a complex normed space. One can check that CNAlgBdLinOps(X) is complex normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, complex algebra-like, and complex linear space-like.

Let X be a non empty normed complex algebra structure. We say that X is Banach Algebra-like1 if and only if:

(Def. 9) For all elements x, y of X holds $\|x \cdot y\| \leqslant \|x\| \cdot \|y\|$.

We say that X is Banach Algebra-like2 if and only if:

(Def. 10) $\|\mathbf{1}_X\| = 1$.

We say that X is Banach Algebra-like3 if and only if:

(Def. 11) For every complex number a and for all elements x, y of X holds $a \cdot (x \cdot y) = x \cdot (a \cdot y)$.

Let X be a normed complex algebra. We say that X is Banach Algebra-like if and only if the condition (Def. 12) is satisfied.

(Def. 12) X is Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left unital, left distributive, and complete.

One can verify that every normed complex algebra which is Banach Algebra-like is also Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left distributive, left unital, and complete and every normed complex algebra which is Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left distributive, left unital, and complete is also Banach Algebra-like.

Let X be a non trivial complex Banach space. One can verify that CNAlgBdLinOps(X) is Banach Algebra-like.

One can check that there exists a normed complex algebra which is Banach Algebra-like.

A complex Banach algebra is a Banach Algebra-like normed complex algebra.

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Formulas and Identities of Trigonometric Functions

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The articles [2], [5], [1], [6], [3], and [4] provide the terminology and notation for this paper.

In this paper t_1, t_2, t_3, t_4 are real numbers.

One can prove the following propositions:

(1) If $\cos t_1 \neq 0$, then $\operatorname{cosec} t_1 = \frac{\sec t_1}{\tan t_1}$.

(2) If $\sin t_1 \neq 0$, then $\cos t_1 = \sin t_1 \cdot \cot t_1$.

(3) If $\sin t_2 \neq 0$ and $\sin t_3 \neq 0$ and $\sin t_4 \neq 0$, then $\sin(t_2 + t_3 + t_4) = \sin t_2 \cdot \sin t_3 \cdot \sin t_4 \cdot ((\cot t_3 \cdot \cot t_4 + \cot t_2 \cdot \cot t_4 + \cot t_2 \cdot \cot t_3) - 1)$.

(4) If $\sin t_2 \neq 0$ and $\sin t_3 \neq 0$ and $\sin t_4 \neq 0$, then $\cos(t_2 + t_3 + t_4) = -\sin t_2 \cdot \sin t_3 \cdot \sin t_4 \cdot ((\cot t_2 + \cot t_3 + \cot t_4) - \cot t_2 \cdot \cot t_3 \cdot \cot t_4)$.

(5) $\sin(2 \cdot t_1) = 2 \cdot \sin t_1 \cdot \cos t_1$.

(6) If $\cos t_1 \neq 0$, then $\sin(2 \cdot t_1) = \frac{2 \cdot \tan t_1}{1 + (\tan t_1)^{\mathbf{2}}}$.

(7) $\cos(2 \cdot t_1) = (\cos t_1)^{\mathbf{2}} - (\sin t_1)^{\mathbf{2}}$ and $\cos(2 \cdot t_1) = 2 \cdot (\cos t_1)^{\mathbf{2}} - 1$ and $\cos(2 \cdot t_1) = 1 - 2 \cdot (\sin t_1)^{\mathbf{2}}$.

(8) If $\cos t_1 \neq 0$, then $\cos(2 \cdot t_1) = \frac{1 - (\tan t_1)^{\mathbf{2}}}{1 + (\tan t_1)^{\mathbf{2}}}$.

(9) If $\cos t_1 \neq 0$, then $\tan(2 \cdot t_1) = \frac{2 \cdot \tan t_1}{1 - (\tan t_1)^{\mathbf{2}}}$.

(10) If $\sin t_1 \neq 0$, then $\cot(2 \cdot t_1) = \frac{(\cot t_1)^{\mathbf{2}} - 1}{2 \cdot \cot t_1}$.

(11) If $\cos t_1 \neq 0$, then $(\sec t_1)^{\mathbf{2}} = 1 + (\tan t_1)^{\mathbf{2}}$.

(12) $\cot t_1 = \frac{1}{\tan t_1}$.

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(13) If $\cos t_1 \neq 0$ and $\sin t_1 \neq 0$, then $\sec(2 \cdot t_1) = \frac{(\sec t_1)^{\mathbf{2}}}{1-(\tan t_1)^{\mathbf{2}}}$ and $\sec(2 \cdot t_1) = \frac{\cot t_1 + \tan t_1}{\cot t_1 - \tan t_1}$.

(14) If $\sin t_1 \neq 0$, then $(\operatorname{cosec} t_1)^{\mathbf{2}} = 1 + (\cot t_1)^{\mathbf{2}}$.

(15) If $\cos t_1 \neq 0$ and $\sin t_1 \neq 0$, then $\operatorname{cosec}(2 \cdot t_1) = \frac{\sec t_1 \cdot \operatorname{cosec} t_1}{2}$ and $\operatorname{cosec}(2 \cdot t_1) = \frac{\tan t_1 + \cot t_1}{2}$.

(16) $\sin(3 \cdot t_1) = -4 \cdot (\sin t_1)^3 + 3 \cdot \sin t_1$.

(17) $\cos(3 \cdot t_1) = 4 \cdot (\cos t_1)^3 - 3 \cdot \cos t_1$.

(18) If $\cos t_1 \neq 0$, then $\tan(3 \cdot t_1) = \frac{3 \cdot \tan t_1 - (\tan t_1)^3}{1 - 3 \cdot (\tan t_1)^{\mathbf{2}}}$.

(19) If $\sin t_1 \neq 0$, then $\cot(3 \cdot t_1) = \frac{(\cot t_1)^3 - 3 \cdot \cot t_1}{3 \cdot (\cot t_1)^{\mathbf{2}} - 1}$.

(20) $(\sin t_1)^{\mathbf{2}} = \frac{1 - \cos(2 \cdot t_1)}{2}$.

(21) $(\cos t_1)^{\mathbf{2}} = \frac{1 + \cos(2 \cdot t_1)}{2}$.

(22) $(\sin t_1)^3 = \frac{3 \cdot \sin t_1 - \sin(3 \cdot t_1)}{4}$.

(23) $(\cos t_1)^3 = \frac{3 \cdot \cos t_1 + \cos(3 \cdot t_1)}{4}$.

(24) $(\sin t_1)^4 = \frac{(3 - 4 \cdot \cos(2 \cdot t_1)) + \cos(4 \cdot t_1)}{8}$.

(25) $(\cos t_1)^4 = \frac{3 + 4 \cdot \cos(2 \cdot t_1) + \cos(4 \cdot t_1)}{8}$.

(26) $\sin(\frac{t_1}{2}) = \sqrt{\frac{1 - \cos t_1}{2}}$ or $\sin(\frac{t_1}{2}) = -\sqrt{\frac{1 - \cos t_1}{2}}$.

(27) $\cos(\frac{t_1}{2}) = \sqrt{\frac{1 + \cos t_1}{2}}$ or $\cos(\frac{t_1}{2}) = -\sqrt{\frac{1 + \cos t_1}{2}}$.

(28) If $\sin(\frac{t_1}{2}) \neq 0$, then $\tan(\frac{t_1}{2}) = \frac{1 - \cos t_1}{\sin t_1}$.

(29) If $\cos(\frac{t_1}{2}) \neq 0$, then $\tan(\frac{t_1}{2}) = \frac{\sin t_1}{1 + \cos t_1}$.

(30) $\tan(\frac{t_1}{2}) = \sqrt{\frac{1 - \cos t_1}{1 + \cos t_1}}$ or $\tan(\frac{t_1}{2}) = -\sqrt{\frac{1 - \cos t_1}{1 + \cos t_1}}$.

(31) If $\cos(\frac{t_1}{2}) \neq 0$, then $\cot(\frac{t_1}{2}) = \frac{1 + \cos t_1}{\sin t_1}$.

(32) If $\sin(\frac{t_1}{2}) \neq 0$, then $\cot(\frac{t_1}{2}) = \frac{\sin t_1}{1 - \cos t_1}$.

(33) $\cot(\frac{t_1}{2}) = \sqrt{\frac{1 + \cos t_1}{1 - \cos t_1}}$ or $\cot(\frac{t_1}{2}) = -\sqrt{\frac{1 + \cos t_1}{1 - \cos t_1}}$.

(34) If $\sin(\frac{t_1}{2}) \neq 0$ and $\cos(\frac{t_1}{2}) \neq 0$ and $1 - (\tan(\frac{t_1}{2}))^{\mathbf{2}} \neq 0$, then $\sec(\frac{t_1}{2}) = \sqrt{\frac{2 \cdot \sec t_1}{\sec t_1 + 1}}$ or $\sec(\frac{t_1}{2}) = -\sqrt{\frac{2 \cdot \sec t_1}{\sec t_1 + 1}}$.

(35) If $\sin(\frac{t_1}{2}) \neq 0$ and $\cos(\frac{t_1}{2}) \neq 0$ and $1 - (\tan(\frac{t_1}{2}))^{\mathbf{2}} \neq 0$, then $\operatorname{cosec}(\frac{t_1}{2}) = \sqrt{\frac{2 \cdot \sec t_1}{\sec t_1 - 1}}$ or $\operatorname{cosec}(\frac{t_1}{2}) = -\sqrt{\frac{2 \cdot \sec t_1}{\sec t_1 - 1}}$.

Let us consider t_1. The functor $\coth t_1$ yielding a real number is defined as follows:

(Def. 1) $\coth t_1 = \frac{\cosh t_1}{\sinh t_1}$.

Let us consider t_1. The functor $\operatorname{sech} t_1$ yielding a real number is defined by:

(Def. 2) $\operatorname{sech} t_1 = \frac{1}{\cosh t_1}$.

Let us consider t_1. The functor $\operatorname{cosech} t_1$ yields a real number and is defined as follows:

(Def. 3) $\operatorname{cosech} t_1 = \frac{1}{\sinh t_1}$.

We now state a number of propositions:

(36) $\coth t_1 = \frac{\exp t_1 + \exp(-t_1)}{\exp t_1 - \exp(-t_1)}$ and $\operatorname{sech} t_1 = \frac{2}{\exp t_1 + \exp(-t_1)}$ and $\operatorname{cosech} t_1 = \frac{2}{\exp t_1 - \exp(-t_1)}$.

(37) If $\exp t_1 - \exp(-t_1) \neq 0$, then $\tanh t_1 \cdot \coth t_1 = 1$.

(38) $(\operatorname{sech} t_1)^{\mathbf{2}} + (\tanh t_1)^{\mathbf{2}} = 1$.

(39) If $\sinh t_1 \neq 0$, then $(\coth t_1)^{\mathbf{2}} - (\operatorname{cosech} t_1)^{\mathbf{2}} = 1$.

(40) If $\sinh t_2 \neq 0$ and $\sinh t_3 \neq 0$, then $\coth(t_2 + t_3) = \frac{1 + \coth t_2 \cdot \coth t_3}{\coth t_2 + \coth t_3}$.

(41) If $\sinh t_2 \neq 0$ and $\sinh t_3 \neq 0$, then $\coth(t_2 - t_3) = \frac{1 - \coth t_2 \cdot \coth t_3}{\coth t_2 - \coth t_3}$.

(42) If $\sinh t_2 \neq 0$ and $\sinh t_3 \neq 0$, then $\coth t_2 + \coth t_3 = \frac{\sinh(t_2 + t_3)}{\sinh t_2 \cdot \sinh t_3}$ and $\coth t_2 - \coth t_3 = -\frac{\sinh(t_2 - t_3)}{\sinh t_2 \cdot \sinh t_3}$.

(43) $\sinh(3 \cdot t_1) = 3 \cdot \sinh t_1 + 4 \cdot (\sinh t_1)^3$.

(44) $\cosh(3 \cdot t_1) = 4 \cdot (\cosh t_1)^3 - 3 \cdot \cosh t_1$.

(45) If $\sinh t_1 \neq 0$, then $\coth(2 \cdot t_1) = \frac{1 + (\coth t_1)^{\mathbf{2}}}{2 \cdot \coth t_1}$.

(46) If $t_1 > 0$, then $\sinh t_1 \geqslant 0$.

(47) If $t_1 < 0$, then $\sinh t_1 \leqslant 0$.

(48) $\cosh(\frac{t_1}{2}) = \sqrt{\frac{\cosh t_1 + 1}{2}}$.

(49) If $\sinh(\frac{t_1}{2}) \neq 0$, then $\tanh(\frac{t_1}{2}) = \frac{\cosh t_1 - 1}{\sinh t_1}$.

(50) If $\cosh(\frac{t_1}{2}) \neq 0$, then $\tanh(\frac{t_1}{2}) = \frac{\sinh t_1}{\cosh t_1 + 1}$.

(51) If $\sinh(\frac{t_1}{2}) \neq 0$, then $\coth(\frac{t_1}{2}) = \frac{\sinh t_1}{\cosh t_1 - 1}$.

(52) If $\cosh(\frac{t_1}{2}) \neq 0$, then $\coth(\frac{t_1}{2}) = \frac{\cosh t_1 + 1}{\sinh t_1}$.

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Solving Roots of the Special Polynomial Equation with Real Coefficients

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The papers [5], [4], [2], [3], and [1] provide the terminology and notation for this paper.

We follow the rules: x, y, a, b, c, p, q are real numbers and m, n are natural numbers.

We now state a number of propositions:

(1) If $a \neq 0$ and $\frac{b}{a} < 0$ and $\frac{c}{a} > 0$ and $\Delta(a,b,c) \geqslant 0$, then $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} > 0$ and $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} > 0$.

(2) If $a \neq 0$ and $\frac{b}{a} > 0$ and $\frac{c}{a} > 0$ and $\Delta(a,b,c) \geqslant 0$, then $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} < 0$ and $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} < 0$.

(3) If $a \neq 0$ and $\frac{c}{a} < 0$, then $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} > 0$ and $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} < 0$ or $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} < 0$ and $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} > 0$.

(4) If $a > 0$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $x^n = a$, then $x = \sqrt[n]{a}$ or $x = -\sqrt[n]{a}$.

(5) If $a \neq 0$ and $\mathrm{Poly2}(a,b,0,x) = 0$, then $x = 0$ or $x = -\frac{b}{a}$.

(6) If $a \neq 0$ and $\mathrm{Poly2}(a,0,0,x) = 0$, then $x = 0$.

(7) If $a \neq 0$ and there exists m such that $n = 2 \cdot m + 1$ and $\Delta(a,b,c) \geqslant 0$ and $\mathrm{Poly2}(a,b,c,x^n) = 0$, then $x = \sqrt[n]{\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}}$ or $x = \sqrt[n]{\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}}$.

(8) Suppose $a \neq 0$ and $\frac{b}{a} < 0$ and $\frac{c}{a} > 0$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $\Delta(a,b,c) \geqslant 0$ and $\mathrm{Poly2}(a,b,c,x^n) = 0$. Then

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$x = \sqrt[n]{\frac{-b+\sqrt{\Delta(a,b,c)}}{2\cdot a}}$ or $x = -\sqrt[n]{\frac{-b+\sqrt{\Delta(a,b,c)}}{2\cdot a}}$ or $x = \sqrt[n]{\frac{-b-\sqrt{\Delta(a,b,c)}}{2\cdot a}}$ or $x = -\sqrt[n]{\frac{-b-\sqrt{\Delta(a,b,c)}}{2\cdot a}}$.

(9) If $a \neq 0$ and there exists m such that $n = 2\cdot m+1$ and $\mathrm{Poly2}(a, b, 0, x^n) = 0$, then $x = 0$ or $x = \sqrt[n]{-\frac{b}{a}}$.

(10) If $a \neq 0$ and $\frac{b}{a} < 0$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $\mathrm{Poly2}(a, b, 0, x^n) = 0$, then $x = 0$ or $x = \sqrt[n]{-\frac{b}{a}}$ or $x = -\sqrt[n]{-\frac{b}{a}}$.

(11) $a^3 + b^3 = (a + b) \cdot ((a^{\mathbf{2}} - a \cdot b) + b^{\mathbf{2}})$ and $a^5 + b^5 = (a + b) \cdot ((((a^4 - a^3 \cdot b) + a^2 \cdot b^2) - a \cdot b^3) + b^4)$.

(12) Suppose $a \neq 0$ and $b^{\mathbf{2}} - 2 \cdot a \cdot b - 3 \cdot a^{\mathbf{2}} \geqslant 0$ and $\mathrm{Poly3}(a, b, b, a, x) = 0$. Then $x = -1$ or $x = \frac{(a-b)+\sqrt{b^{\mathbf{2}}-2\cdot a\cdot b-3\cdot a^{\mathbf{2}}}}{2\cdot a}$ or $x = \frac{a-b-\sqrt{b^{\mathbf{2}}-2\cdot a\cdot b-3\cdot a^{\mathbf{2}}}}{2\cdot a}$.

Let a, b, c, d, e, f, x be real numbers. The functor $\mathrm{Poly}_5(a, b, c, d, e, f, x)$ is defined by:

(Def. 1) $\mathrm{Poly}_5(a, b, c, d, e, f, x) = a \cdot x^5 + b \cdot x^4 + c \cdot x^3 + d \cdot x^{\mathbf{2}} + e \cdot x + f$.

We now state a number of propositions:

(13) Suppose $a \neq 0$ and $(b^{\mathbf{2}} + 2 \cdot a \cdot b + 5 \cdot a^{\mathbf{2}}) - 4 \cdot a \cdot c > 0$ and $\mathrm{Poly}_5(a, b, c, c, b, a, x) = 0$. Let y_1, y_2 be real numbers. Suppose $y_1 = \frac{(a-b)+\sqrt{(b^{\mathbf{2}}+2\cdot a\cdot b+5\cdot a^{\mathbf{2}})-4\cdot a\cdot c}}{2\cdot a}$ and $y_2 = \frac{a-b-\sqrt{(b^{\mathbf{2}}+2\cdot a\cdot b+5\cdot a^{\mathbf{2}})-4\cdot a\cdot c}}{2\cdot a}$. Then $x = -1$ or $x = \frac{y_1+\sqrt{\Delta(1,-y_1,1)}}{2}$ or $x = \frac{y_2+\sqrt{\Delta(1,-y_2,1)}}{2}$ or $x = \frac{y_1-\sqrt{\Delta(1,-y_1,1)}}{2}$ or $x = \frac{y_2-\sqrt{\Delta(1,-y_2,1)}}{2}$.

(14) Suppose $x + y = p$ and $x \cdot y = q$ and $p^{\mathbf{2}} - 4 \cdot q \geqslant 0$. Then $x = \frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}$ and $y = \frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}$ or $x = \frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}$ and $y = \frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}$.

(15) Suppose $x^n + y^n = p$ and $x^n \cdot y^n = q$ and $p^{\mathbf{2}} - 4 \cdot q \geqslant 0$ and there exists m such that $n = 2 \cdot m + 1$. Then $x = \sqrt[n]{\frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ or $x = \sqrt[n]{\frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$.

(16) Suppose $x^n+y^n = p$ and $x^n \cdot y^n = q$ and $p^{\mathbf{2}}-4\cdot q \geqslant 0$ and $p > 0$ and $q > 0$ and there exists m such that $n = 2\cdot m$ and $m \geqslant 1$. Then $x = \sqrt[n]{\frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ or $x = -\sqrt[n]{\frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ or $x = \sqrt[n]{\frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ and $y = -\sqrt[n]{\frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ or $x = -\sqrt[n]{\frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ and $y = -\sqrt[n]{\frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ or $x = \sqrt[n]{\frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ or $x = -\sqrt[n]{\frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ or $x = \sqrt[n]{\frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ and $y = -\sqrt[n]{\frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ or $x = -\sqrt[n]{\frac{p-\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$ and $y = -\sqrt[n]{\frac{p+\sqrt{p^{\mathbf{2}}-4\cdot q}}{2}}$.

(18)[1] Suppose $x^n + y^n = a$ and $x^n - y^n = b$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $a > 0$ and $a + b > 0$ and $a - b > 0$. Then

(i) $x = \sqrt[n]{\frac{a+b}{2}}$ and $y = \sqrt[n]{\frac{a-b}{2}}$, or

(ii) $x = \sqrt[n]{\frac{a+b}{2}}$ and $y = -\sqrt[n]{\frac{a-b}{2}}$, or

(iii) $x = -\sqrt[n]{\frac{a+b}{2}}$ and $y = \sqrt[n]{\frac{a-b}{2}}$, or

(iv) $x = -\sqrt[n]{\frac{a+b}{2}}$ and $y = -\sqrt[n]{\frac{a-b}{2}}$.

(19) If $a \cdot x^n + b \cdot y^n = p$ and $x \cdot y = 0$ and there exists m such that $n = 2 \cdot m + 1$ and $a \cdot b \neq 0$, then $x = 0$ and $y = \sqrt[n]{\frac{p}{b}}$ or $x = \sqrt[n]{\frac{p}{a}}$ and $y = 0$.

(20) Suppose $a \cdot x^n + b \cdot y^n = p$ and $x \cdot y = 0$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $\frac{p}{b} > 0$ and $\frac{p}{a} > 0$ and $a \cdot b \neq 0$. Then $x = 0$ and $y = \sqrt[n]{\frac{p}{b}}$ or $x = 0$ and $y = -\sqrt[n]{\frac{p}{b}}$ or $x = \sqrt[n]{\frac{p}{a}}$ and $y = 0$ or $x = -\sqrt[n]{\frac{p}{a}}$ and $y = 0$.

(21) If $a \cdot x^n = p$ and $x \cdot y = q$ and there exists m such that $n = 2 \cdot m + 1$ and $p \cdot a \neq 0$, then $x = \sqrt[n]{\frac{p}{a}}$ and $y = q \cdot \sqrt[n]{\frac{a}{p}}$.

(22) Suppose $a \cdot x^n = p$ and $x \cdot y = q$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $\frac{p}{a} > 0$ and $a \neq 0$. Then $x = \sqrt[n]{\frac{p}{a}}$ and $y = q \cdot \sqrt[n]{\frac{a}{p}}$ or $x = -\sqrt[n]{\frac{p}{a}}$ and $y = -q \cdot \sqrt[n]{\frac{a}{p}}$.

(24)[2] For all real numbers a, x such that $a > 0$ and $a \neq 1$ and $a^x = 1$ holds $x = 0$.

(25) For all real numbers a, x such that $a > 0$ and $a \neq 1$ and $a^x = a$ holds $x = 1$.

(27)[3] For all real numbers a, b, x such that $a > 0$ and $a \neq 1$ and $x > 0$ and $\log_a x = 0$ holds $x = 1$.

(28) For all real numbers a, b, x such that $a > 0$ and $a \neq 1$ and $x > 0$ and $\log_a x = 1$ holds $x = a$.

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[1] The proposition (17) has been removed.

[2] The proposition (23) has been removed.

[3] The proposition (26) has been removed.

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Algebraic Properties of Homotopies

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The notation and terminology used here are introduced in the following papers: [21], [9], [25], [1], [20], [14], [24], [22], [2], [5], [27], [6], [7], [18], [11], [19], [10], [17], [26], [8], [15], [23], [12], [4], [3], [16], and [13].

1. Preliminaries

The scheme *ExFunc3CondD* deals with a non empty set $\mathcal{A}$, three unary functors $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ yielding sets, and three unary predicates $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, and states that:

> There exists a function f such that $\mathrm{dom}\, f = \mathcal{A}$ and for every element c of $\mathcal{A}$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ and if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$ and if $\mathcal{R}[c]$, then $f(c) = \mathcal{H}(c)$

provided the parameters meet the following conditions:

- For every element c of $\mathcal{A}$ holds if $\mathcal{P}[c]$, then not $\mathcal{Q}[c]$ and if $\mathcal{P}[c]$, then not $\mathcal{R}[c]$ and if $\mathcal{Q}[c]$, then not $\mathcal{R}[c]$, and
- For every element c of $\mathcal{A}$ holds $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ or $\mathcal{R}[c]$.

Let n be a natural number. Observe that every element of $\mathcal{E}^n_{\mathrm{T}}$ is function-like and relation-like.

Let n be a natural number. Observe that every element of $\mathcal{E}^n_{\mathrm{T}}$ is finite sequence-like.

We now state a number of propositions:

(1) The carrier of $[\!:\mathbb{I}, \mathbb{I}:\!] = [\!:[0,1], [0,1]:\!]$.

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(2) For every real number x such that $x \leqslant \frac{1}{2}$ holds $2 \cdot x - 1 \leqslant 1 - 2 \cdot x$.

(3) For every real number x such that $x \geqslant \frac{1}{2}$ holds $2 \cdot x - 1 \geqslant 1 - 2 \cdot x$.

(4) For all real numbers x, a, b, c, d such that $a \neq b$ holds $\frac{d-c}{b-a} \cdot (x-a) + c = (1 - \frac{x-a}{b-a}) \cdot c + \frac{x-a}{b-a} \cdot d$.

(5) For all real numbers a, b, x such that $a \leqslant x$ and $x \leqslant b$ holds $\frac{x-a}{b-a} \in$ the carrier of $[0, 1]_{\mathrm{T}}$.

(6) For every point x of $\mathbb{I}$ such that $x \leqslant \frac{1}{2}$ holds $2 \cdot x$ is a point of $\mathbb{I}$.

(7) For every point x of $\mathbb{I}$ such that $x \geqslant \frac{1}{2}$ holds $2 \cdot x - 1$ is a point of $\mathbb{I}$.

(8) For all points p, q of $\mathbb{I}$ holds $p \cdot q$ is a point of $\mathbb{I}$.

(9) For every point x of $\mathbb{I}$ holds $\frac{1}{2} \cdot x$ is a point of $\mathbb{I}$.

(10) For every point x of $\mathbb{I}$ such that $x \geqslant \frac{1}{2}$ holds $x - \frac{1}{4}$ is a point of $\mathbb{I}$.

(12)[3] $\mathrm{id}_{\mathbb{I}}$ is a path from $0_{\mathbb{I}}$ to $1_{\mathbb{I}}$.

(13) For all points a, b, c, d of $\mathbb{I}$ such that $a \leqslant b$ and $c \leqslant d$ holds $[\![[a,b], [c,d]]\!]$ is a compact non empty subset of $[\![\mathbb{I}, \mathbb{I}]\!]$.

2. Affine Maps

One can prove the following four propositions:

(14) Let S, T be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $S = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant 2 \cdot p_{\mathbf{1}} - 1\}$ and $T = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant p_{\mathbf{1}}\}$. Then $(\mathrm{AffineMap}(1, 0, \frac{1}{2}, \frac{1}{2}))^{\circ} S = T$.

(15) Let S, T be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $S = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$ and $T = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant p_{\mathbf{1}}\}$. Then $(\mathrm{AffineMap}(1, 0, \frac{1}{2}, \frac{1}{2}))^{\circ} S = T$.

(16) Let S, T be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $S = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ and $T = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant -p_{\mathbf{1}}\}$. Then $(\mathrm{AffineMap}(1, 0, \frac{1}{2}, -\frac{1}{2}))^{\circ} S = T$.

(17) Let S, T be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $S = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ and $T = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}\}$. Then $(\mathrm{AffineMap}(1, 0, \frac{1}{2}, -\frac{1}{2}))^{\circ} S = T$.

3. Real-Membered Structures

Let T be a 1-sorted structure. We say that T is real-membered if and only if:

(Def. 1) The carrier of T is real-membered.

We now state the proposition

[3]The proposition (11) has been removed.

(18) For every non empty 1-sorted structure T holds T is real-membered iff every element of T is real.

Let us mention that $\mathbb{I}$ is real-membered.

One can verify that there exists a 1-sorted structure which is non empty and real-membered and there exists a topological space which is non empty and real-membered.

Let T be a real-membered 1-sorted structure. Note that every element of T is real.

Let T be a real-membered topological structure. Note that every subspace of T is real-membered.

Let S, T be real-membered non empty topological spaces and let p be an element of $[: S, T :]$. One can check that $p_{\mathbf{1}}$ is real and $p_{\mathbf{2}}$ is real.

Let T be a non empty subspace of $[: \mathbb{I}, \mathbb{I} :]$ and let x be a point of T. One can check that $x_{\mathbf{1}}$ is real and $x_{\mathbf{2}}$ is real.

One can check that $\mathbb{R}^{\mathbf{1}}$ is real-membered.

4. Closed Subsets of Euclidean Topological Spaces

The following propositions are true:

(19) $\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(20) $\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(21) $\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(22) $\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(23) $\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}} \ \wedge \ p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(24) There exists a map f from $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$ into $\mathcal{E}^2_{\mathrm{T}}$ such that for all real numbers x, y holds $f(\langle x, y \rangle) = \langle x, y \rangle$.

(25) $\{p; p$ ranges over points of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$: $p_{\mathbf{2}} \leqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ is a closed subset of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$.

(26) $\{p; p$ ranges over points of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$: $p_{\mathbf{2}} \leqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed subset of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$.

(27) $\{p; p$ ranges over points of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}} \ \wedge \ p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed subset of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$.

(28) $\{p; p$ ranges over points of $[: \mathbb{I}, \mathbb{I} :]$: $p_{\mathbf{2}} \leqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ is a closed non empty subset of $[: \mathbb{I}, \mathbb{I} :]$.

(29) $\{p; p$ ranges over points of $[: \mathbb{I}, \mathbb{I} :]$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}} \ \wedge \ p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed non empty subset of $[: \mathbb{I}, \mathbb{I} :]$.

(30) $\{p; p$ ranges over points of $[: \mathbb{I}, \mathbb{I} :]$: $p_{\mathbf{2}} \leqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed non empty subset of $[: \mathbb{I}, \mathbb{I} :]$.

(31) Let S, T be non empty topological spaces and p be a point of $[\!: S, T :\!]$. Then $p_{\mathbf{1}}$ is a point of S and $p_{\mathbf{2}}$ is a point of T.

(32) For all subsets A, B of $[\!: \mathbb{I}, \mathbb{I} :\!]$ such that $A = [\!: [0, \frac{1}{2}], [0,1] :\!]$ and $B = [\!: [\frac{1}{2}, 1], [0,1] :\!]$ holds $\Omega_{[\!: \mathbb{I}, \mathbb{I} :\!] \restriction A} \cup \Omega_{[\!: \mathbb{I}, \mathbb{I} :\!] \restriction B} = \Omega_{[\!: \mathbb{I}, \mathbb{I} :\!]}$.

(33) For all subsets A, B of $[\!: \mathbb{I}, \mathbb{I} :\!]$ such that $A = [\!: [0, \frac{1}{2}], [0,1] :\!]$ and $B = [\!: [\frac{1}{2}, 1], [0,1] :\!]$ holds $\Omega_{[\!: \mathbb{I}, \mathbb{I} :\!] \restriction A} \cap \Omega_{[\!: \mathbb{I}, \mathbb{I} :\!] \restriction B} = [\!: \{\frac{1}{2}\}, [0,1] :\!]$.

5. Compact Spaces

Let T be a topological structure. Note that $\emptyset_T$ is compact.

Let T be a topological structure. Observe that there exists a subset of T which is empty and compact.

Next we state three propositions:

(34) For every topological structure T holds $\emptyset$ is an empty compact subset of T.

(35) Let T be a topological structure and a, b be real numbers. If $a > b$, then $[a, b]$ is an empty compact subset of T.

(36) For all points a, b, c, d of $\mathbb{I}$ holds $[\!: [a,b], [c,d] :\!]$ is a compact subset of $[\!: \mathbb{I}, \mathbb{I} :\!]$.

6. Continuous Maps

Let a, b, c, d be real numbers. The functor $\mathrm{L}_{01}(a,b,c,d)$ yielding a map from $[a, b]_{\mathrm{T}}$ into $[c, d]_{\mathrm{T}}$ is defined by:

(Def. 2) $\mathrm{L}_{01}(a,b,c,d) = \mathrm{L}_{01}(c_{[c,d]_{\mathrm{T}}}, d_{[c,d]_{\mathrm{T}}}) \cdot \mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}})$.

The following propositions are true:

(37) For all real numbers a, b, c, d such that $a < b$ and $c < d$ holds $(\mathrm{L}_{01}(a,b,c,d))(a) = c$ and $(\mathrm{L}_{01}(a,b,c,d))(b) = d$.

(38) For all real numbers a, b, c, d such that $a < b$ and $c \leqslant d$ holds $\mathrm{L}_{01}(a,b,c,d)$ is a continuous map from $[a, b]_{\mathrm{T}}$ into $[c, d]_{\mathrm{T}}$.

(39) Let a, b, c, d be real numbers. Suppose $a < b$ and $c \leqslant d$. Let x be a real number. If $a \leqslant x$ and $x \leqslant b$, then $(\mathrm{L}_{01}(a,b,c,d))(x) = \frac{d-c}{b-a} \cdot (x - a) + c$.

(40) Let f_1, f_2 be maps from $[\!: \mathbb{I}, \mathbb{I} :\!]$ into $\mathbb{I}$. Suppose f_1 is continuous and f_2 is continuous and for every point p of $[\!: \mathbb{I}, \mathbb{I} :\!]$ holds $f_1(p) \cdot f_2(p)$ is a point of $\mathbb{I}$. Then there exists a map g from $[\!: \mathbb{I}, \mathbb{I} :\!]$ into $\mathbb{I}$ such that

(i) for every point p of $[\!: \mathbb{I}, \mathbb{I} :\!]$ and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 \cdot r_2$, and

(ii) g is continuous.

(41) Let f_1, f_2 be maps from $[: \mathbb{I}, \mathbb{I} :]$ into $\mathbb{I}$. Suppose f_1 is continuous and f_2 is continuous and for every point p of $[: \mathbb{I}, \mathbb{I} :]$ holds $f_1(p) + f_2(p)$ is a point of $\mathbb{I}$. Then there exists a map g from $[: \mathbb{I}, \mathbb{I} :]$ into $\mathbb{I}$ such that

(i) for every point p of $[: \mathbb{I}, \mathbb{I} :]$ and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 + r_2$, and

(ii) g is continuous.

(42) Let f_1, f_2 be maps from $[: \mathbb{I}, \mathbb{I} :]$ into $\mathbb{I}$. Suppose f_1 is continuous and f_2 is continuous and for every point p of $[: \mathbb{I}, \mathbb{I} :]$ holds $f_1(p) - f_2(p)$ is a point of $\mathbb{I}$. Then there exists a map g from $[: \mathbb{I}, \mathbb{I} :]$ into $\mathbb{I}$ such that

(i) for every point p of $[: \mathbb{I}, \mathbb{I} :]$ and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 - r_2$, and

(ii) g is continuous.

7. Paths

We follow the rules: T denotes a non empty topological space and a, b, c, d denote points of T.

The following three propositions are true:

(43) For every path P from a to b such that P is continuous holds $P \cdot \mathrm{L}_{01}(1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}})$ is a continuous map from $\mathbb{I}$ into T.

(44) Let X be a non empty topological structure, a, b be points of X, and P be a path from a to b. If $P(0) = a$ and $P(1) = b$, then $(P \cdot \mathrm{L}_{01}(1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}}))(0) = b$ and $(P \cdot \mathrm{L}_{01}(1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}}))(1) = a$.

(45) Let P be a path from a to b. Suppose P is continuous and $P(0) = a$ and $P(1) = b$. Then $-P$ is continuous and $(-P)(0) = b$ and $(-P)(1) = a$.

Let T be a topological structure and let a, b be points of T. We say that a, b are connected if and only if:

(Def. 3) There exists a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$.

Let T be a non empty topological space and let a, b be points of T. Let us notice that the predicate a, b are connected is reflexive and symmetric.

We now state several propositions:

(46) If a, b are connected and b, c are connected, then a, c are connected.

(47) For every arcwise connected topological structure T and for all points a, b of T holds a, b are connected.

(48) For every path A from a to a holds A, A are homotopic.

(49) If a, b are connected, then for every path A from a to b holds A, A are homotopic.

(50) If a, b are connected, then for every path A from a to b holds $A = --A$.

(51) Let T be a non empty arcwise connected topological space, a, b be points of T, and A be a path from a to b. Then $A = --A$.

(52) If a, b are connected, then every path from a to b is continuous.

8. Reexamination of a Path Concept

Let T be a non empty arcwise connected topological space, let a, b, c be points of T, let P be a path from a to b, and let Q be a path from b to c. Then $P + Q$ can be characterized by the condition:

(Def. 4) For every point t of $\mathbb{I}$ holds if $t \leqslant \frac{1}{2}$, then $(P + Q)(t) = P(2 \cdot t)$ and if $\frac{1}{2} \leqslant t$, then $(P + Q)(t) = Q(2 \cdot t - 1)$.

Let T be a non empty arcwise connected topological space, let a, b be points of T, and let P be a path from a to b. Then $-P$ can be characterized by the condition:

(Def. 5) For every point t of $\mathbb{I}$ holds $(-P)(t) = P(1 - t)$.

9. Reparametrizations

Let T be a non empty topological space, let a, b be points of T, let P be a path from a to b, and let f be a continuous map from $\mathbb{I}$ into $\mathbb{I}$. Let us assume that $f(0) = 0$ and $f(1) = 1$ and a, b are connected. The functor $\mathrm{RePar}(P, f)$ yields a path from a to b and is defined by:

(Def. 6) $\mathrm{RePar}(P, f) = P \cdot f$.

Next we state two propositions:

(53) Let P be a path from a to b and f be a continuous map from $\mathbb{I}$ into $\mathbb{I}$. Suppose $f(0) = 0$ and $f(1) = 1$ and a, b are connected. Then $\mathrm{RePar}(P, f)$, P are homotopic.

(54) Let T be a non empty arcwise connected topological space, a, b be points of T, P be a path from a to b, and f be a continuous map from $\mathbb{I}$ into $\mathbb{I}$. If $f(0) = 0$ and $f(1) = 1$, then $\mathrm{RePar}(P, f)$, P are homotopic.

The map $1^{\mathrm{st}}\mathrm{RP}$ from $\mathbb{I}$ into $\mathbb{I}$ is defined as follows:

(Def. 7) For every point t of $\mathbb{I}$ holds if $t \leqslant \frac{1}{2}$, then $(1^{\mathrm{st}}\mathrm{RP})(t) = 2 \cdot t$ and if $t > \frac{1}{2}$, then $(1^{\mathrm{st}}\mathrm{RP})(t) = 1$.

Let us note that $1^{\mathrm{st}}\mathrm{RP}$ is continuous.

One can prove the following proposition

(55) $(1^{\mathrm{st}}\mathrm{RP})(0) = 0$ and $(1^{\mathrm{st}}\mathrm{RP})(1) = 1$.

The map $2^{\mathrm{nd}}\mathrm{RP}$ from $\mathbb{I}$ into $\mathbb{I}$ is defined by:

(Def. 8) For every point t of $\mathbb{I}$ holds if $t \leqslant \frac{1}{2}$, then $(2^{\mathrm{nd}}\mathrm{RP})(t) = 0$ and if $t > \frac{1}{2}$, then $(2^{\mathrm{nd}}\mathrm{RP})(t) = 2 \cdot t - 1$.

One can verify that $2^{\mathrm{nd}}\mathrm{RP}$ is continuous.

One can prove the following proposition

(56) $(2^{\mathrm{nd}}\mathrm{RP})(0) = 0$ and $(2^{\mathrm{nd}}\mathrm{RP})(1) = 1$.

The map $3^{\mathrm{rd}}\mathrm{RP}$ from $\mathbb{I}$ into $\mathbb{I}$ is defined by the condition (Def. 9).

(Def. 9) Let x be a point of $\mathbb{I}$. Then

- (i) if $x \leqslant \frac{1}{2}$, then $(3^{\mathrm{rd}}\mathrm{RP})(x) = \frac{1}{2} \cdot x$,
- (ii) if $x > \frac{1}{2}$ and $x \leqslant \frac{3}{4}$, then $(3^{\mathrm{rd}}\mathrm{RP})(x) = x - \frac{1}{4}$, and
- (iii) if $x > \frac{3}{4}$, then $(3^{\mathrm{rd}}\mathrm{RP})(x) = 2 \cdot x - 1$.

Let us note that $3^{\mathrm{rd}}\mathrm{RP}$ is continuous.

We now state four propositions:

(57) $(3^{\mathrm{rd}}\mathrm{RP})(0) = 0$ and $(3^{\mathrm{rd}}\mathrm{RP})(1) = 1$.

(58) Let P be a path from a to b and Q be a constant path from b to b. If a, b are connected, then $\mathrm{RePar}(P, 1^{\mathrm{st}}\mathrm{RP}) = P + Q$.

(59) Let P be a path from a to b and Q be a constant path from a to a. If a, b are connected, then $\mathrm{RePar}(P, 2^{\mathrm{nd}}\mathrm{RP}) = Q + P$.

(60) Let P be a path from a to b, Q be a path from b to c, and R be a path from c to d. Suppose a, b are connected and b, c are connected and c, d are connected. Then $\mathrm{RePar}(P + Q + R, 3^{\mathrm{rd}}\mathrm{RP}) = P + (Q + R)$.

10. Decomposition of the Unit Square

The subset LowerLeftUnitTriangle of $[\!:\mathbb{I}, \mathbb{I}:\!]$ is defined as follows:

(Def. 10) For every set x holds $x \in$ LowerLeftUnitTriangle iff there exist points a, b of $\mathbb{I}$ such that $x = \langle a, b \rangle$ and $b \leqslant 1 - 2 \cdot a$.

We introduce IAA as a synonym of LowerLeftUnitTriangle.

The subset UpperUnitTriangle of $[\!:\mathbb{I}, \mathbb{I}:\!]$ is defined by:

(Def. 11) For every set x holds $x \in$ UpperUnitTriangle iff there exist points a, b of $\mathbb{I}$ such that $x = \langle a, b \rangle$ and $b \geqslant 1 - 2 \cdot a$ and $b \geqslant 2 \cdot a - 1$.

We introduce IBB as a synonym of UpperUnitTriangle.

The subset LowerRightUnitTriangle of $[\!:\mathbb{I}, \mathbb{I}:\!]$ is defined as follows:

(Def. 12) For every set x holds $x \in$ LowerRightUnitTriangle iff there exist points a, b of $\mathbb{I}$ such that $x = \langle a, b \rangle$ and $b \leqslant 2 \cdot a - 1$.

We introduce ICC as a synonym of LowerRightUnitTriangle.

The following propositions are true:

(61) $\mathrm{IAA} = \{p; p$ ranges over points of $[\!:\mathbb{I}, \mathbb{I}:\!]$: $p_{\mathbf{2}} \leqslant 1 - 2 \cdot p_{\mathbf{1}}\}$.

(62) $\mathrm{IBB} = \{p; p$ ranges over points of $[\!:\mathbb{I}, \mathbb{I}:\!]$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}} \wedge p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$.

(63) $\mathrm{ICC} = \{p; p$ ranges over points of $[\!:\mathbb{I}, \mathbb{I}:\!]$: $p_{\mathbf{2}} \leqslant 2 \cdot p_{\mathbf{1}} - 1\}$.

One can check the following observations:

- $*$ IAA is closed and non empty,

* IBB is closed and non empty, and
* ICC is closed and non empty.

Next we state a number of propositions:

(64) IAA $\cup$ IBB $\cup$ ICC $= [:[0,1], [0,1]:]$.

(65) IAA $\cap$ IBB $= \{p; p$ ranges over points of $[:\mathbb{I}, \mathbb{I}:]$: $p_{\mathbf{2}} = 1 - 2 \cdot p_{\mathbf{1}}\}$.

(66) ICC $\cap$ IBB $= \{p; p$ ranges over points of $[:\mathbb{I}, \mathbb{I}:]$: $p_{\mathbf{2}} = 2 \cdot p_{\mathbf{1}} - 1\}$.

(67) For every point x of $[:\mathbb{I}, \mathbb{I}:]$ such that $x \in$ IAA holds $x_{\mathbf{1}} \leqslant \frac{1}{2}$.

(68) For every point x of $[:\mathbb{I}, \mathbb{I}:]$ such that $x \in$ ICC holds $x_{\mathbf{1}} \geqslant \frac{1}{2}$.

(69) For every point x of $\mathbb{I}$ holds $\langle 0, x \rangle \in$ IAA.

(70) For every set s such that $\langle 0, s \rangle \in$ IBB holds $s = 1$.

(71) For every set s such that $\langle s, 1 \rangle \in$ ICC holds $s = 1$.

(72) $\langle 0, 1 \rangle \in$ IBB.

(73) For every point x of $\mathbb{I}$ holds $\langle x, 1 \rangle \in$ IBB.

(74) $\langle \frac{1}{2}, 0 \rangle \in$ ICC and $\langle 1, 1 \rangle \in$ ICC.

(75) $\langle \frac{1}{2}, 0 \rangle \in$ IBB.

(76) For every point x of $\mathbb{I}$ holds $\langle 1, x \rangle \in$ ICC.

(77) For every point x of $\mathbb{I}$ such that $x \geqslant \frac{1}{2}$ holds $\langle x, 0 \rangle \in$ ICC.

(78) For every point x of $\mathbb{I}$ such that $x \leqslant \frac{1}{2}$ holds $\langle x, 0 \rangle \in$ IAA.

(79) For every point x of $\mathbb{I}$ such that $x < \frac{1}{2}$ holds $\langle x, 0 \rangle \notin$ IBB and $\langle x, 0 \rangle \notin$ ICC.

(80) IAA $\cap$ ICC $= \{\langle \frac{1}{2}, 0 \rangle\}$.

11. Properties of a Homotopy

We use the following convention: X denotes a non empty arcwise connected topological space and a_1, b_1, c_1, d_1 denote points of X.

One can prove the following propositions:

(81) Let P be a path from a to b, Q be a path from b to c, and R be a path from c to d. Suppose a, b are connected and b, c are connected and c, d are connected. Then $(P + Q) + R$, $P + (Q + R)$ are homotopic.

(82) Let P be a path from a_1 to b_1, Q be a path from b_1 to c_1, and R be a path from c_1 to d_1. Then $(P + Q) + R$, $P + (Q + R)$ are homotopic.

(83) Let P_1, P_2 be paths from a to b and Q_1, Q_2 be paths from b to c. Suppose a, b are connected and b, c are connected and P_1, P_2 are homotopic and Q_1, Q_2 are homotopic. Then $P_1 + Q_1$, $P_2 + Q_2$ are homotopic.

(84) Let P_1, P_2 be paths from a_1 to b_1 and Q_1, Q_2 be paths from b_1 to c_1. Suppose P_1, P_2 are homotopic and Q_1, Q_2 are homotopic. Then $P_1 + Q_1$, $P_2 + Q_2$ are homotopic.

(85) Let P, Q be paths from a to b. Suppose a, b are connected and P, Q are homotopic. Then $-P$, $-Q$ are homotopic.

(86) For all paths P, Q from a_1 to b_1 such that P, Q are homotopic holds $-P$, $-Q$ are homotopic.

(87) Let P, Q, R be paths from a to b. Suppose P, Q are homotopic and Q, R are homotopic. Then P, R are homotopic.

(88) Let P be a path from a to b and Q be a constant path from b to b. If a, b are connected, then $P+Q$, P are homotopic.

(89) For every path P from a_1 to b_1 and for every constant path Q from b_1 to b_1 holds $P+Q$, P are homotopic.

(90) Let P be a path from a to b and Q be a constant path from a to a. If a, b are connected, then $Q+P$, P are homotopic.

(91) For every path P from a_1 to b_1 and for every constant path Q from a_1 to a_1 holds $Q+P$, P are homotopic.

(92) Let P be a path from a to b and Q be a constant path from a to a. If a, b are connected, then $P+-P$, Q are homotopic.

(93) For every path P from a_1 to b_1 and for every constant path Q from a_1 to a_1 holds $P+-P$, Q are homotopic.

(94) Let P be a path from b to a and Q be a constant path from a to a. If b, a are connected, then $-P+P$, Q are homotopic.

(95) For every path P from b_1 to a_1 and for every constant path Q from a_1 to a_1 holds $-P+P$, Q are homotopic.

(96) For all constant paths P, Q from a to a holds P, Q are homotopic.

Let T be a non empty topological space, let a, b be points of T, and let P, Q be paths from a to b. Let us assume that P, Q are homotopic. A map from $[\!:\mathbb{I}, \mathbb{I}:\!]$ into T is said to be a homotopy between P and Q if it satisfies the conditions (Def. 13).

(Def. 13)(i) It is continuous, and

(ii) for every point s of $\mathbb{I}$ holds $\mathrm{it}(s, 0) = P(s)$ and $\mathrm{it}(s, 1) = Q(s)$ and for every point t of $\mathbb{I}$ holds $\mathrm{it}(0, t) = a$ and $\mathrm{it}(1, t) = b$.

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The Fundamental Group

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Summary. This is the next article in a series devoted to the homotopy theory (following [11] and [12]). The concept of fundamental groups of pointed topological spaces has been introduced. Isomorphism of fundamental groups defined with respect to different points belonging to the same component has been stated. Triviality of fundamental group(s) of $\mathbb{R}^n$ has been shown.

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The articles [22], [7], [26], [27], [19], [4], [6], [5], [28], [2], [21], [1], [18], [20], [16], [8], [3], [15], [13], [17], [29], [9], [14], [24], [23], [10], [11], [25], and [12] provide the terminology and notation for this paper.

1. Preliminaries

We adopt the following convention: p, q, x, y are real numbers and n is a natural number.

Next we state a number of propositions:

(1) Let G, H be groups and h be a homomorphism from G to H. If $h \cdot h^{-1} = \mathrm{id}_H$ and $h^{-1} \cdot h = \mathrm{id}_G$, then h is an isomorphism.

(2) For every subset X of $\mathbb{I}$ and for every point a of $\mathbb{I}$ such that $X =]a, 1]$ holds $X^{\mathrm{c}} = [0, a]$.

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(3) For every subset X of $\mathbb{I}$ and for every point a of $\mathbb{I}$ such that $X = [0, a[$ holds $X^{c} = [a, 1]$.

(4) For every subset X of $\mathbb{I}$ and for every point a of $\mathbb{I}$ such that $X =]a, 1]$ holds X is open.

(5) For every subset X of $\mathbb{I}$ and for every point a of $\mathbb{I}$ such that $X = [0, a[$ holds X is open.

(6) For every element f of $\mathbb{R}^n$ holds $x \cdot -f = -x \cdot f$.

(7) For all elements f, g of $\mathbb{R}^n$ holds $x \cdot (f - g) = x \cdot f - x \cdot g$.

(8) For every element f of $\mathbb{R}^n$ holds $(x - y) \cdot f = x \cdot f - y \cdot f$.

(9) For all elements f, g, h, k of $\mathbb{R}^n$ holds $(f+g)-(h+k) = (f-h)+(g-k)$.

(10) For every element f of $\mathcal{R}^n$ such that $0 \leqslant x$ and $x \leqslant 1$ holds $|x \cdot f| \leqslant |f|$.

(11) For every element f of $\mathcal{R}^n$ and for every point p of $\mathbb{I}$ holds $|p \cdot f| \leqslant |f|$.

(12) Let e_1, e_2, e_3, e_4, e_5, e_6 be points of $\mathcal{E}^n$ and p_1, p_2, p_3, p_4 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $e_1 = p_1$ and $e_2 = p_2$ and $e_3 = p_3$ and $e_4 = p_4$ and $e_5 = p_1+p_3$ and $e_6 = p_2 + p_4$ and $\rho(e_1, e_2) < x$ and $\rho(e_3, e_4) < y$. Then $\rho(e_5, e_6) < x + y$.

(13) Let e_1, e_2, e_5, e_6 be points of $\mathcal{E}^n$ and p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. If $e_1 = p_1$ and $e_2 = p_2$ and $e_5 = y \cdot p_1$ and $e_6 = y \cdot p_2$ and $\rho(e_1, e_2) < x$ and $y \neq 0$, then $\rho(e_5, e_6) < |y| \cdot x$.

(14) Let e_1, e_2, e_3, e_4, e_5, e_6 be points of $\mathcal{E}^n$ and p_1, p_2, p_3, p_4 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $e_1 = p_1$ and $e_2 = p_2$ and $e_3 = p_3$ and $e_4 = p_4$ and $e_5 = x \cdot p_1 + y \cdot p_3$ and $e_6 = x \cdot p_2 + y \cdot p_4$ and $\rho(e_1, e_2) < p$ and $\rho(e_3, e_4) < q$ and $x \neq 0$ and $y \neq 0$. Then $\rho(e_5, e_6) < |x| \cdot p + |y| \cdot q$.

(16)[3] Let X be a non empty topological space and f, g be maps from X into $\mathcal{E}^n_{\mathrm{T}}$. Suppose f is continuous and for every point p of X holds $g(p) = y \cdot f(p)$. Then g is continuous.

(17) Let X be a non empty topological space and f_1, f_2, g be maps from X into $\mathcal{E}^n_{\mathrm{T}}$. Suppose f_1 is continuous and f_2 is continuous and for every point p of X holds $g(p) = x \cdot f_1(p) + y \cdot f_2(p)$. Then g is continuous.

(18) Let F be a map from $[\!: \mathcal{E}^n_{\mathrm{T}}, \mathbb{I} :\!]$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose that for every point x of $\mathcal{E}^n_{\mathrm{T}}$ and for every point i of $\mathbb{I}$ holds $F(x, i) = (1 - i) \cdot x$. Then F is continuous.

(19) Let F be a map from $[\!: \mathcal{E}^n_{\mathrm{T}}, \mathbb{I} :\!]$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose that for every point x of $\mathcal{E}^n_{\mathrm{T}}$ and for every point i of $\mathbb{I}$ holds $F(x, i) = i \cdot x$. Then F is continuous.

2. Paths

For simplicity, we follow the rules: X denotes a non empty topological space, a, b, c, d, e, f denote points of X, T denotes a non empty arcwise connected

[3]The proposition (15) has been removed.

topological space, and a_1, b_1, c_1, d_1, e_1, f_1 denote points of T.

One can prove the following propositions:

(20) Suppose a, b are connected and b, c are connected. Let A be a path from a to b and B be a path from b to c. Then A, $A + B + -B$ are homotopic.

(21) For every path A from a_1 to b_1 and for every path B from b_1 to c_1 holds A, $A + B + -B$ are homotopic.

(22) Suppose a, b are connected and c, b are connected. Let A be a path from a to b and B be a path from c to b. Then A, $A + -B + B$ are homotopic.

(23) For every path A from a_1 to b_1 and for every path B from c_1 to b_1 holds A, $A + -B + B$ are homotopic.

(24) Suppose a, b are connected and c, a are connected. Let A be a path from a to b and B be a path from c to a. Then A, $-B + B + A$ are homotopic.

(25) For every path A from a_1 to b_1 and for every path B from c_1 to a_1 holds A, $-B + B + A$ are homotopic.

(26) Suppose a, b are connected and a, c are connected. Let A be a path from a to b and B be a path from a to c. Then A, $B + -B + A$ are homotopic.

(27) For every path A from a_1 to b_1 and for every path B from a_1 to c_1 holds A, $B + -B + A$ are homotopic.

(28) Suppose a, b are connected and c, b are connected. Let A, B be paths from a to b and C be a path from b to c. If $A + C$, $B + C$ are homotopic, then A, B are homotopic.

(29) Let A, B be paths from a_1 to b_1 and C be a path from b_1 to c_1. If $A+C$, $B + C$ are homotopic, then A, B are homotopic.

(30) Suppose a, b are connected and a, c are connected. Let A, B be paths from a to b and C be a path from c to a. If $C + A$, $C + B$ are homotopic, then A, B are homotopic.

(31) Let A, B be paths from a_1 to b_1 and C be a path from c_1 to a_1. If $C+A$, $C + B$ are homotopic, then A, B are homotopic.

(32) Suppose a, b are connected and b, c are connected and c, d are connected and d, e are connected. Let A be a path from a to b, B be a path from b to c, C be a path from c to d, and D be a path from d to e. Then $A + B + C + D$, $A + (B + C) + D$ are homotopic.

(33) Let A be a path from a_1 to b_1, B be a path from b_1 to c_1, C be a path from c_1 to d_1, and D be a path from d_1 to e_1. Then $A + B + C + D$, $A + (B + C) + D$ are homotopic.

(34) Suppose a, b are connected and b, c are connected and c, d are connected and d, e are connected. Let A be a path from a to b, B be a path from b to c, C be a path from c to d, and D be a path from d to e. Then $(A + B + C) + D$, $A + (B + C + D)$ are homotopic.

(35) Let A be a path from a_1 to b_1, B be a path from b_1 to c_1, C be a path from c_1 to d_1, and D be a path from d_1 to e_1. Then $(A + B + C) + D$, $A + (B + C + D)$ are homotopic.

(36) Suppose a, b are connected and b, c are connected and c, d are connected and d, e are connected. Let A be a path from a to b, B be a path from b to c, C be a path from c to d, and D be a path from d to e. Then $(A + (B + C)) + D$, $A + B + (C + D)$ are homotopic.

(37) Let A be a path from a_1 to b_1, B be a path from b_1 to c_1, C be a path from c_1 to d_1, and D be a path from d_1 to e_1. Then $(A + (B + C)) + D$, $A + B + (C + D)$ are homotopic.

(38) Suppose a, b are connected and b, c are connected and b, d are connected. Let A be a path from a to b, B be a path from d to b, and C be a path from b to c. Then $A + -B + B + C$, $A + C$ are homotopic.

(39) Let A be a path from a_1 to b_1, B be a path from d_1 to b_1, and C be a path from b_1 to c_1. Then $A + -B + B + C$, $A + C$ are homotopic.

(40) Suppose a, b are connected and a, c are connected and c, d are connected. Let A be a path from a to b, B be a path from c to d, and C be a path from a to c. Then $A + -A + C + B + -B$, C are homotopic.

(41) Let A be a path from a_1 to b_1, B be a path from c_1 to d_1, and C be a path from a_1 to c_1. Then $A + -A + C + B + -B$, C are homotopic.

(42) Suppose a, b are connected and a, c are connected and d, c are connected. Let A be a path from a to b, B be a path from c to d, and C be a path from a to c. Then $A + (-A + C + B) + -B$, C are homotopic.

(43) Let A be a path from a_1 to b_1, B be a path from c_1 to d_1, and C be a path from a_1 to c_1. Then $A + (-A + C + B) + -B$, C are homotopic.

(44) Suppose that

(i) a, b are connected,

(ii) b, c are connected,

(iii) c, d are connected,

(iv) d, e are connected, and

(v) a, f are connected.

Let A be a path from a to b, B be a path from b to c, C be a path from c to d, D be a path from d to e, and E be a path from f to c. Then $(A + (B + C)) + D$, $A + B + -E + (E + C + D)$ are homotopic.

(45) Let A be a path from a_1 to b_1, B be a path from b_1 to c_1, C be a path from c_1 to d_1, D be a path from d_1 to e_1, and E be a path from f_1 to c_1. Then $(A + (B + C)) + D$, $A + B + -E + (E + C + D)$ are homotopic.

3. The Fundamental Group

Let T be a topological structure and let t be a point of T. A loop of t is a path from t to t.

Let T be a non empty topological structure and let t be a point of T. The functor $\mathrm{Loops}(t)$ is defined by:

(Def. 1) For every set x holds $x \in \mathrm{Loops}(t)$ iff x is a loop of t.

Let T be a non empty topological structure and let t be a point of T. Observe that $\mathrm{Loops}(t)$ is non empty.

Let X be a non empty topological space and let a be a point of X. The functor $\mathrm{EqRel}(X, a)$ yielding a binary relation on $\mathrm{Loops}(a)$ is defined by:

(Def. 2) For all loops P, Q of a holds $\langle P, Q\rangle \in \mathrm{EqRel}(X, a)$ iff P, Q are homotopic.

Let X be a non empty topological space and let a be a point of X. One can check that $\mathrm{EqRel}(X, a)$ is non empty, total, symmetric, and transitive.

We now state two propositions:

(46) For all loops P, Q of a holds $Q \in [P]_{\mathrm{EqRel}(X,a)}$ iff P, Q are homotopic.

(47) For all loops P, Q of a holds $[P]_{\mathrm{EqRel}(X,a)} = [Q]_{\mathrm{EqRel}(X,a)}$ iff P, Q are homotopic.

Let X be a non empty topological space and let a be a point of X. The functor $\mathrm{FundamentalGroup}(X, a)$ yielding a strict groupoid is defined by the conditions (Def. 3).

(Def. 3)(i) The carrier of $\mathrm{FundamentalGroup}(X, a) = \mathrm{Classes}\,\mathrm{EqRel}(X, a)$, and

(ii) for all elements x, y of $\mathrm{FundamentalGroup}(X, a)$ there exist loops P, Q of a such that $x = [P]_{\mathrm{EqRel}(X,a)}$ and $y = [Q]_{\mathrm{EqRel}(X,a)}$ and (the multiplication of $\mathrm{FundamentalGroup}(X, a))(x, y) = [P + Q]_{\mathrm{EqRel}(X,a)}$.

We introduce $\pi_1(X, a)$ as a synonym of $\mathrm{FundamentalGroup}(X, a)$.

Let X be a non empty topological space and let a be a point of X. One can verify that $\pi_1(X, a)$ is non empty.

Next we state the proposition

(48) For every set x holds $x \in$ the carrier of $\pi_1(X, a)$ iff there exists a loop P of a such that $x = [P]_{\mathrm{EqRel}(X,a)}$.

Let X be a non empty topological space and let a be a point of X. Note that $\pi_1(X, a)$ is associative and group-like.

Let T be a non empty topological space, let x_0, x_1 be points of T, and let P be a path from x_0 to x_1. Let us assume that x_0, x_1 are connected. The functor $\pi_1\text{-iso}(P)$ yielding a map from $\pi_1(T, x_1)$ into $\pi_1(T, x_0)$ is defined by:

(Def. 4) For every loop Q of x_1 holds $(\pi_1\text{-iso}(P))([Q]_{\mathrm{EqRel}(T,x_1)}) = [P + Q + -P]_{\mathrm{EqRel}(T,x_0)}$.

For simplicity, we follow the rules: x_0, x_1 denote points of X, P, Q denote paths from x_0 to x_1, y_0, y_1 denote points of T, and R, V denote paths from y_0 to y_1.

Next we state three propositions:

(49) If x_0, x_1 are connected and P, Q are homotopic, then π_1-iso(P) = π_1-iso(Q).

(50) If R, V are homotopic, then π_1-iso$(R) = \pi_1$-iso(V).

(51) If x_0, x_1 are connected, then π_1-iso(P) is a homomorphism from $\pi_1(X, x_1)$ to $\pi_1(X, x_0)$.

Let T be a non empty arcwise connected topological space, let x_0, x_1 be points of T, and let P be a path from x_0 to x_1. Then π_1-iso(P) is a homomorphism from $\pi_1(T, x_1)$ to $\pi_1(T, x_0)$.

The following propositions are true:

(52) If x_0, x_1 are connected, then π_1-iso(P) is one-to-one.

(53) If x_0, x_1 are connected, then π_1-iso(P) is onto.

Let T be a non empty arcwise connected topological space, let x_0, x_1 be points of T, and let P be a path from x_0 to x_1. One can verify that π_1-iso(P) is one-to-one and onto.

One can prove the following propositions:

(54) If x_0, x_1 are connected, then $(\pi_1\text{-iso}(P))^{-1} = \pi_1$-iso$(-P)$.

(55) $(\pi_1\text{-iso}(R))^{-1} = \pi_1$-iso$(-R)$.

(56) If x_0, x_1 are connected, then for every homomorphism h from $\pi_1(X, x_1)$ to $\pi_1(X, x_0)$ such that $h = \pi_1$-iso(P) holds h is an isomorphism.

(57) π_1-iso(R) is an isomorphism.

(58) If x_0, x_1 are connected, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.

(59) $\pi_1(T, y_0)$ and $\pi_1(T, y_1)$ are isomorphic.

4. Euclidean Topological Space

Let n be a natural number, let a, b be points of $\mathcal{E}^n_{\mathrm{T}}$, and let P, Q be paths from a to b. The functor RealHomotopy(P, Q) yields a map from $[\!: \mathbb{I}, \mathbb{I} :\!]$ into $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 5) For all elements s, t of $\mathbb{I}$ holds $(\text{RealHomotopy}(P, Q))(s, t) = (1 - t) \cdot P(s) + t \cdot Q(s)$.

The following proposition is true

(60) For all points a, b of $\mathcal{E}^n_{\mathrm{T}}$ and for all paths P, Q from a to b holds P, Q are homotopic.

Let n be a natural number, let a, b be points of $\mathcal{E}^n_{\mathrm{T}}$, and let P, Q be paths from a to b. Then RealHomotopy(P, Q) is a homotopy between P and Q.

Let n be a natural number, let a, b be points of $\mathcal{E}^n_{\rm T}$, and let P, Q be paths from a to b. One can check that every homotopy between P and Q is continuous.

Next we state the proposition

(61) For every point a of $\mathcal{E}^n_{\rm T}$ and for every loop C of a holds the carrier of $\pi_1(\mathcal{E}^n_{\rm T}, a) = \{[C]_{{\rm EqRel}(\mathcal{E}^n_{\rm T}, a)}\}$.

Let n be a natural number and let a be a point of $\mathcal{E}^n_{\rm T}$. Note that $\pi_1(\mathcal{E}^n_{\rm T}, a)$ is trivial.

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The Continuous Functions on Normed Linear Spaces

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Summary. In this article, the basic properties of the continuous function on normed linear spaces are described.

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The articles [16], [19], [20], [2], [21], [4], [9], [3], [1], [11], [15], [5], [17], [18], [10], [7], [8], [6], [13], [22], [12], and [14] provide the notation and terminology for this paper.

We use the following convention: n is a natural number, x, X, X_1 are sets, and s, r, p are real numbers.

Let S, T be 1-sorted structures. A partial function from S to T is a partial function from the carrier of S to the carrier of T.

For simplicity, we adopt the following rules: S, T denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, s_1 denotes a sequence of S, x_0, x_1, x_2 denote points of S, and Y denotes a subset of S.

Let R_1 be a real linear space and let S_1 be a sequence of R_1. The functor $-S_1$ yields a sequence of R_1 and is defined as follows:

(Def. 1) For every n holds $(-S_1)(n) = -S_1(n)$.

Next we state two propositions:

(1) For all sequences s_2, s_3 of S holds $s_2 - s_3 = s_2 + -s_3$.

(2) For every sequence s_4 of S holds $-s_4 = (-1) \cdot s_4$.

Let us consider S, T and let f be a partial function from S to T. The functor $\|f\|$ yielding a partial function from the carrier of S to $\mathbb{R}$ is defined as follows:

(Def. 2) $\mathrm{dom}\|f\| = \mathrm{dom}\, f$ and for every point c of S such that $c \in \mathrm{dom}\|f\|$ holds $\|f\|(c) = \|f_c\|$.

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Let us consider S, x_0. A subset of S is called a neighbourhood of x_0 if:

(Def. 3) There exists a real number g such that $0 < g$ and $\{y; y$ ranges over points of S: $\|y - x_0\| < g\} \subseteq$ it.

The following two propositions are true:

(3) For every real number g such that $0 < g$ holds $\{y; y$ ranges over points of S: $\|y - x_0\| < g\}$ is a neighbourhood of x_0.

(4) For every neighbourhood N of x_0 holds $x_0 \in N$.

Let us consider S and let X be a subset of S. We say that X is compact if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let s_1 be a sequence of S. Suppose $\operatorname{rng} s_1 \subseteq X$. Then there exists a sequence s_5 of S such that s_5 is a subsequence of s_1 and convergent and $\lim s_5 \in X$.

Let us consider S and let X be a subset of S. We say that X is closed if and only if:

(Def. 5) For every sequence s_1 of S such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent holds $\lim s_1 \in X$.

Let us consider S and let X be a subset of S. We say that X is open if and only if:

(Def. 6) X^{c} is closed.

Let us consider S, T, let us consider f, and let s_4 be a sequence of S. Let us assume that $\operatorname{rng} s_4 \subseteq \operatorname{dom} f$. The functor $f \cdot s_4$ yields a sequence of T and is defined as follows:

(Def. 7) $f \cdot s_4 = (f$ **qua** function$)\ \cdot (s_4)$.

Let us consider S, let f be a partial function from the carrier of S to $\mathbb{R}$, and let s_4 be a sequence of S. Let us assume that $\operatorname{rng} s_4 \subseteq \operatorname{dom} f$. The functor $f \cdot s_4$ yields a sequence of real numbers and is defined as follows:

(Def. 8) $f \cdot s_4 = (f$ **qua** function$)\ \cdot (s_4)$.

Let us consider S, T and let us consider f, x_0. We say that f is continuous in x_0 if and only if:

(Def. 9) $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider S, let f be a partial function from the carrier of S to $\mathbb{R}$, and let us consider x_0. We say that f is continuous in x_0 if and only if:

(Def. 10) $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

The scheme *SeqPointNormSpChoice* deals with a non empty normed structure $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a sequence s_1 of $\mathcal{A}$ such that for every natural number n holds $\mathcal{P}[n, s_1(n)]$

provided the following condition is met:

- For every natural number n there exists a point r of $\mathcal{A}$ such that $\mathcal{P}[n, r]$.

The following propositions are true:

(5) For every sequence s_4 of S and for every partial function h from S to T such that $\operatorname{rng} s_4 \subseteq \operatorname{dom} h$ holds $s_4(n) \in \operatorname{dom} h$.

(6) For every sequence s_4 of S and for every set x holds $x \in \operatorname{rng} s_4$ iff there exists n such that $x = s_4(n)$.

(7) For all sequences s_4, s_2 of S such that s_2 is a subsequence of s_4 holds $\operatorname{rng} s_2 \subseteq \operatorname{rng} s_4$.

(8) For all f, s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and for every n holds $(f \cdot s_1)(n) = f_{s_1(n)}$.

(9) Let f be a partial function from the carrier of S to $\mathbb{R}$ and given s_1. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$, then for every n holds $(f \cdot s_1)(n) = f_{s_1(n)}$.

(10) Let h be a partial function from S to T, s_4 be a sequence of S, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} s_4 \subseteq \operatorname{dom} h$, then $(h \cdot s_4) \cdot N_1 = h \cdot (s_4 \cdot N_1)$.

(11) Let h be a partial function from the carrier of S to $\mathbb{R}$, s_4 be a sequence of S, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} s_4 \subseteq \operatorname{dom} h$, then $(h \cdot s_4) \cdot N_1 = h \cdot (s_4 \cdot N_1)$.

(12) Let h be a partial function from S to T and s_2, s_3 be sequences of S. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$ and s_3 is a subsequence of s_2, then $h \cdot s_3$ is a subsequence of $h \cdot s_2$.

(13) Let h be a partial function from the carrier of S to $\mathbb{R}$ and s_2, s_3 be sequences of S. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$ and s_3 is a subsequence of s_2, then $h \cdot s_3$ is a subsequence of $h \cdot s_2$.

(14) f is continuous in x_0 if and only if the following conditions are satisfied:
  (i) $x_0 \in \operatorname{dom} f$, and
  (ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(15) Let f be a partial function from the carrier of S to $\mathbb{R}$. Then f is continuous in x_0 if and only if the following conditions are satisfied:
  (i) $x_0 \in \operatorname{dom} f$, and
  (ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(16) Let given f, x_0. Then f is continuous in x_0 if and only if the following conditions are satisfied:
  (i) $x_0 \in \operatorname{dom} f$, and
  (ii) for every neighbourhood N_2 of f_{x_0} there exists a neighbourhood N of x_0 such that for every x_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_2$.

(17) Let given f, x_0. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_2 of f_{x_0} there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_2$.

(18) If $x_0 \in \operatorname{dom} f$ and there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$, then f is continuous in x_0.

(19) Let h_1, h_2 be partial functions from S to T and s_4 be a sequence of S. If $\operatorname{rng} s_4 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1 + h_2) \cdot s_4 = h_1 \cdot s_4 + h_2 \cdot s_4$ and $(h_1 - h_2) \cdot s_4 = h_1 \cdot s_4 - h_2 \cdot s_4$.

(20) Let h be a partial function from S to T, s_4 be a sequence of S, and r be a real number. If $\operatorname{rng} s_4 \subseteq \operatorname{dom} h$, then $(r\,h) \cdot s_4 = r \cdot (h \cdot s_4)$.

(21) Let h be a partial function from S to T and s_4 be a sequence of S. If $\operatorname{rng} s_4 \subseteq \operatorname{dom} h$, then $\|h \cdot s_4\| = \|h\| \cdot s_4$ and $-h \cdot s_4 = (-h) \cdot s_4$.

(22) If f_1 is continuous in x_0 and f_2 is continuous in x_0, then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0.

(23) If f is continuous in x_0, then $r\,f$ is continuous in x_0.

(24) If f is continuous in x_0, then $\|f\|$ is continuous in x_0 and $-f$ is continuous in x_0.

Let us consider S, T and let us consider f, X. We say that f is continuous on X if and only if:

(Def. 11) $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f{\restriction}X$ is continuous in x_0.

Let us consider S, let f be a partial function from the carrier of S to $\mathbb{R}$, and let us consider X. We say that f is continuous on X if and only if:

(Def. 12) $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f{\restriction}X$ is continuous in x_0.

One can prove the following propositions:

(25) Let given X, f. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 \in X$ holds $f \cdot s_1$ is convergent and $f_{\lim s_1} = \lim(f \cdot s_1)$.

(26) f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(27) Let f be a partial function from the carrier of S to $\mathbb{R}$. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(28) f is continuous on X iff $f{\restriction}X$ is continuous on X.

(29) Let f be a partial function from the carrier of S to $\mathbb{R}$. Then f is continuous on X if and only if $f{\restriction}X$ is continuous on X.

(30) If f is continuous on X and $X_1 \subseteq X$, then f is continuous on X_1.

(31) If $x_0 \in \operatorname{dom} f$, then f is continuous on $\{x_0\}$.

(32) For all X, f_1, f_2 such that f_1 is continuous on X and f_2 is continuous on X holds $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X.

(33) Let given X, X_1, f_1, f_2. Suppose f_1 is continuous on X and f_2 is continuous on X_1. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

(34) For all r, X, f such that f is continuous on X holds $r\,f$ is continuous on X.

(35) If f is continuous on X, then $\|f\|$ is continuous on X and $-f$ is continuous on X.

(36) Suppose f is total and for all x_1, x_2 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists x_0 such that f is continuous in x_0. Then f is continuous on the carrier of S.

(37) For every f such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds $\operatorname{rng} f$ is compact.

(38) Let f be a partial function from the carrier of S to $\mathbb{R}$. If $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$, then $\operatorname{rng} f$ is compact.

(39) If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y, then $f^\circ Y$ is compact.

(40) Let f be a partial function from the carrier of S to $\mathbb{R}$. Suppose $\operatorname{dom} f \neq \emptyset$ and $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$. Then there exist x_1, x_2 such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $f_{x_1} = \sup \operatorname{rng} f$ and $f_{x_2} = \inf \operatorname{rng} f$.

(41) Let given f. Suppose $\operatorname{dom} f \neq \emptyset$ and $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$. Then there exist x_1, x_2 such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $\|f\|_{x_1} = \sup \operatorname{rng} \|f\|$ and $\|f\|_{x_2} = \inf \operatorname{rng} \|f\|$.

(42) $\|f\|{\restriction}X = \|f{\restriction}X\|$.

(43) Let given f, Y. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y. Then there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $\|f\|_{x_1} = \sup(\|f\|^\circ Y)$ and $\|f\|_{x_2} = \inf(\|f\|^\circ Y)$.

(44) Let f be a partial function from the carrier of S to $\mathbb{R}$ and given Y. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y. Then there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $f_{x_1} = \sup(f^\circ Y)$

and $f_{x_2} = \inf(f^\circ Y)$.

Let us consider S, T and let us consider X, f. We say that f is Lipschitzian on X if and only if:

(Def. 13) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ holds $\|f_{x_1} - f_{x_2}\| \leqslant r \cdot \|x_1 - x_2\|$.

Let us consider S, let us consider X, and let f be a partial function from the carrier of S to $\mathbb{R}$. We say that f is Lipschitzian on X if and only if:

(Def. 14) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ holds $|f_{x_1} - f_{x_2}| \leqslant r \cdot \|x_1 - x_2\|$.

The following propositions are true:

(45) If f is Lipschitzian on X and $X_1 \subseteq X$, then f is Lipschitzian on X_1.

(46) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1, then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(47) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1, then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(48) If f is Lipschitzian on X, then $p\,f$ is Lipschitzian on X.

(49) If f is Lipschitzian on X, then $-f$ is Lipschitzian on X and $\|f\|$ is Lipschitzian on X.

(50) If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is Lipschitzian on X.

(51) id_Y is Lipschitzian on Y.

(52) If f is Lipschitzian on X, then f is continuous on X.

(53) Let f be a partial function from the carrier of S to $\mathbb{R}$. If f is Lipschitzian on X, then f is continuous on X.

(54) For every f such that there exists a point r of T such that $\operatorname{rng} f = \{r\}$ holds f is continuous on $\operatorname{dom} f$.

(55) If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is continuous on X.

(56) For every partial function f from S to S such that for every x_0 such that $x_0 \in \operatorname{dom} f$ holds $f_{x_0} = x_0$ holds f is continuous on $\operatorname{dom} f$.

(57) For every partial function f from S to S such that $f = \operatorname{id}_{\operatorname{dom} f}$ holds f is continuous on $\operatorname{dom} f$.

(58) For every partial function f from S to S such that $Y \subseteq \operatorname{dom} f$ and $f{\restriction}Y = \operatorname{id}_Y$ holds f is continuous on Y.

(59) Let f be a partial function from S to S, r be a real number, and p be a point of S. Suppose $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f_{x_0} = r \cdot x_0 + p$. Then f is continuous on X.

(60) Let f be a partial function from the carrier of S to $\mathbb{R}$. If for every x_0 such that $x_0 \in \operatorname{dom} f$ holds $f_{x_0} = \|x_0\|$, then f is continuous on $\operatorname{dom} f$.

(61) Let f be a partial function from the carrier of S to $\mathbb{R}$. If $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f_{x_0} = \|x_0\|$, then f is continuous on X.

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The Uniform Continuity of Functions on Normed Linear Spaces

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Summary. In this article, the basic properties of uniform continuity of functions on normed linear spaces are described.

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The notation and terminology used in this paper are introduced in the following articles: [15], [18], [19], [1], [20], [3], [2], [7], [14], [16], [9], [13], [4], [17], [6], [5], [11], [21], [10], [12], and [8].

1. The Uniform Continuity of Functions on Normed Linear Spaces

For simplicity, we follow the rules: X, X_1 are sets, s, r, p are real numbers, S, T are real normed spaces, f, f_1, f_2 are partial functions from S to T, x_1, x_2 are points of S, and Y is a subset of S.

Let us consider X, S, T and let us consider f. We say that f is uniformly continuous on X if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

Let us consider X, S and let f be a partial function from the carrier of S to $\mathbb{R}$. We say that f is uniformly continuous on X if and only if the conditions (Def. 2) are satisfied.

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(Def. 2)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $|f_{x_1} - f_{x_2}| < r$.

The following propositions are true:

(1) If f is uniformly continuous on X and $X_1 \subseteq X$, then f is uniformly continuous on X_1.

(2) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1, then $f_1 + f_2$ is uniformly continuous on $X \cap X_1$.

(3) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1, then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

(4) If f is uniformly continuous on X, then $p\,f$ is uniformly continuous on X.

(5) If f is uniformly continuous on X, then $-f$ is uniformly continuous on X.

(6) If f is uniformly continuous on X, then $\|f\|$ is uniformly continuous on X.

(7) If f is uniformly continuous on X, then f is continuous on X.

(8) Let f be a partial function from the carrier of S to $\mathbb{R}$. If f is uniformly continuous on X, then f is continuous on X.

(9) If f is Lipschitzian on X, then f is uniformly continuous on X.

(10) For all f, Y such that Y is compact and f is continuous on Y holds f is uniformly continuous on Y.

(11) If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y, then $f^\circ Y$ is compact.

(12) Let f be a partial function from the carrier of S to $\mathbb{R}$ and given Y. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y. Then there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $f_{x_1} = \sup(f^\circ Y)$ and $f_{x_2} = \inf(f^\circ Y)$.

(13) If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is uniformly continuous on X.

2. The Contraction Mapping Principle on Normed Linear Spaces

Let M be a real Banach space. A function from the carrier of M into the carrier of M is said to be a contraction of M if:

(Def. 3) There exists a real number L such that $0 < L$ and $L < 1$ and for all points x, y of M holds $\|\mathrm{it}(x) - \mathrm{it}(y)\| \leqslant L \cdot \|x - y\|$.

The following two propositions are true:

(14) Let X be a real Banach space and f be a function from X into X. Suppose f is a contraction of X. Then there exists a point x_3 of X such that $f(x_3) = x_3$ and for every point x of X such that $f(x) = x$ holds $x_3 = x$.

(15) Let X be a real Banach space and f be a function from X into X. Given a natural number n_0 such that f^{n_0} is a contraction of X. Then there exists a point x_3 of X such that $f(x_3) = x_3$ and for every point x of X such that $f(x) = x$ holds $x_3 = x$.

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Series on Complex Banach Algebra

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The articles [22], [24], [25], [5], [6], [3], [2], [21], [11], [1], [23], [4], [15], [16], [17], [14], [12], [13], [19], [18], [10], [8], [9], [7], and [20] provide the notation and terminology for this paper.

1. Basic Properties of Sequences of Norm Space

Let X be a non empty complex normed space structure and let s_1 be a sequence of X. The functor $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ yielding a sequence of X is defined as follows:

(Def. 1) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(0) = s_1(0)$ and for every natural number n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) + s_1(n+1)$.

One can prove the following proposition

(1) Let X be an add-associative right zeroed right complementable non empty complex normed space structure and s_1 be a sequence of X. Suppose that for every natural number n holds $s_1(n) = 0_X$. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) = 0_X$.

Let X be a complex normed space and let s_1 be a sequence of X. We say that s_1 is summable if and only if:

(Def. 2) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent.

Let X be a complex normed space. One can verify that there exists a sequence of X which is summable.

Let X be a complex normed space and let s_1 be a sequence of X. The functor $\sum s_1$ yields an element of X and is defined by:

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(Def. 3) $\sum s_1 = \lim((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})$.

Let X be a complex normed space and let s_1 be a sequence of X. We say that s_1 is norm-summable if and only if:

(Def. 4) $\|s_1\|$ is summable.

The following propositions are true:

(2) For every complex normed space X and for every sequence s_1 of X and for every natural number m holds $0 \leqslant \|s_1\|(m)$.

(3) For every complex normed space X and for all elements x, y, z of X holds $\|x - y\| = \|(x - z) + (z - y)\|$.

(4) Let X be a complex normed space and s_1 be a sequence of X. Suppose s_1 is convergent. Let s be a real number. Suppose $0 < s$. Then there exists a natural number n such that for every natural number m if $n \leqslant m$, then $\|s_1(m) - s_1(n)\| < s$.

(5) Let X be a complex normed space and s_1 be a sequence of X. Then s_1 is Cauchy sequence by norm if and only if for every real number p such that $p > 0$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $\|s_1(m) - s_1(n)\| < p$.

(6) Let X be a complex normed space and s_1 be a sequence of X. Suppose that for every natural number n holds $s_1(n) = 0_X$. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(m) = 0$.

Let X be a complex normed space and let s_1 be a sequence of X. Let us observe that s_1 is constant if and only if:

(Def. 5) There exists an element r of X such that for every natural number n holds $s_1(n) = r$.

Let X be a complex normed space, let s_1 be a sequence of X, and let k be a natural number. The functor $s_1 \uparrow k$ yielding a sequence of X is defined as follows:

(Def. 6) For every natural number n holds $(s_1 \uparrow k)(n) = s_1(n + k)$.

Let X be a complex normed space and let s_1, s_2 be sequences of X. We say that s_1 is a subsequence of s_2 if and only if:

(Def. 7) There exists an increasing sequence N_1 of naturals such that $s_1 = s_2 \cdot N_1$.

Next we state a number of propositions:

(7) For every complex normed space X and for every sequence s_1 of X holds $s_1 \uparrow 0 = s_1$.

(8) For every complex normed space X and for every sequence s_1 of X and for all natural numbers k, m holds $s_1 \uparrow k \uparrow m = s_1 \uparrow m \uparrow k$.

(9) For every complex normed space X and for every sequence s_1 of X and for all natural numbers k, m holds $s_1 \uparrow k \uparrow m = s_1 \uparrow (k + m)$.

(10) Let X be a complex normed space and s_1, s_2 be sequences of X. If s_2 is a subsequence of s_1 and s_1 is convergent, then s_2 is convergent.

(11) Let X be a complex normed space and s_1, s_2 be sequences of X. If s_2 is a subsequence of s_1 and s_1 is convergent, then $\lim s_2 = \lim s_1$.

(12) Let X be a complex normed space, s_1 be a sequence of X, and k be a natural number. Then $s_1 \uparrow k$ is a subsequence of s_1.

(13) Let X be a complex normed space, s_1, s_2 be sequences of X, and k be a natural number. If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim(s_1 \uparrow k) = \lim s_1$.

(14) Let X be a complex normed space and s_1, s_2 be sequences of X. Suppose s_1 is convergent and there exists a natural number k such that $s_1 = s_2 \uparrow k$. Then s_2 is convergent.

(15) Let X be a complex normed space and s_1, s_2 be sequences of X. Suppose s_1 is convergent and there exists a natural number k such that $s_1 = s_2 \uparrow k$. Then $\lim s_2 = \lim s_1$.

(16) For every complex normed space X and for every sequence s_1 of X such that s_1 is constant holds s_1 is convergent.

(17) Let X be a complex normed space and s_1 be a sequence of X. If for every natural number n holds $s_1(n) = 0_X$, then s_1 is norm-summable.

Let X be a complex normed space. Observe that there exists a sequence of X which is norm-summable.

The following three propositions are true:

(18) Let X be a complex normed space and s be a sequence of X. If s is summable, then s is convergent and $\lim s = 0_X$.

(19) For every complex normed space X and for all sequences s_3, s_4 of X holds $(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa}(s_4)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_3 + s_4)(\alpha))_{\kappa\in\mathbb{N}}$.

(20) For every complex normed space X and for all sequences s_3, s_4 of X holds $(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa}(s_4)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_3 - s_4)(\alpha))_{\kappa\in\mathbb{N}}$.

Let X be a complex normed space and let s_1 be a norm-summable sequence of X. Observe that $\|s_1\|$ is summable.

Let X be a complex normed space. One can check that every sequence of X which is summable is also convergent.

The following two propositions are true:

(21) Let X be a complex normed space and s_2, s_5 be sequences of X. If s_2 is summable and s_5 is summable, then s_2+s_5 is summable and $\sum(s_2+s_5) = \sum s_2 + \sum s_5$.

(22) Let X be a complex normed space and s_2, s_5 be sequences of X. If s_2 is summable and s_5 is summable, then s_2-s_5 is summable and $\sum(s_2-s_5) = \sum s_2 - \sum s_5$.

Let X be a complex normed space and let s_2, s_5 be summable sequences of X. One can check that $s_2 + s_5$ is summable and $s_2 - s_5$ is summable.

The following propositions are true:

(23) For every complex normed space X and for every sequence s_1 of X and for every complex number z holds $(\sum_{\alpha=0}^{\kappa}(z \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}} = z \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

(24) Let X be a complex normed space, s_1 be a summable sequence of X, and z be a complex number. Then $z \cdot s_1$ is summable and $\sum(z \cdot s_1) = z \cdot \sum s_1$.

Let X be a complex normed space, let z be a complex number, and let s_1 be a summable sequence of X. One can check that $z \cdot s_1$ is summable.

Next we state two propositions:

(25) Let X be a complex normed space and s, s_3 be sequences of X. If for every natural number n holds $s_3(n) = s(0)$, then $(\sum_{\alpha=0}^{\kappa}(s \uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}} \uparrow 1 - s_3$.

(26) Let X be a complex normed space and s be a sequence of X. If s is summable, then for every natural number n holds $s \uparrow n$ is summable.

Let X be a complex normed space, let s_1 be a summable sequence of X, and let n be a natural number. Observe that $s_1 \uparrow n$ is summable.

We now state the proposition

(27) Let X be a complex normed space and s_1 be a sequence of X. Then $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded if and only if s_1 is norm-summable.

Let X be a complex normed space and let s_1 be a norm-summable sequence of X. Note that $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded.

The following propositions are true:

(28) Let X be a complex Banach space and s_1 be a sequence of X. Then s_1 is summable if and only if for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)\| < p$.

(29) Let X be a complex normed space, s be a sequence of X, and n, m be natural numbers. If $n \leqslant m$, then $\|(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n)\| \leqslant |(\sum_{\alpha=0}^{\kappa}\|s\|(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|s\|(\alpha))_{\kappa\in\mathbb{N}}(n)|$.

(30) For every complex Banach space X and for every sequence s_1 of X such that s_1 is norm-summable holds s_1 is summable.

(31) Let X be a complex normed space, r_1 be a sequence of real numbers, and s_5 be a sequence of X. Suppose r_1 is summable and there exists a natural number m such that for every natural number n such that $m \leqslant n$ holds $\|s_5(n)\| \leqslant r_1(n)$. Then s_5 is norm-summable.

(32) Let X be a complex normed space and s_2, s_5 be sequences of X. Suppose for every natural number n holds $0 \leqslant \|s_2\|(n)$ and $\|s_2\|(n) \leqslant \|s_5\|(n)$ and s_5 is norm-summable. Then s_2 is norm-summable and $\sum\|s_2\| \leqslant \sum\|s_5\|$.

(33) Let X be a complex normed space and s_1 be a sequence of X. Suppose that

(i) for every natural number n holds $\|s_1\|(n) > 0$, and

(ii) there exists a natural number m such that for every natural number n such that $n \geqslant m$ holds $\frac{\|s_1\|(n+1)}{\|s_1\|(n)} \geqslant 1$.

Then s_1 is not norm-summable.

(34) Let X be a complex normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose for every natural number n holds $r_1(n) = \sqrt[n]{\|s_1\|(n)}$ and r_1 is convergent and $\lim r_1 < 1$. Then s_1 is norm-summable.

(35) Let X be a complex normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose that

(i) for every natural number n holds $r_1(n) = \sqrt[n]{\|s_1\|(n)}$, and

(ii) there exists a natural number m such that for every natural number n such that $m \leqslant n$ holds $r_1(n) \geqslant 1$.

Then $\|s_1\|$ is not summable.

(36) Let X be a complex normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose for every natural number n holds $r_1(n) = \sqrt[n]{\|s_1\|(n)}$ and r_1 is convergent and $\lim r_1 > 1$. Then s_1 is not norm-summable.

(37) Let X be a complex normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose $\|s_1\|$ is non-increasing and for every natural number n holds $r_1(n) = 2^n \cdot \|s_1\|(2^n)$. Then s_1 is norm-summable if and only if r_1 is summable.

(38) Let X be a complex normed space, s_1 be a sequence of X, and p be a real number. Suppose $p > 1$ and for every natural number n such that $n \geqslant 1$ holds $\|s_1\|(n) = \frac{1}{n^p}$. Then s_1 is norm-summable.

(39) Let X be a complex normed space, s_1 be a sequence of X, and p be a real number. Suppose $p \leqslant 1$ and for every natural number n such that $n \geqslant 1$ holds $\|s_1\|(n) = \frac{1}{n^p}$. Then s_1 is not norm-summable.

(40) Let X be a complex normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose for every natural number n holds $s_1(n) \neq 0_X$ and $r_1(n) = \frac{\|s_1\|(n+1)}{\|s_1\|(n)}$ and r_1 is convergent and $\lim r_1 < 1$. Then s_1 is norm-summable.

(41) Let X be a complex normed space and s_1 be a sequence of X. Suppose that

(i) for every natural number n holds $s_1(n) \neq 0_X$, and

(ii) there exists a natural number m such that for every natural number n such that $n \geqslant m$ holds $\frac{\|s_1\|(n+1)}{\|s_1\|(n)} \geqslant 1$.

Then s_1 is not norm-summable.

Let X be a complex Banach space. One can check that every sequence of X which is norm-summable is also summable.

2. Basic Properties of Sequence of Banach Algebra

The scheme *ExNCBCASeq* deals with a non empty normed complex algebra structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

We now state the proposition

(42) Let X be a complex Banach algebra, x, y, z be elements of X, and a, b be complex numbers. Then $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x + 0_X = x$ and there exists an element t of X such that $x + t = 0_X$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $1_{\mathbb{C}} \cdot x = x$ and $0_{\mathbb{C}} \cdot x = 0_X$ and $a \cdot 0_X = 0_X$ and $(-1_{\mathbb{C}}) \cdot x = -x$ and $x \cdot \mathbf{1}_X = x$ and $\mathbf{1}_X \cdot x = x$ and $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ and $a \cdot (x \cdot y) = (a \cdot x) \cdot y$ and $a \cdot (x + y) = a \cdot x + a \cdot y$ and $(a + b) \cdot x = a \cdot x + b \cdot x$ and $(a \cdot b) \cdot x = a \cdot (b \cdot x)$ and $(a \cdot b) \cdot (x \cdot y) = a \cdot x \cdot (b \cdot y)$ and $a \cdot (x \cdot y) = x \cdot (a \cdot y)$ and $0_X \cdot x = 0_X$ and $x \cdot 0_X = 0_X$ and $x \cdot (y - z) = x \cdot y - x \cdot z$ and $(y - z) \cdot x = y \cdot x - z \cdot x$ and $(x+y)-z = x+(y-z)$ and $(x-y)+z = x-(y-z)$ and $x-y-z = x-(y+z)$ and $x+y = (x-z)+(z+y)$ and $x-y = (x-z)+(z-y)$ and $x = (x-y)+y$ and $x = y - (y - x)$ and $\|x\| = 0$ iff $x = 0_X$ and $\|a \cdot x\| = |a| \cdot \|x\|$ and $\|x + y\| \leqslant \|x\| + \|y\|$ and $\|x \cdot y\| \leqslant \|x\| \cdot \|y\|$ and $\|\mathbf{1}_X\| = 1$ and X is complete.

Let X be a non empty normed complex algebra structure, let S be a sequence of X, and let a be an element of X. The functor $a \cdot S$ yields a sequence of X and is defined by:

(Def. 8) For every natural number n holds $(a \cdot S)(n) = a \cdot S(n)$.

Let X be a non empty normed complex algebra structure, let S be a sequence of X, and let a be an element of X. The functor $S \cdot a$ yields a sequence of X and is defined by:

(Def. 9) For every natural number n holds $(S \cdot a)(n) = S(n) \cdot a$.

Let X be a non empty normed complex algebra structure and let s_2, s_5 be sequences of X. The functor $s_2 \cdot s_5$ yielding a sequence of X is defined by:

(Def. 10) For every natural number n holds $(s_2 \cdot s_5)(n) = s_2(n) \cdot s_5(n)$.

Let X be a complex Banach algebra and let x be an element of X. Let us assume that x is invertible. The functor x^{-1} yields an element of X and is defined as follows:

(Def. 11) $x \cdot x^{-1} = \mathbf{1}_X$ and $x^{-1} \cdot x = \mathbf{1}_X$.

Let X be a complex Banach algebra and let z be an element of X. The functor $(z^\kappa)_{\kappa\in\mathbb{N}}$ yielding a sequence of X is defined as follows:

(Def. 12) $(z^\kappa)_{\kappa\in\mathbb{N}}(0) = \mathbf{1}_X$ and for every natural number n holds $(z^\kappa)_{\kappa\in\mathbb{N}}(n+1) = (z^\kappa)_{\kappa\in\mathbb{N}}(n) \cdot z$.

Let X be a complex Banach algebra, let z be an element of X, and let n be a natural number. The functor $z_{\mathbb{N}}^n$ yielding an element of X is defined as follows:

(Def. 13) $z_{\mathbb{N}}^n = (z^\kappa)_{\kappa\in\mathbb{N}}(n)$.

The following propositions are true:

(43) For every complex Banach algebra X and for every element z of X holds $z_{\mathbb{N}}^0 = \mathbf{1}_X$.

(44) For every complex Banach algebra X and for every element z of X such that $\|z\| < 1$ holds $(z^\kappa)_{\kappa\in\mathbb{N}}$ is summable and norm-summable.

(45) Let X be a complex Banach algebra and x be a point of X. If $\|\mathbf{1}_X - x\| < 1$, then $((\mathbf{1}_X - x)^\kappa)_{\kappa\in\mathbb{N}}$ is summable and $((\mathbf{1}_X - x)^\kappa)_{\kappa\in\mathbb{N}}$ is norm-summable.

(46) For every complex Banach algebra X and for every point x of X such that $\|\mathbf{1}_X - x\| < 1$ holds x is invertible and $x^{-1} = \sum(((\mathbf{1}_X - x)^\kappa)_{\kappa\in\mathbb{N}})$.

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Exponential Function on Complex Banach Algebra

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Summary. This article is an extension of [18].

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The papers [23], [24], [4], [5], [2], [20], [21], [9], [1], [22], [13], [15], [16], [12], [10], [11], [17], [14], [25], [3], [7], [6], [19], and [8] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: X denotes a complex Banach algebra, w, z, z_1, z_2 denote elements of X, k, l, m, n denote natural numbers, s_1, s_2, s_3, s, s' denote sequences of X, and r_1 denotes a sequence of real numbers.

Let X be a non empty normed complex algebra structure and let x, y be elements of X. We say that x, y are commutative if and only if:

(Def. 1) $x \cdot y = y \cdot x$.

Let us note that the predicate x, y are commutative is symmetric.

One can prove the following propositions:

(1) If s_2 is convergent and s_3 is convergent and $\lim(s_2 - s_3) = 0_X$, then $\lim s_2 = \lim s_3$.

(2) For every z such that for every natural number n holds $s(n) = z$ holds $\lim s = z$.

(3) If s is convergent and s' is convergent, then $s \cdot s'$ is convergent.

(4) If s is convergent, then $z \cdot s$ is convergent.

(5) If s is convergent, then $s \cdot z$ is convergent.

(6) If s is convergent, then $\lim(z \cdot s) = z \cdot \lim s$.

(7) If s is convergent, then $\lim(s \cdot z) = \lim s \cdot z$.

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(8) If s is convergent and s' is convergent, then $\lim(s \cdot s') = \lim s \cdot \lim s'$.

(9) $(\sum_{\alpha=0}^{\kappa}(z \cdot s_1)(\alpha))_{\kappa \in \mathbb{N}} = z \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}$ and $(\sum_{\alpha=0}^{\kappa}(s_1 \cdot z)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}} \cdot z$.

(10) $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}(k)\| \leqslant (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa \in \mathbb{N}}(k)$.

(11) If for every n such that $n \leqslant m$ holds $s_2(n) = s_3(n)$, then $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa \in \mathbb{N}}(m) = (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(12) If for every n holds $\|s_1(n)\| \leqslant r_1(n)$ and r_1 is convergent and $\lim r_1 = 0$, then s_1 is convergent and $\lim s_1 = 0_X$.

Let us consider X, z. The functor $z\,\mathrm{ExpSeq}$ yields a sequence of X and is defined as follows:

(Def. 2) For every n holds $z\,\mathrm{ExpSeq}(n) = \frac{1_{\mathbb{C}}}{n!_{\mathbb{C}}} \cdot z_{\mathbb{N}}^{n}$.

The scheme *ExNormSpace CASE* deals with a non empty complex Banach algebra $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

> For every k there exists a sequence s_1 of $\mathcal{A}$ such that for every n holds if $n \leqslant k$, then $s_1(n) = \mathcal{F}(k, n)$ and if $n > k$, then $s_1(n) = 0_{\mathcal{A}}$

for all values of the parameters.

Let us consider X, s_1. The functor $\mathrm{Shift}\, s_1$ yielding a sequence of X is defined by:

(Def. 3) $(\mathrm{Shift}\, s_1)(0) = 0_X$ and for every natural number k holds $(\mathrm{Shift}\, s_1)(k + 1) = s_1(k)$.

Let us consider n, X, z, w. The functor $\mathrm{Expan}(n, z, w)$ yielding a sequence of X is defined by:

(Def. 4) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Expan}(n, z, w))(k) = (\mathrm{Coef}\, n)(k) \cdot z_{\mathbb{N}}^{k} \cdot w_{\mathbb{N}}^{n -' k}$ and if $n < k$, then $(\mathrm{Expan}(n, z, w))(k) = 0_X$.

Let us consider n, X, z, w. The functor $\mathrm{Expan\_e}(n, z, w)$ yields a sequence of X and is defined as follows:

(Def. 5) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Expan\_e}(n, z, w))(k) = (\mathrm{Coef\_e}\, n)(k) \cdot z_{\mathbb{N}}^{k} \cdot w_{\mathbb{N}}^{n -' k}$ and if $n < k$, then $(\mathrm{Expan\_e}(n, z, w))(k) = 0_X$.

Let us consider n, X, z, w. The functor $\mathrm{Alfa}(n, z, w)$ yielding a sequence of X is defined by:

(Def. 6) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Alfa}(n, z, w))(k) = z\,\mathrm{ExpSeq}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n -' k)$ and if $n < k$, then $(\mathrm{Alfa}(n, z, w))(k) = 0_X$.

Let us consider X, z, w, n. The functor $\mathrm{Conj}(n, z, w)$ yields a sequence of X and is defined as follows:

(Def. 7) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Conj}(n, z, w))(k) = z\,\mathrm{ExpSeq}(k) \cdot ((\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n -' k))$ and if $n < k$, then $(\mathrm{Conj}(n, z, w))(k) = 0_X$.

Next we state several propositions:

(13) $z\,\mathrm{ExpSeq}(n+1) = \frac{1_{\mathbb{C}}}{(n+1)+0i} \cdot z \cdot z\,\mathrm{ExpSeq}(n)$ and $z\,\mathrm{ExpSeq}(0) = \mathbf{1}_X$ and $\|z\,\mathrm{ExpSeq}(n)\| \leqslant \|z\|\,\mathrm{ExpSeq}(n)$.

(14) If $0 < k$, then $(\mathrm{Shift}\,s_1)(k) = s_1(k -' 1)$.

(15) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Shift}\,s_1)(\alpha))_{\kappa\in\mathbb{N}}(k) + s_1(k)$.

(16) For all z, w such that z, w are commutative holds $(z + w)_{\mathbb{N}}^{n} = (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan}(n, z, w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(17) $\mathrm{Expan\_e}(n, z, w) = \frac{1_{\mathbb{C}}}{n!_{\mathbb{C}}} \cdot \mathrm{Expan}(n, z, w)$.

(18) For all z, w such that z, w are commutative holds $\frac{1_{\mathbb{C}}}{n!_{\mathbb{C}}} \cdot (z + w)_{\mathbb{N}}^{n} = (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan\_e}(n, z, w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(19) $0_X\,\mathrm{ExpSeq}$ is norm-summable and $\sum(0_X\,\mathrm{ExpSeq}) = \mathbf{1}_X$.

Let us consider X and let z be an element of X. One can check that $z\,\mathrm{ExpSeq}$ is norm-summable.

We now state a number of propositions:

(20) $z\,\mathrm{ExpSeq}(0) = \mathbf{1}_X$ and $(\mathrm{Expan}(0, z, w))(0) = \mathbf{1}_X$.

(21) If $l \leqslant k$, then $(\mathrm{Alfa}(k + 1, z, w))(l) = (\mathrm{Alfa}(k, z, w))(l) + (\mathrm{Expan\_e}(k + 1, z, w))(l)$.

(22) $(\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(k+1, z, w))(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(k, z, w))(\alpha))_{\kappa\in\mathbb{N}}(k) + (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan\_e}(k + 1, z, w))(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(23) $z\,\mathrm{ExpSeq}(k) = (\mathrm{Expan\_e}(k, z, w))(k)$.

(24) For all z, w such that z, w are commutative holds $(\sum_{\alpha=0}^{\kappa} z + w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(n, z, w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(25) For all z, w such that z, w are commutative holds $(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) - (\sum_{\alpha=0}^{\kappa} z + w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Conj}(k, z, w))(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(26) $0 \leqslant \|z\|\,\mathrm{ExpSeq}(n)$.

(27) $\|(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)\| \leqslant (\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)$ and $(\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) \leqslant \sum(\|z\|\,\mathrm{ExpSeq})$ and $\|(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)\| \leqslant \sum(\|z\|\,\mathrm{ExpSeq})$.

(28) $1 \leqslant \sum(\|z\|\,\mathrm{ExpSeq})$.

(29) $|(\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)$ and if $n \leqslant m$, then $|(\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(30) $|(\sum_{\alpha=0}^{\kappa}\|\mathrm{Conj}(k, z, w)\|(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|\mathrm{Conj}(k, z, w)\|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(31) For every real number p such that $p > 0$ there exists n such that for every k such that $n \leqslant k$ holds $|(\sum_{\alpha=0}^{\kappa}\|\mathrm{Conj}(k, z, w)\|(\alpha))_{\kappa\in\mathbb{N}}(k)| < p$.

(32) For every s_1 such that for every k holds $s_1(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Conj}(k, z, w))(\alpha))_{\kappa\in\mathbb{N}}(k)$ holds s_1 is convergent and $\lim s_1 = 0_X$.

Let us consider X. The functor $\exp X$ yields a function from the carrier of X into the carrier of X and is defined by:

(Def. 8) For every element z of the carrier of X holds $(\exp X)(z) = \sum(z\,\mathrm{ExpSeq})$.

Let us consider X, z. The functor $\exp z$ yielding an element of X is defined as follows:

(Def. 9) $\exp z = (\exp X)(z)$.

The following propositions are true:

(33) For every z holds $\exp z = \sum(z\,\mathrm{ExpSeq})$.

(34) Let given z_1, z_2. Suppose z_1, z_2 are commutative. Then $\exp(z_1 + z_2) = \exp z_1 \cdot \exp z_2$ and $\exp(z_2 + z_1) = \exp z_2 \cdot \exp z_1$ and $\exp(z_1 + z_2) = \exp(z_2 + z_1)$ and $\exp z_1$, $\exp z_2$ are commutative.

(35) For all z_1, z_2 such that z_1, z_2 are commutative holds $z_1 \cdot \exp z_2 = \exp z_2 \cdot z_1$.

(36) $\exp(0_X) = \mathbf{1}_X$.

(37) $\exp z \cdot \exp(-z) = \mathbf{1}_X$ and $\exp(-z) \cdot \exp z = \mathbf{1}_X$.

(38) $\exp z$ is invertible and $(\exp z)^{-1} = \exp(-z)$ and $\exp(-z)$ is invertible and $(\exp(-z))^{-1} = \exp z$.

(39) For every z and for all complex numbers s, t holds $s \cdot z$, $t \cdot z$ are commutative.

(40) Let given z and s, t be complex numbers. Then $\exp(s \cdot z) \cdot \exp(t \cdot z) = \exp((s+t) \cdot z)$ and $\exp(t \cdot z) \cdot \exp(s \cdot z) = \exp((t+s) \cdot z)$ and $\exp((s+t) \cdot z) = \exp((t+s) \cdot z)$ and $\exp(s \cdot z)$, $\exp(t \cdot z)$ are commutative.

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The Fundamental Group of Convex Subspaces of $\mathcal{E}^n_{\rm T}$

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Summary. The triviality of the fundamental group of subspaces of $\mathcal{E}^n_{\rm T}$ and $\mathbb{R}^{\mathbf{1}}$ have been shown.

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The notation and terminology used in this paper have been introduced in the following articles: [20], [6], [23], [1], [17], [24], [4], [5], [3], [2], [19], [11], [16], [22], [21], [18], [14], [8], [7], [15], [13], [9], [10], and [12].

1. Convex subspaces of $\mathcal{E}^n_{\rm T}$

In this paper n denotes a natural number and a, b denote real numbers.

Let us consider n. One can verify that there exists a subset of $\mathcal{E}^n_{\rm T}$ which is non empty and convex.

Let n be a natural number and let T be a subspace of $\mathcal{E}^n_{\rm T}$. We say that T is convex if and only if:

(Def. 1) Ω_T is a convex subset of $\mathcal{E}^n_{\rm T}$.

Let n be a natural number. Note that every non empty subspace of $\mathcal{E}^n_{\rm T}$ which is convex is also arcwise connected.

Let n be a natural number. One can verify that there exists a subspace of $\mathcal{E}^n_{\rm T}$ which is strict, non empty, and convex.

The following proposition is true

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(1) Let X be a non empty topological space, Y be a non empty subspace of X, x_1, x_2 be points of X, y_1, y_2 be points of Y, and f be a path from y_1 to y_2. Suppose $x_1 = y_1$ and $x_2 = y_2$ and y_1, y_2 are connected. Then f is a path from x_1 to x_2.

Let n be a natural number, let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, let a, b be points of T, and let P, Q be paths from a to b. The functor ConvexHomotopy(P, Q) yielding a map from $[:\mathbb{I}, \mathbb{I}:]$ into T is defined as follows:

(Def. 2) For all elements s, t of $\mathbb{I}$ and for all points a_1, b_1 of $\mathcal{E}^n_{\mathrm{T}}$ such that $a_1 = P(s)$ and $b_1 = Q(s)$ holds $(\mathrm{ConvexHomotopy}(P,Q))(s, t) = (1-t) \cdot a_1 + t \cdot b_1$.

Next we state the proposition

(2) Let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, a, b be points of T, and P, Q be paths from a to b. Then P, Q are homotopic.

Let n be a natural number, let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, let a, b be points of T, and let P, Q be paths from a to b. Then ConvexHomotopy(P, Q) is a homotopy between P and Q.

Let n be a natural number, let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, let a, b be points of T, and let P, Q be paths from a to b. Note that every homotopy between P and Q is continuous.

We now state the proposition

(3) Let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, a be a point of T, and C be a loop of a. Then the carrier of $\pi_1(T, a) = \{[C]_{\mathrm{EqRel}(T,a)}\}$.

Let n be a natural number, let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, and let a be a point of T. Observe that $\pi_1(T, a)$ is trivial.

2. Convex subspaces of $\mathbb{R}^1$

We now state the proposition

(4) $\mathrm{Proj}(|[a]|, 1) = a$.

One can verify that every subspace of $\mathbb{R}^1$ is real-membered.

Next we state three propositions:

(5) If $a \leqslant b$, then $[a, b] = \{(1-l) \cdot a + l \cdot b; l$ ranges over real numbers: $0 \leqslant l \wedge l \leqslant 1\}$.

(6) Let F be a map from $[:\mathbb{R}^1, \mathbb{I}:]$ into $\mathbb{R}^1$. Suppose that for every point x of $\mathbb{R}^1$ and for every point i of $\mathbb{I}$ holds $F(x, i) = (1-i) \cdot x$. Then F is continuous.

(7) Let F be a map from $[:\mathbb{R}^1, \mathbb{I}:]$ into $\mathbb{R}^1$. Suppose that for every point x of $\mathbb{R}^1$ and for every point i of $\mathbb{I}$ holds $F(x, i) = i \cdot x$. Then F is continuous.

Let P be a subset of $\mathbb{R}^1$. We say that P is convex if and only if:

(Def. 3) For all points a, b of $\mathbb{R}^1$ such that $a \in P$ and $b \in P$ holds $[a, b] \subseteq P$.

One can check that there exists a subset of $\mathbb{R}^{\mathbf{1}}$ which is non empty and convex and every subset of $\mathbb{R}^{\mathbf{1}}$ which is empty is also convex.

We now state four propositions:

(8) $[a, b]$ is a convex subset of $\mathbb{R}^{\mathbf{1}}$.

(9) $]a, b[$ is a convex subset of $\mathbb{R}^{\mathbf{1}}$.

(10) $[a, b[$ is a convex subset of $\mathbb{R}^{\mathbf{1}}$.

(11) $]a, b]$ is a convex subset of $\mathbb{R}^{\mathbf{1}}$.

Let T be a subspace of $\mathbb{R}^{\mathbf{1}}$. We say that T is convex if and only if:

(Def. 4) Ω_T is a convex subset of $\mathbb{R}^{\mathbf{1}}$.

Let us note that there exists a subspace of $\mathbb{R}^{\mathbf{1}}$ which is strict, non empty, and convex.

$\mathbb{R}^{\mathbf{1}}$ is a strict convex subspace of $\mathbb{R}^{\mathbf{1}}$.

The following proposition is true

(12) For every non empty convex subspace T of $\mathbb{R}^{\mathbf{1}}$ and for all points a, b of T holds $[a, b] \subseteq$ the carrier of T.

Let us note that every non empty subspace of $\mathbb{R}^{\mathbf{1}}$ which is convex is also arcwise connected.

One can prove the following propositions:

(13) If $a \leqslant b$, then $[a,\, b]_{\mathrm{T}}$ is convex.

(14) $\mathbb{I}$ is convex.

(15) If $a \leqslant b$, then $[a,\, b]_{\mathrm{T}}$ is arcwise connected.

Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$, let a, b be points of T, and let P, Q be paths from a to b. The functor R1Homotopy(P, Q) yields a map from $[\!:\mathbb{I},\, \mathbb{I}:\!]$ into T and is defined by:

(Def. 5) For all elements s, t of $\mathbb{I}$ holds $(\mathrm{R1Homotopy}(P, Q))(s,\, t) = (1 - t) \cdot P(s) + t \cdot Q(s)$.

Next we state the proposition

(16) Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$, a, b be points of T, and P, Q be paths from a to b. Then P, Q are homotopic.

Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$, let a, b be points of T, and let P, Q be paths from a to b. Then R1Homotopy(P, Q) is a homotopy between P and Q.

Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$, let a, b be points of T, and let P, Q be paths from a to b. Note that every homotopy between P and Q is continuous.

The following proposition is true

(17) Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$, a be a point of T, and C be a loop of a. Then the carrier of $\pi_1(T, a) = \{[C]_{\mathrm{EqRel}(T,a)}\}$.

Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$ and let a be a point of T. Observe that $\pi_1(T,a)$ is trivial.

One can prove the following four propositions:

(18) If $a \leqslant b$, then for all points x, y of $[a, b]_{\rm T}$ and for all paths P, Q from x to y holds P, Q are homotopic.

(19) If $a \leqslant b$, then for every point x of $[a, b]_{\rm T}$ and for every loop C of x holds the carrier of $\pi_1([a, b]_{\rm T}, x) = \{[C]_{{\rm EqRel}([a, b]_{\rm T}, x)}\}$.

(20) For all points x, y of $\mathbb{I}$ and for all paths P, Q from x to y holds P, Q are homotopic.

(21) For every point x of $\mathbb{I}$ and for every loop C of x holds the carrier of $\pi_1(\mathbb{I}, x) = \{[C]_{{\rm EqRel}(\mathbb{I}, x)}\}$.

Let x be a point of $\mathbb{I}$. Observe that $\pi_1(\mathbb{I}, x)$ is trivial.

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Intersections of Intervals and Balls in $\mathcal{E}^n_{\mathrm{T}}$

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The terminology and notation used in this paper are introduced in the following papers: [17], [19], [1], [4], [16], [8], [14], [2], [3], [5], [18], [13], [7], [9], [6], [15], [11], [12], and [10].

1. Preliminaries

For simplicity, we follow the rules: n denotes a natural number, a, b, r denote real numbers, x, y, z denote points of $\mathcal{E}^n_{\mathrm{T}}$, and e denotes a point of $\mathcal{E}^n$.

The following propositions are true:

(1) $x - y - z = x - z - y$.

(2) If $x + y = x + z$, then $y = z$.

(3) If n is non empty, then $x \neq x + 1.\mathrm{REAL}\, n$.

(4) For every set x such that $x = (1 - r) \cdot y + r \cdot z$ holds $x = y$ iff $r = 0$ or $y = z$ and $x = z$ iff $r = 1$ or $y = z$.

(5) For every finite sequence f of elements of $\mathbb{R}$ holds $|f|^{\mathbf{2}} = \sum^2 f$.

(6) For every non empty metric space M and for all points z_1, z_2, z_3 of M such that $z_1 \neq z_2$ and $z_1 \in \overline{\mathrm{Ball}}(z_3, r)$ and $z_2 \in \overline{\mathrm{Ball}}(z_3, r)$ holds $r > 0$.

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2. Subsets of $\mathcal{E}^n_{\mathrm{T}}$

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a real number. The functor $\mathrm{Ball}(x,r)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 1) $\mathrm{Ball}(x,r) = \{p; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|p-x| < r\}$.

The functor $\overline{\mathrm{Ball}}(x,r)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 2) $\overline{\mathrm{Ball}}(x,r) = \{p; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|p-x| \leqslant r\}$.

The functor $\mathrm{Sphere}(x,r)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 3) $\mathrm{Sphere}(x,r) = \{p; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|p-x| = r\}$.

We now state a number of propositions:

(7) $y \in \mathrm{Ball}(x,r)$ iff $|y-x| < r$.

(8) $y \in \overline{\mathrm{Ball}}(x,r)$ iff $|y-x| \leqslant r$.

(9) $y \in \mathrm{Sphere}(x,r)$ iff $|y-x| = r$.

(10) If $y \in \mathrm{Ball}(0_{\mathcal{E}^n_{\mathrm{T}}}, r)$, then $|y| < r$.

(11) If $y \in \overline{\mathrm{Ball}}(0_{\mathcal{E}^n_{\mathrm{T}}}, r)$, then $|y| \leqslant r$.

(12) If $y \in \mathrm{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), r)$, then $|y| = r$.

(13) If $x = e$, then $\mathrm{Ball}(e,r) = \mathrm{Ball}(x,r)$.

(14) If $x = e$, then $\overline{\mathrm{Ball}}(e,r) = \overline{\mathrm{Ball}}(x,r)$.

(15) If $x = e$, then $\mathrm{Sphere}(e,r) = \mathrm{Sphere}(x,r)$.

(16) $\mathrm{Ball}(x,r) \subseteq \overline{\mathrm{Ball}}(x,r)$.

(17) $\mathrm{Sphere}(x,r) \subseteq \overline{\mathrm{Ball}}(x,r)$.

(18) $\mathrm{Ball}(x,r) \cup \mathrm{Sphere}(x,r) = \overline{\mathrm{Ball}}(x,r)$.

(19) $\mathrm{Ball}(x,r)$ misses $\mathrm{Sphere}(x,r)$.

Let us consider n, x and let r be a non positive real number. One can check that $\mathrm{Ball}(x,r)$ is empty.

Let us consider n, x and let r be a positive real number. Note that $\mathrm{Ball}(x,r)$ is non empty.

One can prove the following propositions:

(20) If $\mathrm{Ball}(x,r)$ is non empty, then $r > 0$.

(21) If $\mathrm{Ball}(x,r)$ is empty, then $r \leqslant 0$.

Let us consider n, x and let r be a negative real number. Observe that $\overline{\mathrm{Ball}}(x,r)$ is empty.

Let us consider n, x and let r be a non negative real number. Observe that $\overline{\mathrm{Ball}}(x,r)$ is non empty.

The following three propositions are true:

(22) If $\overline{\mathrm{Ball}}(x,r)$ is non empty, then $r \geqslant 0$.

(23) If $\overline{\mathrm{Ball}}(x,r)$ is empty, then $r < 0$.

(24) If $a + b = 1$ and $|a| + |b| = 1$ and $b \neq 0$ and $x \in \overline{\mathrm{Ball}}(z, r)$ and $y \in \mathrm{Ball}(z, r)$, then $a \cdot x + b \cdot y \in \mathrm{Ball}(z, r)$.

Let us consider n, x, r. One can check the following observations:

* $\mathrm{Ball}(x, r)$ is open and Bounded,
* $\overline{\mathrm{Ball}}(x, r)$ is closed and Bounded, and
* $\mathrm{Sphere}(x, r)$ is closed and Bounded.

Let us consider n, x, r. Observe that $\mathrm{Ball}(x, r)$ is convex and $\overline{\mathrm{Ball}}(x, r)$ is convex.

Let n be a natural number and let f be a map from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. We say that f is homogeneous if and only if:

(Def. 4) For every real number r and for every point x of $\mathcal{E}^n_{\mathrm{T}}$ holds $f(r \cdot x) = r \cdot f(x)$.

We say that f is additive if and only if:

(Def. 5) For all points x, y of $\mathcal{E}^n_{\mathrm{T}}$ holds $f(x + y) = f(x) + f(y)$.

Let us consider n. One can verify that $(\mathcal{E}^n_{\mathrm{T}}) \longmapsto 0_{\mathcal{E}^n_{\mathrm{T}}}$ is homogeneous and additive.

Let us consider n. Observe that there exists a map from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$ which is homogeneous, additive, and continuous.

Let a, c be real numbers. One can check that $\mathrm{AffineMap}(a, 0, c, 0)$ is homogeneous and additive.

One can prove the following proposition

(25) For every homogeneous additive map f from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$ and for every convex subset X of $\mathcal{E}^n_{\mathrm{T}}$ holds $f^\circ X$ is convex.

In the sequel p, q are points of $\mathcal{E}^n_{\mathrm{T}}$.

Let n be a natural number and let p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{HL}(p, q)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 6) $\mathrm{HL}(p, q) = \{(1 - l) \cdot p + l \cdot q; l$ ranges over real numbers: $0 \leqslant l\}$.

One can prove the following proposition

(26) For every set x holds $x \in \mathrm{HL}(p, q)$ iff there exists a real number l such that $x = (1 - l) \cdot p + l \cdot q$ and $0 \leqslant l$.

Let us consider n, p, q. One can verify that $\mathrm{HL}(p, q)$ is non empty.

The following propositions are true:

(27) $p \in \mathrm{HL}(p, q)$.

(28) $q \in \mathrm{HL}(p, q)$.

(29) $\mathrm{HL}(p, p) = \{p\}$.

(30) If $x \in \mathrm{HL}(p, q)$, then $\mathrm{HL}(p, x) \subseteq \mathrm{HL}(p, q)$.

(31) If $x \in \mathrm{HL}(p, q)$ and $x \neq p$, then $\mathrm{HL}(p, q) = \mathrm{HL}(p, x)$.

(32) $\mathcal{L}(p, q) \subseteq \mathrm{HL}(p, q)$.

Let us consider n, p, q. Note that $\mathrm{HL}(p, q)$ is convex.

One can prove the following propositions:

(33) If $y \in \mathrm{Sphere}(x,r)$ and $z \in \mathrm{Ball}(x,r)$, then $\mathcal{L}(y,z) \cap \mathrm{Sphere}(x,r) = \{y\}$.

(34) If $y \in \mathrm{Sphere}(x,r)$ and $z \in \mathrm{Sphere}(x,r)$, then $\mathcal{L}(y,z) \setminus \{y,z\} \subseteq \mathrm{Ball}(x,r)$.

(35) If $y \in \mathrm{Sphere}(x,r)$ and $z \in \mathrm{Sphere}(x,r)$, then $\mathcal{L}(y,z) \cap \mathrm{Sphere}(x,r) = \{y,z\}$.

(36) If $y \in \mathrm{Sphere}(x,r)$ and $z \in \mathrm{Sphere}(x,r)$, then $\mathrm{HL}(y,z) \cap \mathrm{Sphere}(x,r) = \{y,z\}$.

(37) If $y \neq z$ and $y \in \mathrm{Ball}(x,r)$, then there exists a point e of $\mathcal{E}^n_{\mathrm{T}}$ such that $\{e\} = \mathrm{HL}(y,z) \cap \mathrm{Sphere}(x,r)$.

(38) If $y \neq z$ and $y \in \mathrm{Sphere}(x,r)$ and $z \in \overline{\mathrm{Ball}}(x,r)$, then there exists a point e of $\mathcal{E}^n_{\mathrm{T}}$ such that $e \neq y$ and $\{y,e\} = \mathrm{HL}(y,z) \cap \mathrm{Sphere}(x,r)$.

Let us consider n, x and let r be a negative real number. Observe that $\mathrm{Sphere}(x,r)$ is empty.

Let n be a non empty natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a non negative real number. Observe that $\mathrm{Sphere}(x,r)$ is non empty.

Next we state two propositions:

(39) If $\mathrm{Sphere}(x,r)$ is non empty, then $r \geqslant 0$.

(40) If n is non empty and $\mathrm{Sphere}(x,r)$ is empty, then $r < 0$.

3. Subsets of $\mathcal{E}^2_{\mathrm{T}}$

In the sequel s, t are points of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(41) $(a \cdot s + b \cdot t)_{\mathbf{1}} = a \cdot s_{\mathbf{1}} + b \cdot t_{\mathbf{1}}$.

(42) $(a \cdot s + b \cdot t)_{\mathbf{2}} = a \cdot s_{\mathbf{2}} + b \cdot t_{\mathbf{2}}$.

(43) $t \in \mathrm{Circle}(a,b,r)$ iff $|t - [a,b]| = r$.

(44) $t \in \mathrm{ClosedInsideOfCircle}(a,b,r)$ iff $|t - [a,b]| \leqslant r$.

(45) $t \in \mathrm{InsideOfCircle}(a,b,r)$ iff $|t - [a,b]| < r$.

Let a, b be real numbers and let r be a positive real number. Observe that $\mathrm{InsideOfCircle}(a,b,r)$ is non empty.

Let a, b be real numbers and let r be a non negative real number. Observe that $\mathrm{ClosedInsideOfCircle}(a,b,r)$ is non empty.

We now state a number of propositions:

(46) $\mathrm{Circle}(a,b,r) \subseteq \mathrm{ClosedInsideOfCircle}(a,b,r)$.

(47) For every point x of $\mathcal{E}^2$ such that $x = [a,b]$ holds $\overline{\mathrm{Ball}}(x,r) = \mathrm{ClosedInsideOfCircle}(a,b,r)$.

(48) For every point x of $\mathcal{E}^2$ such that $x = [a,b]$ holds $\mathrm{Ball}(x,r) = \mathrm{InsideOfCircle}(a,b,r)$.

(49) For every point x of $\mathcal{E}^2$ such that $x = [a,b]$ holds $\mathrm{Sphere}(x,r) = \mathrm{Circle}(a,b,r)$.

(50) $\mathrm{Ball}([a,b],r) = \mathrm{InsideOfCircle}(a,b,r)$.

(51) $\overline{\mathrm{Ball}}([a,b],r) = \mathrm{ClosedInsideOfCircle}(a,b,r)$.

(52) $\mathrm{Sphere}([a,b],r) = \mathrm{Circle}(a,b,r)$.

(53) $\mathrm{InsideOfCircle}(a,b,r) \subseteq \mathrm{ClosedInsideOfCircle}(a,b,r)$.

(54) $\mathrm{InsideOfCircle}(a,b,r)$ misses $\mathrm{Circle}(a,b,r)$.

(55) $\mathrm{InsideOfCircle}(a,b,r) \cup \mathrm{Circle}(a,b,r) = \mathrm{ClosedInsideOfCircle}(a,b,r)$.

(56) If $s \in \mathrm{Sphere}((0_{\mathcal{E}^2_{\mathrm{T}}}),r)$, then $(s_{\mathbf{1}})^{\mathbf{2}} + (s_{\mathbf{2}})^{\mathbf{2}} = r^{\mathbf{2}}$.

(57) If $s \neq t$ and $s \in \mathrm{ClosedInsideOfCircle}(a,b,r)$ and $t \in \mathrm{ClosedInsideOfCircle}(a,b,r)$, then $r > 0$.

(58) If $s \neq t$ and $s \in \mathrm{InsideOfCircle}(a,b,r)$, then there exists a point e of $\mathcal{E}^2_{\mathrm{T}}$ such that $\{e\} = \mathrm{HL}(s,t) \cap \mathrm{Circle}(a,b,r)$.

(59) If $s \in \mathrm{Circle}(a,b,r)$ and $t \in \mathrm{InsideOfCircle}(a,b,r)$, then $\mathcal{L}(s,t) \cap \mathrm{Circle}(a,b,r) = \{s\}$.

(60) If $s \in \mathrm{Circle}(a,b,r)$ and $t \in \mathrm{Circle}(a,b,r)$, then $\mathcal{L}(s,t) \setminus \{s,t\} \subseteq \mathrm{InsideOfCircle}(a,b,r)$.

(61) If $s \in \mathrm{Circle}(a,b,r)$ and $t \in \mathrm{Circle}(a,b,r)$, then $\mathcal{L}(s,t) \cap \mathrm{Circle}(a,b,r) = \{s,t\}$.

(62) If $s \in \mathrm{Circle}(a,b,r)$ and $t \in \mathrm{Circle}(a,b,r)$, then $\mathrm{HL}(s,t) \cap \mathrm{Circle}(a,b,r) = \{s,t\}$.

(63) If $s \neq t$ and $s \in \mathrm{Circle}(a,b,r)$ and $t \in \mathrm{ClosedInsideOfCircle}(a,b,r)$, then there exists a point e of $\mathcal{E}^2_{\mathrm{T}}$ such that $e \neq s$ and $\{s,e\} = \mathrm{HL}(s,t) \cap \mathrm{Circle}(a,b,r)$.

Let a, b, r be real numbers. Observe that $\mathrm{InsideOfCircle}(a,b,r)$ is convex and $\mathrm{ClosedInsideOfCircle}(a,b,r)$ is convex.

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Some Properties of Fibonacci Numbers

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Summary. We formalized some basic properties of the Fibonacci numbers using definitions and lemmas from [7] and [23], e.g. Cassini's and Catalan's identities. We also showed the connections between Fibonacci numbers and Pythagorean triples as defined in [31]. The main result of this article is a proof of Carmichael's Theorem on prime divisors of prime-generated Fibonacci numbers. According to it, if we look at the prime factors of a Fibonacci number generated by a prime number, none of them have appeared as a factor in any earlier Fibonacci number. We plan to develop the full proof of the Carmichael Theorem following [33].

MML Identifier: `FIB_NUM2`.

The papers [26], [3], [4], [30], [24], [1], [28], [29], [2], [18], [13], [27], [32], [9], [10], [7], [12], [8], [17], [21], [19], [22], [25], [6], [20], [11], [23], [15], [31], [14], [16], and [5] provide the terminology and notation for this paper.

1. Preliminaries

In this paper n, k, r, m, i, j denote natural numbers.

We now state a number of propositions:

(1) For every non empty natural number n holds $(n -' 1) + 2 = n + 1$.

(2) For every odd integer n and for every non empty real number m holds $(-m)^n = -m^n$.

(3) For every odd integer n holds $(-1)^n = -1$.

(4) For every even integer n and for every non empty real number m holds $(-m)^n = m^n$.

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(5) For every even integer n holds $(-1)^n = 1$.

(6) For every non empty real number m and for every integer n holds $((-1) \cdot m)^n = (-1)^n \cdot m^n$.

(7) For every non empty real number a holds $a^{k+m} = a^k \cdot a^m$.

(8) For every non empty real number k and for every odd integer m holds $(k^m)^n = k^{m \cdot n}$.

(9) $((-1)^{-n})^{\mathbf{2}} = 1$.

(10) For every non empty real number a holds $a^{-k} \cdot a^{-m} = a^{-k-m}$.

(11) $(-1)^{-2 \cdot n} = 1$.

(12) For every non empty real number a holds $a^k \cdot a^{-k} = 1$.

Let n be an odd integer. One can verify that $-n$ is odd.

Let n be an even integer. Note that $-n$ is even.

One can prove the following two propositions:

(13) $(-1)^{-n} = (-1)^n$.

(14) For all natural numbers k, m, m_1, n_1 such that $k \mid m$ and $k \mid n$ holds $k \mid m \cdot m_1 + n \cdot n_1$.

One can check that there exists a set which is finite, non empty, and natural-membered and has non empty elements.

Let f be a function from $\mathbb{N}$ into $\mathbb{N}$ and let A be a finite natural-membered set with non empty elements. Note that $f \restriction A$ is finite subsequence-like.

One can prove the following proposition

(15) For every finite subsequence p holds $\operatorname{rng} \operatorname{Seq} p \subseteq \operatorname{rng} p$.

Let f be a function from $\mathbb{N}$ into $\mathbb{N}$ and let A be a finite natural-membered set with non empty elements. The functor $\operatorname{Prefix}(f, A)$ yields a finite sequence of elements of $\mathbb{N}$ and is defined as follows:

(Def. 1) $\operatorname{Prefix}(f, A) = \operatorname{Seq}(f \restriction A)$.

The following proposition is true

(16) For every natural number k such that $k \neq 0$ holds if $k + m \leqslant n$, then $m < n$.

Let us mention that $\mathbb{N}$ is lower bounded.

Let us mention that $\{1, 2, 3\}$ is natural-membered and has non empty elements.

Let us note that $\{1, 2, 3, 4\}$ is natural-membered and has non empty elements.

The following propositions are true:

(17) For all sets x, y such that $0 < i$ and $i < j$ holds $\{\langle i, x \rangle, \langle j, y \rangle\}$ is a finite subsequence.

(18) For all sets x, y and for every finite subsequence q such that $i < j$ and $q = \{\langle i, x \rangle, \langle j, y \rangle\}$ holds $\operatorname{Seq} q = \langle x, y \rangle$.

Let n be a natural number. Observe that $\mathrm{Seg}\, n$ has non empty elements.

Let A be a set with non empty elements. Note that every subset of A has non empty elements.

Let A be a set with non empty elements and let B be a set. Observe that $A \cap B$ has non empty elements and $B \cap A$ has non empty elements.

We now state four propositions:

(19) For every natural number k and for every set a such that $k \geqslant 1$ holds $\{\langle k, a\rangle\}$ is a finite subsequence.

(20) Let i, k be natural numbers, y be a set, and f be a finite subsequence. If $f = \{\langle 1, y\rangle\}$, then $\mathrm{Shift}^i\, f = \{\langle 1 + i, y\rangle\}$.

(21) Let q be a finite subsequence and k, n be natural numbers. Suppose $\mathrm{dom}\, q \subseteq \mathrm{Seg}\, k$ and $n > k$. Then there exists a finite sequence p such that $q \subseteq p$ and $\mathrm{dom}\, p = \mathrm{Seg}\, n$.

(22) For every finite subsequence q there exists a finite sequence p such that $q \subseteq p$.

2. Fibonacci Numbers

In this article we present several logical schemes. The scheme *Fib Ind 1* concerns a unary predicate $\mathcal{P}$, and states that:

For every non empty natural number k holds $\mathcal{P}[k]$

provided the parameters have the following properties:

- $\mathcal{P}[1]$,
- $\mathcal{P}[2]$, and
- For every non empty natural number k such that $\mathcal{P}[k]$ and $\mathcal{P}[k+1]$ holds $\mathcal{P}[k + 2]$.

The scheme *Fib Ind 2* concerns a unary predicate $\mathcal{P}$, and states that:

For every non trivial natural number k holds $\mathcal{P}[k]$

provided the parameters meet the following conditions:

- $\mathcal{P}[2]$,
- $\mathcal{P}[3]$, and
- For every non trivial natural number k such that $\mathcal{P}[k]$ and $\mathcal{P}[k+1]$ holds $\mathcal{P}[k + 2]$.

Next we state a number of propositions:

(23) $\mathrm{Fib}(2) = 1$.

(24) $\mathrm{Fib}(3) = 2$.

(25) $\mathrm{Fib}(4) = 3$.

(26) $\mathrm{Fib}(n + 2) = \mathrm{Fib}(n) + \mathrm{Fib}(n + 1)$.

(27) $\mathrm{Fib}(n + 3) = \mathrm{Fib}(n + 2) + \mathrm{Fib}(n + 1)$.

(28) $\mathrm{Fib}(n + 4) = \mathrm{Fib}(n + 2) + \mathrm{Fib}(n + 3)$.

(29) $\mathrm{Fib}(n+5) = \mathrm{Fib}(n+3) + \mathrm{Fib}(n+4)$.

(30) $\mathrm{Fib}(n+2) = \mathrm{Fib}(n+3) - \mathrm{Fib}(n+1)$.

(31) $\mathrm{Fib}(n+1) = \mathrm{Fib}(n+2) - \mathrm{Fib}(n)$.

(32) $\mathrm{Fib}(n) = \mathrm{Fib}(n+2) - \mathrm{Fib}(n+1)$.

3. Cassini's and Catalan's Identities

The following propositions are true:

(33) $\mathrm{Fib}(n) \cdot \mathrm{Fib}(n+2) - \mathrm{Fib}(n+1)^{\mathbf{2}} = (-1)^{n+1}$.

(34) For every non empty natural number n holds $\mathrm{Fib}(n -' 1) \cdot \mathrm{Fib}(n+1) - \mathrm{Fib}(n)^{\mathbf{2}} = (-1)^n$.

(35) $\tau > 0$.

(36) $\overline{\tau} = (-\tau)^{-1}$.

(37) $(-\tau)^{(-1) \cdot n} = ((-\tau)^{-1})^n$.

(38) $-\frac{1}{\tau} = \overline{\tau}$.

(39) $((\tau^r)^{\mathbf{2}} - 2 \cdot (-1)^r) + (\tau^{-r})^{\mathbf{2}} = (\tau^r - \overline{\tau}^r)^{\mathbf{2}}$.

(40) For all non empty natural numbers n, r such that $r \leqslant n$ holds $\mathrm{Fib}(n)^{\mathbf{2}} - \mathrm{Fib}(n+r) \cdot \mathrm{Fib}(n -' r) = (-1)^{n -' r} \cdot \mathrm{Fib}(r)^{\mathbf{2}}$.

(41) $\mathrm{Fib}(n)^{\mathbf{2}} + \mathrm{Fib}(n+1)^{\mathbf{2}} = \mathrm{Fib}(2 \cdot n + 1)$.

(42) For every non empty natural number k holds $\mathrm{Fib}(n+k) = \mathrm{Fib}(k) \cdot \mathrm{Fib}(n+1) + \mathrm{Fib}(k -' 1) \cdot \mathrm{Fib}(n)$.

(43) For every non empty natural number n holds $\mathrm{Fib}(n) \mid \mathrm{Fib}(n \cdot k)$.

(44) For every non empty natural number k such that $k \mid n$ holds $\mathrm{Fib}(k) \mid \mathrm{Fib}(n)$.

(45) $\mathrm{Fib}(n) \leqslant \mathrm{Fib}(n+1)$.

(46) For every natural number n such that $n > 1$ holds $\mathrm{Fib}(n) < \mathrm{Fib}(n+1)$.

(47) For all natural numbers m, n such that $m \geqslant n$ holds $\mathrm{Fib}(m) \geqslant \mathrm{Fib}(n)$.

(48) For every natural number k such that $k > 1$ holds if $k < n$, then $\mathrm{Fib}(k) < \mathrm{Fib}(n)$.

(49) $\mathrm{Fib}(k) = 1$ iff $k = 1$ or $k = 2$.

(50) Let k, n be natural numbers. Suppose $n > 1$ and $k \neq 0$ and $k \neq 1$ and $k \neq 1$ and $n \neq 2$ or $k \neq 2$ and $n \neq 1$. Then $\mathrm{Fib}(k) = \mathrm{Fib}(n)$ if and only if $k = n$.

(51) Let n be a natural number. Suppose $n > 1$ and $n \neq 4$. Suppose n is non prime. Then there exists a non empty natural number k such that $k \neq 1$ and $k \neq 2$ and $k \neq n$ and $k \mid n$.

(52) For every natural number n such that $n > 1$ and $n \neq 4$ holds if $\mathrm{Fib}(n)$ is prime, then n is prime.

4. Sequence of Fibonacci Numbers

The function FIB from $\mathbb{N}$ into $\mathbb{N}$ is defined as follows:

(Def. 2) For every natural number k holds $\text{FIB}(k) = \text{Fib}(k)$.

The subset $\mathbb{N}_{\text{even}}$ of $\mathbb{N}$ is defined by:

(Def. 3) $\mathbb{N}_{\text{even}} = \{2 \cdot k : k$ ranges over natural numbers$\}$.

The subset $\mathbb{N}_{\text{odd}}$ of $\mathbb{N}$ is defined as follows:

(Def. 4) $\mathbb{N}_{\text{odd}} = \{2 \cdot k + 1 : k$ ranges over natural numbers$\}$.

One can prove the following two propositions:

(53) For every natural number k holds $2 \cdot k \in \mathbb{N}_{\text{even}}$ and $2 \cdot k + 1 \notin \mathbb{N}_{\text{even}}$.

(54) For every natural number k holds $2 \cdot k + 1 \in \mathbb{N}_{\text{odd}}$ and $2 \cdot k \notin \mathbb{N}_{\text{odd}}$.

Let n be a natural number. The functor $\text{EvenFibs}(n)$ yielding a finite sequence of elements of $\mathbb{N}$ is defined by:

(Def. 5) $\text{EvenFibs}(n) = \text{Prefix}(\text{FIB}, \mathbb{N}_{\text{even}} \cap \text{Seg}\, n)$.

The functor $\text{OddFibs}(n)$ yields a finite sequence of elements of $\mathbb{N}$ and is defined by:

(Def. 6) $\text{OddFibs}(n) = \text{Prefix}(\text{FIB}, \mathbb{N}_{\text{odd}} \cap \text{Seg}\, n)$.

We now state a number of propositions:

(55) $\text{EvenFibs}(0) = \emptyset$.

(56) $\text{Seq}(\text{FIB} \restriction \{2\}) = \langle 1 \rangle$.

(57) $\text{EvenFibs}(2) = \langle 1 \rangle$.

(58) $\text{EvenFibs}(4) = \langle 1, 3 \rangle$.

(59) For every natural number k holds $\mathbb{N}_{\text{even}} \cap \text{Seg}(2 \cdot k + 2) \cup \{2 \cdot k + 4\} = \mathbb{N}_{\text{even}} \cap \text{Seg}(2 \cdot k + 4)$.

(60) For every natural number k holds $\text{FIB} \restriction (\mathbb{N}_{\text{even}} \cap \text{Seg}(2 \cdot k + 2)) \cup \{\langle\!\langle 2 \cdot k + 4, \text{FIB}(2 \cdot k + 4) \rangle\!\rangle\} = \text{FIB} \restriction (\mathbb{N}_{\text{even}} \cap \text{Seg}(2 \cdot k + 4))$.

(61) For every natural number n holds $\text{EvenFibs}(2 \cdot n + 2) = \text{EvenFibs}(2 \cdot n) \frown \langle \text{Fib}(2 \cdot n + 2) \rangle$.

(62) $\text{OddFibs}(1) = \langle 1 \rangle$.

(63) $\text{OddFibs}(3) = \langle 1, 2 \rangle$.

(64) For every natural number k holds $\mathbb{N}_{\text{odd}} \cap \text{Seg}(2 \cdot k + 3) \cup \{2 \cdot k + 5\} = \mathbb{N}_{\text{odd}} \cap \text{Seg}(2 \cdot k + 5)$.

(65) For every natural number k holds $\text{FIB} \restriction (\mathbb{N}_{\text{odd}} \cap \text{Seg}(2 \cdot k + 3)) \cup \{\langle\!\langle 2 \cdot k + 5, \text{FIB}(2 \cdot k + 5) \rangle\!\rangle\} = \text{FIB} \restriction (\mathbb{N}_{\text{odd}} \cap \text{Seg}(2 \cdot k + 5))$.

(66) For every natural number n holds $\text{OddFibs}(2 \cdot n + 3) = \text{OddFibs}(2 \cdot n + 1) \frown \langle \text{Fib}(2 \cdot n + 3) \rangle$.

(67) For every natural number n holds $\sum \text{EvenFibs}(2 \cdot n + 2) = \text{Fib}(2 \cdot n + 3) - 1$.

(68) For every natural number n holds $\sum \text{OddFibs}(2 \cdot n + 1) = \text{Fib}(2 \cdot n + 2)$.

5. Carmichael's Theorem on Prime Divisors

One can prove the following three propositions:

(69) For every natural number n holds $\mathrm{Fib}(n)$ and $\mathrm{Fib}(n+1)$ are relative prime.

(70) For every non empty natural number n and for every natural number m such that $m \neq 1$ holds if $m \mid \mathrm{Fib}(n)$, then $m \nmid \mathrm{Fib}(n -' 1)$.

(71) Let n be a non empty natural number. Suppose m is prime and n is prime and $m \mid \mathrm{Fib}(n)$. Let r be a natural number. If $r < n$ and $r \neq 0$, then $m \nmid \mathrm{Fib}(r)$.

6. Fibonacci Numbers and Pythagorean Triples

We now state the proposition

(72) For every non empty natural number n holds $\{\mathrm{Fib}(n) \cdot \mathrm{Fib}(n+3), 2 \cdot \mathrm{Fib}(n+1) \cdot \mathrm{Fib}(n+2), \mathrm{Fib}(n+1)^{\mathbf{2}} + \mathrm{Fib}(n+2)^{\mathbf{2}}\}$ is a Pythagorean triple.

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The Hall Marriage Theorem

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Summary. The Marriage Theorem, as credited to Philip Hall [7], gives the necessary and sufficient condition allowing us to select a distinct element from each of a finite collection $\{A_i\}$ of n finite subsets. This selection, called a set of different representatives (SDR), exists if and only if the marriage condition (or Hall condition) is satisfied:

$$\forall_{J\subseteq\{1,\ldots,n\}} |\bigcup_{i\in J} A_i| \geqslant |J|.$$

The proof which is given in this article (according to Richard Rado, 1967) is based on the lemma that for finite sequences with non-trivial elements which satisfy Hall property there exists a reduction (see Def. 5) such that Hall property again holds (see Th. 29 for details).

MML Identifier: `HALLMAR1`.

The notation and terminology used here are introduced in the following papers: [9], [5], [10], [11], [4], [8], [2], [6], [1], and [3].

1. Preliminaries

One can prove the following proposition

(1) For all finite sets X, Y holds $\operatorname{card}(X\cup Y)+\operatorname{card}(X\cap Y) = \operatorname{card} X+\operatorname{card} Y$.

In this article we present several logical schemes. The scheme *Regr11* deals with a natural number $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every natural number k such that $1 \leqslant k$ and $k \leqslant \mathcal{A}$ holds $\mathcal{P}[k]$

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provided the parameters meet the following conditions:

- $\mathcal{P}[\mathcal{A}]$ and $\mathcal{A} \geqslant 2$, and
- For every natural number k such that $1 \leqslant k$ and $k < \mathcal{A}$ and $\mathcal{P}[k+1]$ holds $\mathcal{P}[k]$.

The scheme *Regr2* concerns a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[1]$

provided the parameters meet the following requirements:

- There exists a natural number n such that $n > 1$ and $\mathcal{P}[n]$, and
- For every natural number k such that $k \geqslant 1$ and $\mathcal{P}[k+1]$ holds $\mathcal{P}[k]$.

Let F be a non empty set. One can check that there exists a finite sequence of elements of 2^F which is non empty and non-empty.

We now state the proposition

(2) Let F be a non empty set, f be a non-empty finite sequence of elements of 2^F, and i be a natural number. If $i \in \operatorname{dom} f$, then $f(i) \neq \emptyset$.

Let F be a finite set, let A be a finite sequence of elements of 2^F, and let i be a natural number. Note that $A(i)$ is finite.

2. Union of Finite Sequences

Let F be a set, let A be a finite sequence of elements of 2^F, and let J be a set. The functor $\bigcup_J A$ yields a set and is defined as follows:

(Def. 1) For every set x holds $x \in \bigcup_J A$ iff there exists a set j such that $j \in J$ and $j \in \operatorname{dom} A$ and $x \in A(j)$.

Next we state two propositions:

(3) For every set F and for every finite sequence A of elements of 2^F and for every set J holds $\bigcup_J A \subseteq F$.

(4) Let F be a finite set, A be a finite sequence of elements of 2^F, and J, K be sets. If $J \subseteq K$, then $\bigcup_J A \subseteq \bigcup_K A$.

Let F be a finite set, let A be a finite sequence of elements of 2^F, and let J be a set. One can verify that $\bigcup_J A$ is finite.

The following propositions are true:

(5) Let F be a finite set, A be a finite sequence of elements of 2^F, and i be a natural number. If $i \in \operatorname{dom} A$, then $\bigcup_{\{i\}} A = A(i)$.

(6) Let F be a finite set, A be a finite sequence of elements of 2^F, and i, j be natural numbers. If $i \in \operatorname{dom} A$ and $j \in \operatorname{dom} A$, then $\bigcup_{\{i,j\}} A = A(i) \cup A(j)$.

(7) Let J be a set, F be a finite set, A be a finite sequence of elements of 2^F, and i be a natural number. If $i \in J$ and $i \in \operatorname{dom} A$, then $A(i) \subseteq \bigcup_J A$.

(8) Let J be a set, F be a finite set, i be a natural number, and A be a finite sequence of elements of 2^F. If $i \in J$ and $i \in \operatorname{dom} A$, then $\bigcup_J A = \bigcup_{J \setminus \{i\}} A \cup A(i)$.

(9) Let J_1, J_2 be sets, F be a finite set, i be a natural number, and A be a finite sequence of elements of 2^F. If $i \in \operatorname{dom} A$, then $\bigcup_{\{i\} \cup J_1 \cup J_2} A = A(i) \cup \bigcup_{J_1 \cup J_2} A$.

(10) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x, y be sets. If $x \neq y$ and $x \in A(i)$ and $y \in A(i)$, then $(A(i) \setminus \{x\}) \cup (A(i) \setminus \{y\}) = A(i)$.

3. Cut Operation for Finite Sequences

Let F be a finite set, let A be a finite sequence of elements of 2^F, let i be a natural number, and let x be a set. The functor $\operatorname{Cut}(A, i, x)$ yielding a finite sequence of elements of 2^F is defined by the conditions (Def. 2).

(Def. 2)(i) $\operatorname{dom} \operatorname{Cut}(A, i, x) = \operatorname{dom} A$, and

(ii) for every natural number k such that $k \in \operatorname{dom} \operatorname{Cut}(A, i, x)$ holds if $i = k$, then $(\operatorname{Cut}(A, i, x))(k) = A(k) \setminus \{x\}$ and if $i \neq k$, then $(\operatorname{Cut}(A, i, x))(k) = A(k)$.

The following propositions are true:

(11) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x be a set. If $i \in \operatorname{dom} A$ and $x \in A(i)$, then $\operatorname{card}(\operatorname{Cut}(A, i, x))(i) = \operatorname{card} A(i) - 1$.

(12) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x, J be sets. Then $\bigcup_{J \setminus \{i\}} \operatorname{Cut}(A, i, x) = \bigcup_{J \setminus \{i\}} A$.

(13) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x, J be sets. If $i \notin J$, then $\bigcup_J A = \bigcup_J \operatorname{Cut}(A, i, x)$.

(14) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x, J be sets. If $i \in \operatorname{dom} \operatorname{Cut}(A, i, x)$ and $J \subseteq \operatorname{dom} \operatorname{Cut}(A, i, x)$ and $i \in J$, then $\bigcup_J \operatorname{Cut}(A, i, x) = \bigcup_{J \setminus \{i\}} A \cup (A(i) \setminus \{x\})$.

4. System of Different Representatives and Hall Property

Let F be a finite set, let X be a finite sequence of elements of 2^F, and let A be a set. We say that A is a system of different representatives of X if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exists a finite sequence f of elements of F such that $f = A$ and $\operatorname{dom} X = \operatorname{dom} f$ and for every natural number i such that $i \in \operatorname{dom} f$ holds $f(i) \in X(i)$ and f is one-to-one.

Let F be a finite set and let A be a finite sequence of elements of 2^F. We say that A satisfies Hall condition if and only if:

(Def. 4) For every finite set J such that $J \subseteq \operatorname{dom} A$ holds $\operatorname{card} J \leqslant \operatorname{card} \bigcup_J A$.

Next we state four propositions:

(15) Let F be a finite set and A be a non empty finite sequence of elements of 2^F. If A satisfies Hall condition, then A is non-empty.

(16) Let F be a finite set, A be a finite sequence of elements of 2^F, and i be a natural number. If $i \in \operatorname{dom} A$ and A satisfies Hall condition, then $\operatorname{card} A(i) \geqslant 1$.

(17) Let F be a non empty finite set and A be a non empty finite sequence of elements of 2^F. Suppose for every natural number i such that $i \in \operatorname{dom} A$ holds $\operatorname{card} A(i) = 1$ and A satisfies Hall condition. Then there exists a set which is a system of different representatives of A.

(18) Let F be a finite set and A be a finite sequence of elements of 2^F such that there exists a set which is a system of different representatives of A. Then A satisfies Hall condition.

5. Reductions and Singlifications of Finite Sequences

Let F be a set, let A be a finite sequence of elements of 2^F, and let i be a natural number. A finite sequence of elements of 2^F is said to be a reduction of A at i-th position if:

(Def. 5) $\operatorname{dom} \mathrm{it} = \operatorname{dom} A$ and for every natural number j such that $j \in \operatorname{dom} A$ and $j \neq i$ holds $A(j) = \mathrm{it}(j)$ and $\mathrm{it}(i) \subseteq A(i)$.

Let F be a set and let A be a finite sequence of elements of 2^F. A finite sequence of elements of 2^F is said to be a reduction of A if:

(Def. 6) $\operatorname{dom} \mathrm{it} = \operatorname{dom} A$ and for every natural number i such that $i \in \operatorname{dom} A$ holds $\mathrm{it}(i) \subseteq A(i)$.

Let F be a set, let A be a finite sequence of elements of 2^F, and let i be a natural number. Let us assume that $i \in \operatorname{dom} A$ and $A(i) \neq \emptyset$. A reduction of A is called a singlification of A at i-th position if:

(Def. 7) $\overline{\overline{\mathrm{it}(i)}} = 1$.

One can prove the following propositions:

(19) Let F be a finite set, A be a finite sequence of elements of 2^F, and i be a natural number. Then every reduction of A at i-th position is a reduction of A.

(20) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x be a set. If $i \in \operatorname{dom} A$ and $x \in A(i)$, then $\mathrm{Cut}(A, i, x)$ is a reduction of A at i-th position.

(21) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x be a set. If $i \in \operatorname{dom} A$ and $x \in A(i)$, then $\operatorname{Cut}(A, i, x)$ is a reduction of A.

(22) Let F be a finite set, A be a finite sequence of elements of 2^F, and B be a reduction of A. Then every reduction of B is a reduction of A.

(23) Let F be a non empty finite set, A be a non-empty finite sequence of elements of 2^F, i be a natural number, and B be a singlification of A at i-th position. If $i \in \operatorname{dom} A$, then $B(i) \neq \emptyset$.

(24) Let F be a non empty finite set, A be a non-empty finite sequence of elements of 2^F, i, j be natural numbers, B be a singlification of A at i-th position, and C be a singlification of B at j-th position. Suppose $i \in \operatorname{dom} A$ and $j \in \operatorname{dom} A$ and $C(i) \neq \emptyset$ and $B(j) \neq \emptyset$. Then C is a singlification of A at j-th position and a singlification of A at i-th position.

(25) Let F be a set, A be a finite sequence of elements of 2^F, and i be a natural number. Then A is a reduction of A at i-th position.

(26) For every set F holds every finite sequence A of elements of 2^F is a reduction of A.

Let F be a non empty set and let A be a finite sequence of elements of 2^F. Let us assume that A is non-empty. A reduction of A is called a singlification of A if:

(Def. 8) For every natural number i such that $i \in \operatorname{dom} A$ holds $\overline{\overline{\operatorname{it}(i)}} = 1$.

We now state the proposition

(27) Let F be a non empty finite set, A be a non empty non-empty finite sequence of elements of 2^F, and f be a function. Then f is a singlification of A if and only if the following conditions are satisfied:

(i) $\operatorname{dom} f = \operatorname{dom} A$, and

(ii) for every natural number i such that $i \in \operatorname{dom} A$ holds f is a singlification of A at i-th position.

Let F be a non empty finite set, let A be a non empty finite sequence of elements of 2^F, and let k be a natural number. Note that every singlification of A at k-th position is non empty.

Let F be a non empty finite set and let A be a non empty finite sequence of elements of 2^F. One can check that every singlification of A is non empty.

6. Rado's Proof of the Hall Marriage Theorem

One can prove the following propositions:

(28) Let F be a non empty finite set, A be a non empty finite sequence of elements of 2^F, X be a set, and B be a reduction of A. Suppose X is a

system of different representatives of B. Then X is a system of different representatives of A.

(29) Let F be a finite set and A be a finite sequence of elements of 2^F. Suppose A satisfies Hall condition. Let i be a natural number. If $\mathrm{card}\, A(i) \geqslant 2$, then there exists a set x such that $x \in A(i)$ and $\mathrm{Cut}(A, i, x)$ satisfies Hall condition.

(30) Let F be a finite set, A be a finite sequence of elements of 2^F, and i be a natural number. If $i \in \mathrm{dom}\, A$ and A satisfies Hall condition, then there exists a singlification of A at i-th position which satisfies Hall condition.

(31) Let F be a non empty finite set and A be a non empty finite sequence of elements of 2^F. If A satisfies Hall condition, then there exists a singlification of A which satisfies Hall condition.

(32) Let F be a non empty finite set and A be a non empty finite sequence of elements of 2^F. Then there exists a set which is a system of different representatives of A if and only if A satisfies Hall condition.

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The Differentiable Functions on Normed Linear Spaces

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Summary. In this article, the basic properties of the differentiable functions on normed linear spaces are described.

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The notation and terminology used in this paper are introduced in the following papers: [20], [23], [4], [24], [6], [5], [19], [3], [10], [1], [18], [7], [21], [22], [11], [8], [9], [25], [13], [15], [16], [17], [12], [14], and [2].

For simplicity, we adopt the following rules: n, k denote natural numbers, x, X, Z denote sets, g, r denote real numbers, S denotes a real normed space, r_1 denotes a sequence of real numbers, s_1, s_2 denote sequences of S, x_0 denotes a point of S, and Y denotes a subset of S.

Next we state several propositions:

(1) For every point x_0 of S and for all neighbourhoods N_1, N_2 of x_0 there exists a neighbourhood N of x_0 such that $N \subseteq N_1$ and $N \subseteq N_2$.

(2) Let X be a subset of S. Suppose X is open. Let r be a point of S. If $r \in X$, then there exists a neighbourhood N of r such that $N \subseteq X$.

(3) Let X be a subset of S. Suppose X is open. Let r be a point of S. If $r \in X$, then there exists g such that $0 < g$ and $\{y; y$ ranges over points of S: $\|y - r\| < g\} \subseteq X$.

(4) Let X be a subset of S. Suppose that for every point r of S such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$. Then X is open.

(5) Let X be a subset of S. Then for every point r of S such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$ if and only if X is open.

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Let S be a zero structure and let f be a sequence of S. We say that f is non-zero if and only if:

(Def. 1) $\operatorname{rng} f \subseteq$ (the carrier of S) $\setminus \{0_S\}$.

We introduce f is non-zero as a synonym of f is non-zero.

We now state two propositions:

(6) s_1 is non-zero iff for every x such that $x \in \mathbb{N}$ holds $s_1(x) \neq 0_S$.

(7) s_1 is non-zero iff for every n holds $s_1(n) \neq 0_S$.

Let R_1 be a real linear space, let S be a sequence of R_1, and let a be a sequence of real numbers. The functor $a\,S$ yields a sequence of R_1 and is defined as follows:

(Def. 2) For every n holds $(a\,S)(n) = a(n) \cdot S(n)$.

Let R_1 be a real linear space, let z be a point of R_1, and let a be a sequence of real numbers. The functor $a \cdot z$ yields a sequence of R_1 and is defined by:

(Def. 3) For every n holds $(a \cdot z)(n) = a(n) \cdot z$.

Next we state a number of propositions:

(8) For all sequences r_2, r_3 of real numbers holds $(r_2 + r_3)\,s_1 = r_2\,s_1 + r_3\,s_1$.

(9) For every sequence r_1 of real numbers and for all sequences s_2, s_3 of S holds $r_1\,(s_2 + s_3) = r_1\,s_2 + r_1\,s_3$.

(10) For every sequence r_1 of real numbers holds $r \cdot (r_1\,s_1) = r_1\,(r \cdot s_1)$.

(11) For all sequences r_2, r_3 of real numbers holds $(r_2 - r_3)\,s_1 = r_2\,s_1 - r_3\,s_1$.

(12) For every sequence r_1 of real numbers and for all sequences s_2, s_3 of S holds $r_1\,(s_2 - s_3) = r_1\,s_2 - r_1\,s_3$.

(13) If r_1 is convergent and s_1 is convergent, then $r_1\,s_1$ is convergent.

(14) If r_1 is convergent and s_1 is convergent, then $\lim(r_1\,s_1) = \lim r_1 \cdot \lim s_1$.

(15) $(s_1 + s_2) \uparrow k = s_1 \uparrow k + s_2 \uparrow k$.

(16) $(s_1 - s_2) \uparrow k = s_1 \uparrow k - s_2 \uparrow k$.

(17) If s_1 is non-zero, then $s_1 \uparrow k$ is non-zero.

(18) $s_1 \uparrow k$ is a subsequence of s_1.

(19) If s_1 is constant and s_2 is a subsequence of s_1, then s_2 is constant.

(20) If s_1 is constant and s_2 is a subsequence of s_1, then $s_1 = s_2$.

Let us consider S and let I_1 be a sequence of S. We say that I_1 is convergent to 0 if and only if:

(Def. 4) I_1 is non-zero and convergent and $\lim I_1 = 0_S$.

The following propositions are true:

(21) Let X be a real normed space and s_1 be a sequence of X. Suppose s_1 is constant. Then s_1 is convergent and for every natural number k holds $\lim s_1 = s_1(k)$.

(22) For every real number r such that $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r} \cdot x_0$ holds s_1 is convergent.

(23) For every real number r such that $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r} \cdot x_0$ holds $\lim s_1 = 0_S$.

(24) Let a be a convergent to 0 sequence of real numbers and z be a point of S. If $z \neq 0_S$, then $a \cdot z$ is convergent to 0.

(25) For every point r of S holds $r \in Y$ iff $r \in$ the carrier of S iff $Y =$ the carrier of S.

For simplicity, we adopt the following rules: S, T denote non trivial real normed spaces, f, f_1, f_2 denote partial functions from S to T, s_4, s_1 denote sequences of S, and x_0 denotes a point of S.

Let S be a non trivial real normed space. Note that there exists a sequence of S which is convergent to 0.

Let us consider S. Note that there exists a sequence of S which is constant.

In the sequel h is a convergent to 0 sequence of S and c is a constant sequence of S.

Let us consider S, T and let I_1 be a partial function from S to T. We say that I_1 is rest-like if and only if:

(Def. 5) I_1 is total and for every h holds $\|h\|^{-1} (I_1 \cdot h)$ is convergent and $\lim(\|h\|^{-1} (I_1 \cdot h)) = 0_T$.

Let us consider S, T. Observe that there exists a partial function from S to T which is rest-like.

Let us consider S, T. A rest of S, T is a rest-like partial function from S to T.

We now state two propositions:

(26) Let R be a partial function from S to T. Suppose R is total. Then R is rest-like if and only if for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every point z of S such that $z \neq 0_S$ and $\|z\| < d$ holds $\|z\|^{-1} \cdot \|R_z\| < r$.

(27) For every rest R of S, T and for every convergent to 0 sequence s of S holds $R \cdot s$ is convergent and $\lim(R \cdot s) = 0_T$.

In the sequel R, R_2, R_3 are rests of S, T and L is a point of RNormSpaceOfBoundedLinearOperators(S, T).

Next we state several propositions:

(28) $\mathrm{rng}(s_1 \uparrow n) \subseteq \mathrm{rng}\, s_1$.

(29) For every partial function h from S to T and for every sequence s_1 of S such that $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, h$ holds $(h \cdot s_1) \uparrow n = h \cdot (s_1 \uparrow n)$.

(30) Let h_1, h_2 be partial functions from S to T and s_1 be a sequence of S. If h_1 is total and h_2 is total, then $(h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1$ and $(h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1$.

(31) Let h be a partial function from S to T, s_1 be a sequence of S, and r be a real number. If h is total, then $(r\,h) \cdot s_1 = r \cdot (h \cdot s_1)$.

(32) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_4 of S such that $\operatorname{rng} s_4 \subseteq \operatorname{dom} f$ and s_4 is convergent and $\lim s_4 = x_0$ and for every n holds $s_4(n) \neq x_0$ holds $f \cdot s_4$ is convergent and $f_{x_0} = \lim(f \cdot s_4)$.

(33) For all R_2, R_3 holds $R_2 + R_3$ is a rest of S, T and $R_2 - R_3$ is a rest of S, T.

(34) For all r, R holds $r\,R$ is a rest of S, T.

Let us consider S, T, let f be a partial function from S to T, and let x_0 be a point of S. We say that f is differentiable in x_0 if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and there exist L, R such that for every point x of S such that $x \in N$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x-x_0}$.

Let us consider S, T, let f be a partial function from S to T, and let x_0 be a point of S. Let us assume that f is differentiable in x_0. The functor $f'(x_0)$ yielding a point of RNormSpaceOfBoundedLinearOperators(S, T) is defined by the condition (Def. 7).

(Def. 7) There exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and there exists R such that for every point x of S such that $x \in N$ holds $f_x - f_{x_0} = f'(x_0)(x - x_0) + R_{x-x_0}$.

Let us consider X, let us consider S, T, and let f be a partial function from S to T. We say that f is differentiable on X if and only if:

(Def. 8) $X \subseteq \operatorname{dom} f$ and for every point x of S such that $x \in X$ holds $f{\restriction}X$ is differentiable in x.

Next we state three propositions:

(35) Let f be a partial function from S to T. If f is differentiable on X, then X is a subset of the carrier of S.

(36) Let f be a partial function from S to T and Z be a subset of S. Suppose Z is open. Then f is differentiable on Z if and only if the following conditions are satisfied:

(i) $Z \subseteq \operatorname{dom} f$, and

(ii) for every point x of S such that $x \in Z$ holds f is differentiable in x.

(37) Let f be a partial function from S to T and Y be a subset of S. If f is differentiable on Y, then Y is open.

Let us consider S, T, let f be a partial function from S to T, and let X be a set. Let us assume that f is differentiable on X. The functor $f'_{\restriction X}$ yielding a partial function from S to RNormSpaceOfBoundedLinearOperators(S,T) is defined by:

(Def. 9) $\mathrm{dom}(f'_{\restriction X}) = X$ and for every point x of S such that $x \in X$ holds $(f'_{\restriction X})_x = f'(x)$.

One can prove the following proposition

(38) Let f be a partial function from S to T and Z be a subset of S. Suppose Z is open and $Z \subseteq \mathrm{dom}\, f$ and there exists a point r of T such that $\mathrm{rng}\, f = \{r\}$. Then f is differentiable on Z and for every point x of S such that $x \in Z$ holds $(f'_{\restriction Z})_x = 0_{\mathrm{RNormSpaceOfBoundedLinearOperators}(S,T)}$.

Let us consider S and let us consider h, n. Observe that $h \uparrow n$ is convergent to 0.

Let us consider S and let us consider c, n. Observe that $c \uparrow n$ is constant.

The following propositions are true:

(39) Let x_0 be a point of S and N be a neighbourhood of x_0. Suppose f is differentiable in x_0 and $N \subseteq \mathrm{dom}\, f$. Let h be a convergent to 0 sequence of S and given c. If $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h+c) \subseteq N$, then $f \cdot (h+c) - f \cdot c$ is convergent and $\lim(f \cdot (h+c) - f \cdot c) = 0_T$.

(40) Let given f_1, f_2, x_0. Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0. Then $f_1 + f_2$ is differentiable in x_0 and $(f_1 + f_2)'(x_0) = {f_1}'(x_0) + {f_2}'(x_0)$.

(41) Let given f_1, f_2, x_0. Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0. Then $f_1 - f_2$ is differentiable in x_0 and $(f_1 - f_2)'(x_0) = {f_1}'(x_0) - {f_2}'(x_0)$.

(42) For all r, f, x_0 such that f is differentiable in x_0 holds $r\, f$ is differentiable in x_0 and $(r\, f)'(x_0) = r \cdot f'(x_0)$.

(43) Let f be a partial function from S to S and Z be a subset of S. Suppose Z is open and $Z \subseteq \mathrm{dom}\, f$ and $f{\restriction}Z = \mathrm{id}_Z$. Then f is differentiable on Z and for every point x of S such that $x \in Z$ holds $(f'_{\restriction Z})_x = \mathrm{id}_{\text{the carrier of } S}$.

(44) Let Z be a subset of S. Suppose Z is open. Let given f_1, f_2. Suppose $Z \subseteq \mathrm{dom}(f_1 + f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 + f_2$ is differentiable on Z and for every point x of S such that $x \in Z$ holds $((f_1 + f_2)'_{\restriction Z})_x = {f_1}'(x) + {f_2}'(x)$.

(45) Let Z be a subset of S. Suppose Z is open. Let given f_1, f_2. Suppose $Z \subseteq \mathrm{dom}(f_1 - f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 - f_2$ is differentiable on Z and for every point x of S such that $x \in Z$ holds $((f_1 - f_2)'_{\restriction Z})_x = {f_1}'(x) - {f_2}'(x)$.

(46) Let Z be a subset of S. Suppose Z is open. Let given r, f. Suppose $Z \subseteq \mathrm{dom}(r\, f)$ and f is differentiable on Z. Then $r\, f$ is differentiable on Z and for every point x of S such that $x \in Z$ holds $((r\, f)'_{\restriction Z})_x = r \cdot f'(x)$.

(47) Let Z be a subset of S. Suppose Z is open. Suppose $Z \subseteq \mathrm{dom}\, f$ and f

is a constant on Z. Then f is differentiable on Z and for every point x of S such that $x \in Z$ holds $(f'_{\restriction Z})_x = 0_{\mathrm{RNormSpaceOfBoundedLinearOperators}(S,T)}$.

(48) Let f be a partial function from S to S, r be a real number, p be a point of S, and Z be a subset of S. Suppose Z is open. Suppose $Z \subseteq \operatorname{dom} f$ and for every point x of S such that $x \in Z$ holds $f_x = r \cdot x + p$. Then f is differentiable on Z and for every point x of S such that $x \in Z$ holds $(f'_{\restriction Z})_x = r \cdot \mathrm{FuncUnit}(S)$.

(49) For every point x_0 of S such that f is differentiable in x_0 holds f is continuous in x_0.

(50) If f is differentiable on X, then f is continuous on X.

(51) For every subset Z of S such that Z is open holds if f is differentiable on X and $Z \subseteq X$, then f is differentiable on Z.

(52) Suppose f is differentiable in x_0. Then there exists a neighbourhood N of x_0 such that

(i) $N \subseteq \operatorname{dom} f$, and

(ii) there exists R such that $R_{0_S} = 0_T$ and R is continuous in 0_S and for every point x of S such that $x \in N$ holds $f_x - f_{x_0} = f'(x_0)(x - x_0) + R_{x-x_0}$.

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Lucas Numbers and Generalized Fibonacci Numbers

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Summary. The recursive definition of Fibonacci sequences [3] is a good starting point for various variants and generalizations. We can here point out e.g. Lucas (with 2 and 1 as opening values) and the so-called generalized Fibonacci numbers (starting with arbitrary integers a and b).

In this paper, we introduce Lucas and G-numbers and we state their basic properties analogous to those proven in [10] and [5].

MML Identifier: `FIB_NUM3`.

The papers [15], [14], [11], [2], [6], [1], [13], [12], [8], [9], [4], [7], [3], and [10] provide the notation and terminology for this paper.

1. Preliminaries

In this paper a, b, k, n denote natural numbers.

The following propositions are true:

(1) For every real number a and for every natural number n such that $a^n = 0$ holds $a = 0$.

(2) For every non negative real number a holds $\sqrt{a} \cdot \sqrt{a} = a$.

(3) For every non empty real number a holds $a^2 = (-a)^2$.

(4) For every non empty natural number k holds $(k -' 1) + 2 = (k + 2) -' 1$.

(5) $(a + b)^2 = a \cdot a + a \cdot b + a \cdot b + b \cdot b$.

(6) For every non empty real number a holds $(a^n)^2 = a^{2 \cdot n}$.

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(7) For all real numbers a, b holds $(a+b)\cdot(a-b)=a^2-b^2$.

(8) For all non empty real numbers a, b holds $(a\cdot b)^n=a^n\cdot b^n$.

Let us mention that τ is positive and $\overline{\tau}$ is negative.

The following propositions are true:

(9) For every natural number n holds $\tau^n+\tau^{n+1}=\tau^{n+2}$.

(10) For every natural number n holds $\overline{\tau}^n+\overline{\tau}^{n+1}=\overline{\tau}^{n+2}$.

2. Lucas Numbers

Let n be a natural number. The functor $\mathrm{Luc}(n)$ yielding a natural number is defined by the condition (Def. 1).

(Def. 1) There exists a function L from $\mathbb{N}$ into $[:\mathbb{N},\mathbb{N}:]$ such that $\mathrm{Luc}(n)=L(n)_{\mathbf{1}}$ and $L(0)=\langle 2, 1\rangle$ and for every natural number n holds $L(n+1)=\langle L(n)_{\mathbf{2}}, L(n)_{\mathbf{1}}+L(n)_{\mathbf{2}}\rangle$.

The following propositions are true:

(11) $\mathrm{Luc}(0)=2$ and $\mathrm{Luc}(1)=1$ and for every natural number n holds $\mathrm{Luc}(n+1+1)=\mathrm{Luc}(n)+\mathrm{Luc}(n+1)$.

(12) For every natural number n holds $\mathrm{Luc}(n+2)=\mathrm{Luc}(n)+\mathrm{Luc}(n+1)$.

(13) For every natural number n holds $\mathrm{Luc}(n+1)+\mathrm{Luc}(n+2)=\mathrm{Luc}(n+3)$.

(14) $\mathrm{Luc}(2)=3$.

(15) $\mathrm{Luc}(3)=4$.

(16) $\mathrm{Luc}(4)=7$.

(17) For every natural number k holds $\mathrm{Luc}(k)\geqslant k$.

(18) For every non empty natural number m holds $\mathrm{Luc}(m+1)\geqslant\mathrm{Luc}(m)$.

Let n be a natural number. Note that $\mathrm{Luc}(n)$ is positive.

Next we state a number of propositions:

(19) For every natural number n holds $2\cdot\mathrm{Luc}(n+2)=\mathrm{Luc}(n)+\mathrm{Luc}(n+3)$.

(20) For every natural number n holds $\mathrm{Luc}(n+1)=\mathrm{Fib}(n)+\mathrm{Fib}(n+2)$.

(21) For every natural number n holds $\mathrm{Luc}(n)=\tau^n+\overline{\tau}^n$.

(22) For every natural number n holds $2\cdot\mathrm{Luc}(n)+\mathrm{Luc}(n+1)=5\cdot\mathrm{Fib}(n+1)$.

(23) For every natural number n holds $\mathrm{Luc}(n+3)-2\cdot\mathrm{Luc}(n)=5\cdot\mathrm{Fib}(n)$.

(24) For every natural number n holds $\mathrm{Luc}(n)+\mathrm{Fib}(n)=2\cdot\mathrm{Fib}(n+1)$.

(25) For every natural number n holds $3\cdot\mathrm{Fib}(n)+\mathrm{Luc}(n)=2\cdot\mathrm{Fib}(n+2)$.

(26) For all natural numbers n, m holds $2\cdot\mathrm{Luc}(n+m)=\mathrm{Luc}(n)\cdot\mathrm{Luc}(m)+5\cdot\mathrm{Fib}(n)\cdot\mathrm{Fib}(m)$.

(27) For every natural number n holds $\mathrm{Luc}(n+3)\cdot\mathrm{Luc}(n)=\mathrm{Luc}(n+2)^2-\mathrm{Luc}(n+1)^2$.

(28) For every natural number n holds $\mathrm{Fib}(2\cdot n)=\mathrm{Fib}(n)\cdot\mathrm{Luc}(n)$.

(29) For every natural number n holds $2 \cdot \mathrm{Fib}(2 \cdot n+1) = \mathrm{Luc}(n+1) \cdot \mathrm{Fib}(n) + \mathrm{Luc}(n) \cdot \mathrm{Fib}(n+1)$.

(30) For every natural number n holds $5 \cdot \mathrm{Fib}(n)^2 - \mathrm{Luc}(n)^2 = 4 \cdot (-1)^{n+1}$.

(31) For every natural number n holds $\mathrm{Fib}(2 \cdot n+1) = \mathrm{Fib}(n+1) \cdot \mathrm{Luc}(n+1) - \mathrm{Fib}(n) \cdot \mathrm{Luc}(n)$.

3. Generalized Fibonacci Numbers

Let a, b, n be natural numbers. The functor $\mathrm{GFib}(a, b, n)$ yielding a natural number is defined by the condition (Def. 2).

(Def. 2) There exists a function L from $\mathbb{N}$ into $[\!:\mathbb{N}, \mathbb{N}:\!]$ such that $\mathrm{GFib}(a, b, n) = L(n)_{\mathbf{1}}$ and $L(0) = \langle a, b \rangle$ and for every natural number n holds $L(n+1) = \langle L(n)_{\mathbf{2}}, L(n)_{\mathbf{1}} + L(n)_{\mathbf{2}} \rangle$.

Next we state a number of propositions:

(32) For all natural numbers a, b holds $\mathrm{GFib}(a, b, 0) = a$ and $\mathrm{GFib}(a, b, 1) = b$ and for every natural number n holds $\mathrm{GFib}(a, b, n+1+1) = \mathrm{GFib}(a, b, n) + \mathrm{GFib}(a, b, n+1)$.

(33) $(\mathrm{GFib}(a, b, k+1) + \mathrm{GFib}(a, b, k+1+1))^2 = \mathrm{GFib}(a, b, k+1)^2 + 2 \cdot \mathrm{GFib}(a, b, k+1) \cdot \mathrm{GFib}(a, b, k+1+1) + \mathrm{GFib}(a, b, k+1+1)^2$.

(34) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n) + \mathrm{GFib}(a, b, n+1) = \mathrm{GFib}(a, b, n+2)$.

(35) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n+1) + \mathrm{GFib}(a, b, n+2) = \mathrm{GFib}(a, b, n+3)$.

(36) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n+2) + \mathrm{GFib}(a, b, n+3) = \mathrm{GFib}(a, b, n+4)$.

(37) For every natural number n holds $\mathrm{GFib}(0, 1, n) = \mathrm{Fib}(n)$.

(38) For every natural number n holds $\mathrm{GFib}(2, 1, n) = \mathrm{Luc}(n)$.

(39) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n) + \mathrm{GFib}(a, b, n+3) = 2 \cdot \mathrm{GFib}(a, b, n+2)$.

(40) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n) + \mathrm{GFib}(a, b, n+4) = 3 \cdot \mathrm{GFib}(a, b, n+2)$.

(41) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n+3) - \mathrm{GFib}(a, b, n) = 2 \cdot \mathrm{GFib}(a, b, n+1)$.

(42) For all non empty natural numbers a, b, n holds $\mathrm{GFib}(a, b, n) = \mathrm{GFib}(a, b, 0) \cdot \mathrm{Fib}(n -' 1) + \mathrm{GFib}(a, b, 1) \cdot \mathrm{Fib}(n)$.

(43) For all natural numbers n, m holds $\mathrm{Fib}(m) \cdot \mathrm{Luc}(n) + \mathrm{Luc}(m) \cdot \mathrm{Fib}(n) = \mathrm{GFib}(\mathrm{Fib}(0), \mathrm{Luc}(0), n+m)$.

(44) For every natural number n holds $\mathrm{Luc}(n) + \mathrm{Luc}(n+3) = 2 \cdot \mathrm{Luc}(n+2)$.

(45) For all natural numbers a, n holds $\mathrm{GFib}(a, a, n) = \frac{\mathrm{GFib}(a,a,0)}{2} \cdot (\mathrm{Fib}(n) + \mathrm{Luc}(n))$.

(46) For all natural numbers a, b, n holds $\mathrm{GFib}(b, a+b, n) = \mathrm{GFib}(a, b, n+1)$.

(47) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n+2) \cdot \mathrm{GFib}(a, b, n) - \mathrm{GFib}(a, b, n+1)^2 = (-1)^n \cdot (\mathrm{GFib}(a, b, 2)^2 - \mathrm{GFib}(a, b, 1) \cdot \mathrm{GFib}(a, b, 3))$.

(48) For all natural numbers a, b, k, n holds $\mathrm{GFib}(\mathrm{GFib}(a, b, k), \mathrm{GFib}(a, b, k+1), n) = \mathrm{GFib}(a, b, n+k)$.

(49) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n+1) = a \cdot \mathrm{Fib}(n) + b \cdot \mathrm{Fib}(n+1)$.

(50) For all natural numbers a, b, n holds $\mathrm{GFib}(0, b, n) = b \cdot \mathrm{Fib}(n)$.

(51) For all natural numbers a, b, n holds $\mathrm{GFib}(a, 0, n+1) = a \cdot \mathrm{Fib}(n)$.

(52) For all natural numbers a, b, c, d, n holds $\mathrm{GFib}(a, b, n) + \mathrm{GFib}(c, d, n) = \mathrm{GFib}(a+c, b+d, n)$.

(53) For all natural numbers a, b, k, n holds $\mathrm{GFib}(k \cdot a, k \cdot b, n) = k \cdot \mathrm{GFib}(a, b, n)$.

(54) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n) = \frac{(a \cdot -\overline{\tau} + b) \cdot \tau^n + (a \cdot \tau - b) \cdot \overline{\tau}^n}{\sqrt{5}}$.

(55) For all natural numbers a, n holds $\mathrm{GFib}(2 \cdot a + 1, 2 \cdot a + 1, n + 1) = (2 \cdot a + 1) \cdot \mathrm{Fib}(n + 2)$.

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The Operation of Addition of Relational Structures

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Summary. The article contains the formalization of the addition operator on relational structures as defined by A. Wroński [8] (as a generalization of Troelstra's sum or Jaśkowski's star addition). The ordering relation of $A \otimes B$ is given by

$$\leqslant_{A \otimes B} = \leqslant_A \cup \leqslant_B \cup (\leqslant_A \circ \leqslant_B),$$

where the carrier is defined as the set-theoretical union of carriers of A and B. Main part – Section 3 – is devoted to the Mizar translation of Theorem 1 (iv–xiii), p. 66 of [8].

MML Identifier: LATSUM_1.

The terminology and notation used in this paper are introduced in the following articles: [4], [6], [7], [5], [2], [3], and [1].

1. Preliminaries

One can prove the following proposition

(1) Let x, y, A, B be sets. Suppose $x \in A \cup B$ and $y \in A \cup B$. Then $x \in A \setminus B$ and $y \in A \setminus B$ or $x \in B$ and $y \in B$ or $x \in A \setminus B$ and $y \in B$ or $x \in B$ and $y \in A \setminus B$.

Let R, S be relational structures. The predicate $R \approx S$ is defined by the condition (Def. 1).

(Def. 1) Let x, y be sets. Suppose $x \in$ (the carrier of R) $\cap$ (the carrier of S) and $y \in$ (the carrier of R) $\cap$ (the carrier of S). Then $\langle x, y \rangle \in$ the internal relation of R if and only if $\langle x, y \rangle \in$ the internal relation of S.

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2. The Wroński's Operation

Let R, S be relational structures. The functor $R \otimes S$ yields a strict relational structure and is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of $R \otimes S$ = (the carrier of R) $\cup$ (the carrier of S), and

(ii) the internal relation of $R \otimes S$ = (the internal relation of R) $\cup$ (the internal relation of S) $\cup$ (the internal relation of R) $\cdot$ (the internal relation of S).

Let R be a relational structure and let S be a non empty relational structure. Observe that $R \otimes S$ is non empty.

Let R be a non empty relational structure and let S be a relational structure. Observe that $R \otimes S$ is non empty.

One can prove the following two propositions:

(2) Let R, S be relational structures. Then

(i) the internal relation of $R \subseteq$ the internal relation of $R \otimes S$, and

(ii) the internal relation of $S \subseteq$ the internal relation of $R \otimes S$.

(3) For all relational structures R, S such that R is reflexive and S is reflexive holds $R \otimes S$ is reflexive.

3. Properties of the Addition

Next we state a number of propositions:

(4) Let R, S be relational structures and a, b be sets. Suppose that

(i) $\langle a, b \rangle \in$ the internal relation of $R \otimes S$,

(ii) $a \in$ the carrier of R,

(iii) $b \in$ the carrier of R,

(iv) $R \approx S$, and

(v) R is transitive.

Then $\langle a, b \rangle \in$ the internal relation of R.

(5) Let R, S be relational structures and a, b be sets. Suppose that

(i) $\langle a, b \rangle \in$ the internal relation of $R \otimes S$,

(ii) $a \in$ the carrier of S,

(iii) $b \in$ the carrier of S,

(iv) $R \approx S$, and

(v) S is transitive.

Then $\langle a, b \rangle \in$ the internal relation of S.

(6) Let R, S be relational structures and a, b be sets. Then

(i) if $\langle a, b \rangle \in$ the internal relation of R, then $\langle a, b \rangle \in$ the internal relation of $R \otimes S$, and

(ii) if $\langle a, b \rangle \in$ the internal relation of S, then $\langle a, b \rangle \in$ the internal relation of $R \otimes S$.

(7) Let R, S be non empty relational structures and x be an element of $R \otimes S$. Then $x \in$ the carrier of R or $x \in$ (the carrier of S) $\setminus$ (the carrier of R).

(8) Let R, S be non empty relational structures, x, y be elements of R, and a, b be elements of $R \otimes S$. Suppose $x = a$ and $y = b$ and $R \approx S$ and R is transitive. Then $x \leqslant y$ if and only if $a \leqslant b$.

(9) Let R, S be non empty relational structures, a, b be elements of $R \otimes S$, and c, d be elements of S. Suppose $a = c$ and $b = d$ and $R \approx S$ and S is transitive. Then $a \leqslant b$ if and only if $c \leqslant d$.

(10) Let R, S be antisymmetric reflexive transitive non empty relational structures with l.u.b.'s and x be a set. If $x \in$ the carrier of R, then x is an element of $R \otimes S$.

(11) Let R, S be antisymmetric reflexive transitive non empty relational structures with l.u.b.'s and x be a set. If $x \in$ the carrier of S, then x is an element of $R \otimes S$.

(12) Let R, S be non empty relational structures and x be a set. Suppose $x \in$ (the carrier of R) $\cap$ (the carrier of S). Then x is an element of R.

(13) Let R, S be non empty relational structures and x be a set. Suppose $x \in$ (the carrier of R) $\cap$ (the carrier of S). Then x is an element of S.

(14) Let R, S be antisymmetric reflexive transitive non empty relational structures with l.u.b.'s and x, y be elements of $R \otimes S$. Suppose $x \in$ the carrier of R and $y \in$ the carrier of S and $R \approx S$. Then $x \leqslant y$ if and only if there exists an element a of $R \otimes S$ such that $a \in$ (the carrier of R) $\cap$ (the carrier of S) and $x \leqslant a$ and $a \leqslant y$.

(15) Let R, S be non empty relational structures, a, b be elements of R, and c, d be elements of S. Suppose $a = c$ and $b = d$ and $R \approx S$ and R is transitive and S is transitive. Then $a \leqslant b$ if and only if $c \leqslant d$.

(16) Let R be an antisymmetric reflexive transitive non empty relational structure with l.u.b.'s, D be a lower directed subset of R, and x, y be elements of R. If $x \in D$ and $y \in D$, then $x \sqcup y \in D$.

(17) Let R, S be relational structures and a, b be sets. Suppose that

(i) (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R,

(ii) $\langle a, b \rangle \in$ the internal relation of $R \otimes S$, and

(iii) $a \in$ the carrier of S.

Then $b \in$ the carrier of S.

(18) Let R, S be relational structures and a, b be elements of $R \otimes S$. Suppose (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R and $a \leqslant b$ and $a \in$ the carrier of S. Then $b \in$ the carrier of S.

(19) Let R, S be antisymmetric reflexive transitive non empty relational structures with l.u.b.'s, x, y be elements of R, and a, b be elements of

S. Suppose that

(i) (the carrier of R) $\cap$ (the carrier of S) is a lower directed subset of S,

(ii) (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R,

(iii) $R \approx S$,

(iv) $x = a$, and

(v) $y = b$.

Then $x \sqcup y = a \sqcup b$.

(20) Let R, S be lower-bounded antisymmetric reflexive transitive non empty relational structures with l.u.b.'s. Suppose (the carrier of R) $\cap$ (the carrier of S) is a non empty lower directed subset of S. Then $\bot_S \in$ the carrier of R.

(21) Let R, S be relational structures and a, b be sets. Suppose that

(i) (the carrier of R) $\cap$ (the carrier of S) is a lower subset of S,

(ii) $\langle a, b\rangle \in$ the internal relation of $R \otimes S$, and

(iii) $b \in$ the carrier of R.

Then $a \in$ the carrier of R.

(22) Let x, y be sets and R, S be relational structures. Suppose $\langle x, y\rangle \in$ the internal relation of $R \otimes S$ and (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R. Then

(i) $x \in$ the carrier of R and $y \in$ the carrier of R, or

(ii) $x \in$ the carrier of S and $y \in$ the carrier of S, or

(iii) $x \in$ (the carrier of R) $\setminus$ (the carrier of S) and $y \in$ (the carrier of S) $\setminus$ (the carrier of R).

(23) Let R, S be relational structures and a, b be elements of $R \otimes S$. Suppose (the carrier of R) $\cap$ (the carrier of S) is a lower subset of S and $a \leqslant b$ and $b \in$ the carrier of R. Then $a \in$ the carrier of R.

(24) Let R, S be relational structures. Suppose that

(i) $R \approx S$,

(ii) (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R,

(iii) (the carrier of R) $\cap$ (the carrier of S) is a lower subset of S,

(iv) R is transitive and antisymmetric, and

(v) S is transitive and antisymmetric.

Then $R \otimes S$ is antisymmetric.

(25) Let R, S be relational structures. Suppose that

(i) (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R,

(ii) (the carrier of R) $\cap$ (the carrier of S) is a lower subset of S,

(iii) $R \approx S$,

(iv) R is transitive, and

(v) S is transitive.

Then $R \otimes S$ is transitive.

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The Nagata-Smirnov Theorem. Part I[1]

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Summary. In this paper we define a discrete subset family of a topological space and basis sigma locally finite and sigma discrete. First, we prove an auxiliary fact for discrete family and sigma locally finite and sigma discrete basis. We also show the necessary condition for the Nagata Smirnov theorem: every metrizable space is T_3 and has a sigma locally finite basis. Also, we define a sufficient condition for a T_3 topological space to be T_4. We introduce the concept of pseudo metric.

MML Identifier: NAGATA_1.

The terminology and notation used in this paper have been introduced in the following articles: [9], [27], [28], [32], [20], [5], [12], [8], [21], [15], [2], [17], [14], [18], [19], [6], [10], [11], [24], [23], [4], [33], [1], [3], [25], [16], [26], [7], [13], [29], [31], [34], [30], and [22].

In this paper T, T_1 denote non empty topological spaces and P_1 denotes a non empty metric structure.

Let T be a topological space and let F be a family of subsets of T. We say that F is discrete if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let p be a point of T. Then there exists an open subset O of T such that $p \in O$ and for all subsets A, B of T such that $A \in F$ and $B \in F$ holds if O meets A and O meets B, then $A = B$.

Let T be a non empty topological space. Note that there exists a family of subsets of T which is discrete.

Let us consider T. One can check that there exists a family of subsets of T which is empty and discrete.

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For simplicity, we adopt the following convention: F, G, H denote families of subsets of T, A, B denote subsets of T, O, U denote open subsets of T, p denotes a point of T, and x, X denote sets.

The following propositions are true:

(1) For every F such that there exists A such that $F = \{A\}$ holds F is discrete.

(2) For all F, G such that $F \subseteq G$ and G is discrete holds F is discrete.

(3) For all F, G such that F is discrete holds $F \cap G$ is discrete.

(4) For all F, G such that F is discrete holds $F \setminus G$ is discrete.

(5) For all F, G, H such that F is discrete and G is discrete and $F \Cap G = H$ holds H is discrete.

(6) For all F, A, B such that F is discrete and $A \in F$ and $B \in F$ holds $A = B$ or A misses B.

(7) If F is discrete, then for every p there exists O such that $p \in O$ and $\{O\} \Cap F \setminus \{\emptyset\}$ is trivial.

(8) F is discrete if and only if the following conditions are satisfied:

(i) for every p there exists O such that $p \in O$ and $\{O\} \Cap F \setminus \{\emptyset\}$ is trivial, and

(ii) for all A, B such that $A \in F$ and $B \in F$ holds $A = B$ or A misses B.

Let us consider T and let F be a discrete family of subsets of T. Observe that $\operatorname{clf} F$ is discrete.

Next we state three propositions:

(9) For every F such that F is discrete and for all A, B such that $A \in F$ and $B \in F$ holds $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

(10) For every F such that F is discrete holds $\overline{\bigcup F} = \bigcup \operatorname{clf} F$.

(11) For every F such that F is discrete holds F is locally finite.

Let T be a topological space. A family sequence of T is a function from $\mathbb{N}$ into $2^{2^{\text{the carrier of } T}}$.

In the sequel U_1 denotes a family sequence of T, r denotes a real number, n denotes a natural number, and f denotes a function.

Let us consider T, U_1, n. Then $U_1(n)$ is a family of subsets of T.

Let us consider T, U_1. Then $\bigcup U_1$ is a family of subsets of T.

Let T be a non empty topological space and let U_1 be a family sequence of T. We say that U_1 is sigma-discrete if and only if:

(Def. 2) For every natural number n holds $U_1(n)$ is discrete.

Let T be a non empty topological space. Note that there exists a family sequence of T which is sigma-discrete.

Let T be a non empty topological space and let U_1 be a family sequence of T. We say that U_1 is sigma-locally-finite if and only if:

(Def. 3) For every natural number n holds $U_1(n)$ is locally finite.

Let us consider T and let F be a family of subsets of T. We say that F is sigma-discrete if and only if:

(Def. 4) There exists a sigma-discrete family sequence f of T such that $F = \bigcup f$.

Let X be a set. We introduce X is uncountable as an antonym of X is countable.

One can verify that every set which is uncountable is also non empty.

Let T be a non empty topological space. One can check that there exists a family sequence of T which is sigma-locally-finite.

Next we state two propositions:

(12) For every U_1 such that U_1 is sigma-discrete holds U_1 is sigma-locally-finite.

(13) Let A be an uncountable set. Then there exists a family F of subsets of $\{[: A, A :]\}_{\rm top}$ such that F is locally finite and F is not sigma-discrete.

Let T be a non empty topological space and let U_1 be a family sequence of T. We say that U_1 is Basis-sigma-discrete if and only if:

(Def. 5) U_1 is sigma-discrete and $\bigcup U_1$ is a basis of T.

Let T be a non empty topological space and let U_1 be a family sequence of T. We say that U_1 is Basis-sigma-locally finite if and only if:

(Def. 6) U_1 is sigma-locally-finite and $\bigcup U_1$ is a basis of T.

The following propositions are true:

(14) Let r be a real number. Suppose P_1 is a non empty metric space. Let x be an element of P_1. Then $\Omega_{(P_1)} \setminus \overline{\rm Ball}(x, r) \in$ the open set family of P_1.

(15) For every T such that T is metrizable holds T is a T_3 space and a T_1 space.

(16) For every T such that T is metrizable holds there exists a family sequence of T which is Basis-sigma-locally finite.

(17) For every function U from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ such that for every n holds $U(n)$ is open holds $\bigcup U$ is open.

(18) Suppose that for all A, U such that A is closed and U is open and $A \subseteq U$ there exists a function W from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ such that $A \subseteq \bigcup W$ and $\bigcup W \subseteq U$ and for every n holds $\overline{W(n)} \subseteq U$ and $W(n)$ is open. Then T is a T_4 space.

(19) Let given T. Suppose T is a T_3 space. Let B_1 be a family sequence of T. Suppose $\bigcup B_1$ is a basis of T. Let U be a subset of T and p be a point of T. Suppose U is open and $p \in U$. Then there exists a subset O of T such that $p \in O$ and $\overline{O} \subseteq U$ and $O \in \bigcup B_1$.

(20) For every T such that T is a T_3 space and a T_1 space and there exists a family sequence of T which is Basis-sigma-locally finite holds T is a T_4

space.

Let us consider T and let F, G be real maps of T. The functor $F+G$ yielding a real map of T is defined as follows:

(Def. 7) For every element t of T holds $(F+G)(t) = F(t) + G(t)$.

Next we state four propositions:

(21) Let f be a real map of T. Suppose f is continuous. Let F be a real map of $[\!:T, T:\!]$. Suppose that for all elements x, y of the carrier of T holds $F(\langle x, y\rangle) = |f(x) - f(y)|$. Then F is continuous.

(22) For all real maps F, G of T such that F is continuous and G is continuous holds $F+G$ is continuous.

(23) Let A_1 be a binary operation on $\mathbb{R}^{\text{the carrier of } T}$. Suppose that for all real maps f_1, f_2 of T holds $A_1(f_1, f_2) = f_1 + f_2$. Then A_1 is commutative and associative and has a unity.

(24) Let A_1 be a binary operation on $\mathbb{R}^{\text{the carrier of } T}$. Suppose that for all real maps f_1, f_2 of T holds $A_1(f_1, f_2) = f_1 + f_2$. Let m'_1 be an element of $\mathbb{R}^{\text{the carrier of } T}$. If m'_1 is a unity w.r.t. A_1, then m'_1 is continuous.

Let T, T_1 be non empty topological spaces, let S_1 be a function from the carrier of T into $2^{\text{the carrier of } T}$, and let F_1 be a function from the carrier of T into (the carrier of T_1)$^{\text{the carrier of } T}$. The functor $F_1 \approx S_1$ yields a map from T into T_1 and is defined by:

(Def. 8) For every point p of T holds $(F_1 \approx S_1)(p) = F_1(p)(p)$.

The following propositions are true:

(25) Let A_1 be a binary operation on $\mathbb{R}^{\text{the carrier of } T}$. Suppose that for all real maps f_1, f_2 of T holds $A_1(f_1, f_2) = f_1 + f_2$. Let F be a finite sequence of elements of $\mathbb{R}^{\text{the carrier of } T}$. Suppose that for every n such that $0 \neq n$ and $n \leqslant \operatorname{len} F$ holds $F(n)$ is a continuous real map of T. Then $A_1 \odot F$ is a continuous real map of T.

(26) Let F be a function from the carrier of T into (the carrier of T_1)$^{\text{the carrier of } T}$. Suppose that for every point p of T holds $F(p)$ is a continuous map from T into T_1. Let S be a function from the carrier of T into $2^{\text{the carrier of } T}$. Suppose that for every point p of T holds $p \in S(p)$ and $S(p)$ is open and for all points p, q of T such that $p \in S(q)$ holds $F(p)(p) = F(q)(p)$. Then $F \approx S$ is continuous.

In the sequel m denotes a function from $[\!:\text{the carrier of } T, \text{ the carrier of } T:\!]$ into $\mathbb{R}$.

Let us consider X, r and let f be a function from X into $\mathbb{R}$. The functor $\min(r, f)$ yielding a function from X into $\mathbb{R}$ is defined as follows:

(Def. 9) For every x such that $x \in X$ holds $(\min(r, f))(x) = \min(r, f(x))$.

One can prove the following proposition

(27) For every real number r and for every real map f of T such that f is continuous holds $\min(r, f)$ is continuous.

Let X be a set and let f be a function from $[: X, X :]$ into $\mathbb{R}$. We say that f is a pseudometric of if and only if:

(Def. 10) f is Reflexive, symmetric, and triangle.

One can prove the following propositions:

(28) Let f be a function from $[: X, X :]$ into $\mathbb{R}$. Then f is a pseudometric of if and only if for all elements a, b, c of X holds $f(a, a) = 0$ and $f(a, c) \leqslant f(a, b) + f(c, b)$.

(29) For every function f from $[: X, X :]$ into $\mathbb{R}$ such that f is a pseudometric of and for all elements x, y of X holds $f(x, y) \geqslant 0$.

(30) For all r, m such that $r > 0$ and m is a pseudometric of holds $\min(r, m)$ is a pseudometric of.

(31) For all r, m such that $r > 0$ and m is a metric of the carrier of T holds $\min(r, m)$ is a metric of the carrier of T.

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Properties of Groups[1]

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Summary. In this article we formalize theorems from Chapter 1 of [7]. Our article covers Theorems 1.5.4, 1.5.5 (inequality on indices), 1.5.6 (equality of indices), Lemma 1.6.1 and several other supporting theorems needed to complete the formalization.

MML Identifier: GROUP_8.

The articles [1], [12], [5], [19], [20], [3], [4], [13], [16], [6], [14], [15], [10], [8], [17], [18], [11], [2], and [9] provide the terminology and notation for this paper.

For simplicity, we adopt the following rules: G is a strict group, a, b, x, y, z are elements of the carrier of G, H, K are strict subgroups of G, p is a natural number, and A is a subset of the carrier of G.

We now state a number of propositions:

(1) If p is prime and $\mathrm{ord}(G) = p$ and G is finite, then there exists a such that $\mathrm{ord}(a) = p$.

(2) Let a_1, a_2 be elements of the carrier of H and b_1, b_2 be elements of the carrier of G. If $a_1 = b_1$ and $a_2 = b_2$, then $a_1 \cdot a_2 = b_1 \cdot b_2$.

(3) Let a be an element of the carrier of H and b be an element of the carrier of G. If $a = b$, then for every natural number n holds $a^n = b^n$.

(4) Let a be an element of the carrier of H and b be an element of the carrier of G. If $a = b$, then for every integer i holds $a^i = b^i$.

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(5) Let a be an element of the carrier of H and b be an element of the carrier of G. If $a = b$ and G is finite, then $\mathrm{ord}(a) = \mathrm{ord}(b)$.

(6) For every element h of the carrier of G such that $h \in H$ holds $H \cdot h \subseteq$ the carrier of H.

(7) For every a such that $a \neq 1_G$ holds $\mathrm{gr}(\{a\}) \neq \{\mathbf{1}\}_G$.

(8) For every integer m holds $(1_G)^m = 1_G$.

(9) For every integer m holds $a^{m \cdot \mathrm{ord}(a)} = 1_G$.

(10) For every a such that a is not of order 0 and for every integer m holds $a^m = a^{m \bmod \mathrm{ord}(a)}$.

(11) If b is not of order 0, then $\mathrm{gr}(\{b\})$ is finite.

(12) If b is of order 0, then b^{-1} is of order 0.

(13) b is of order 0 iff for every integer n such that $b^n = 1_G$ holds $n = 0$.

(14) Let given G. Given a such that $a \neq 1_G$. Then for every H holds $H = G$ or $H = \{\mathbf{1}\}_G$ if and only if the following conditions are satisfied:
 (i) G is a cyclic group and finite, and
 (ii) there exists a natural number p such that $\mathrm{ord}(G) = p$ and p is prime.

(15) Let x, y, z be elements of the carrier of G and A be a subset of the carrier of G. Then $z \in x \cdot A \cdot y$ if and only if there exists an element a of the carrier of G such that $z = x \cdot a \cdot y$ and $a \in A$.

(16) For every non empty subset A of G and for every element x of the carrier of G holds $\overline{\overline{A}} = \overline{\overline{x^{-1} \cdot A \cdot x}}$.

Let us consider G, H, K. The functor DoubleCosets(H, K) yielding a family of subsets of the carrier of G is defined as follows:

(Def. 1) $A \in \mathrm{DoubleCosets}(H, K)$ iff there exists a such that $A = H \cdot a \cdot K$.

We now state two propositions:

(17) $z \in H \cdot x \cdot K$ iff there exist elements g, h of the carrier of G such that $z = g \cdot x \cdot h$ and $g \in H$ and $h \in K$.

(18) For all H, K holds $H \cdot x \cdot K = H \cdot y \cdot K$ or it is not true that there exists z such that $z \in H \cdot x \cdot K$ and $z \in H \cdot y \cdot K$.

In the sequel B, A denote strict subgroups of G and D denotes a strict subgroup of A.

Let us consider G, A. Observe that the left cosets of A is non empty.

Let us consider G and let H be a subgroup of G. We introduce $[G : H]_{\mathbb{N}}$ as a synonym of $|\bullet : H|_{\mathbb{N}}$.

Next we state several propositions:

(19) If $G = A \sqcup B$ and $D = A \cap B$ and G is finite, then $[G : B]_{\mathbb{N}} \geqslant [A : D]_{\mathbb{N}}$.

(20) If G is finite, then $[G : H]_{\mathbb{N}} > 0$.

(21) Let G be a strict group. Suppose G is finite. Let C be a strict subgroup of G and A, B be strict subgroups of C. Suppose $C = A \sqcup B$. Let D be a

strict subgroup of A. Suppose $D = A \cap B$. Let E be a strict subgroup of B. Suppose $E = A \cap B$. Let F be a strict subgroup of C. Suppose $F = A \cap B$. Suppose the left cosets of B is finite and the left cosets of A is finite and $[A : C]_{\mathbb{N}}$ and $[B : C]_{\mathbb{N}}$ are relative prime. Then $[B : C]_{\mathbb{N}} = [D : A]_{\mathbb{N}}$ and $[A : C]_{\mathbb{N}} = [E : B]_{\mathbb{N}}$.

(22) For every element a of the carrier of G such that $a \in H$ and for every integer j holds $a^j \in H$.

(23) For every strict group G such that $G \neq \{\mathbf{1}\}_G$ there exists an element b of the carrier of G such that $b \neq 1_G$.

(24) Let G be a strict group and a be an element of the carrier of G. Suppose $G = \mathrm{gr}(\{a\})$ and $G \neq \{\mathbf{1}\}_G$. Let H be a strict subgroup of G. If $H \neq \{\mathbf{1}\}_G$, then there exists a natural number k such that $0 < k$ and $a^k \in H$.

(25) Let G be a strict cyclic group. Suppose $G \neq \{\mathbf{1}\}_G$. Let H be a strict subgroup of G. If $H \neq \{\mathbf{1}\}_G$, then H is a cyclic group.

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Catalan Numbers

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Summary. In this paper, we define Catalan sequence (starting from 0) and prove some of its basic properties. The Catalan numbers $(0, 1, 1, 2, 5, 14, 42, \ldots)$ arise in a number of problems in combinatorics. They can be computed e.g. using the formula

$$C_n = \frac{\genfrac{}{}{0pt}{}{2n}{n}}{n+1},$$

their recursive definition is also well known:

$$C_1 = 1, \quad C_n = \Sigma_{i=1}^{n-1} C_i C_{n-i}, \quad n \geqslant 2.$$

Among other things, the Catalan numbers describe the number of ways in which parentheses can be placed in a sequence of numbers to be multiplied, two at a time.

MML Identifier: `CATALAN1`.

The articles [2], [3], [4], [1], [5], [8], [6], and [7] provide the terminology and notation for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For every natural number n such that $n > 1$ holds $n -' 1 \leqslant 2 \cdot n -' 3$.

(2) For every natural number n such that $n \geqslant 1$ holds $n -' 1 \leqslant 2 \cdot n -' 2$.

(3) For every natural number n such that $n > 1$ holds $n < 2 \cdot n -' 1$.

(4) For every natural number n such that $n > 1$ holds $(n -' 2) + 1 = n -' 1$.

(5) For every natural number n such that $n > 1$ holds $\frac{4 \cdot n \cdot n - 2 \cdot n}{n+1} > 1$.

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(6) For every natural number n such that $n > 1$ holds $(2 \cdot n -' 2)! \cdot n \cdot (n+1) < (2 \cdot n)!$.

(7) For every natural number n holds $2 \cdot (2 - \frac{3}{n+1}) < 4$.

2. Definition of Catalan Numbers

Let n be a natural number. The functor Catalan(n) yields a real number and is defined as follows:

(Def. 1) $\mathrm{Catalan}(n) = \frac{\binom{2 \cdot n -' 2}{n -' 1}}{n}$.

The following propositions are true:

(8) For every natural number n such that $n > 1$ holds $\mathrm{Catalan}(n) = \frac{(2 \cdot n -' 2)!}{(n -' 1)! \cdot n!}$.

(9) For every natural number n such that $n > 1$ holds $\mathrm{Catalan}(n) = 4 \cdot \binom{2 \cdot n -' 3}{n -' 1} - \binom{2 \cdot n -' 1}{n -' 1}$.

(10) $\mathrm{Catalan}(0) = 0$.

(11) $\mathrm{Catalan}(1) = 1$.

(12) $\mathrm{Catalan}(2) = 1$.

(13) For every natural number n holds Catalan(n) is an integer.

(14) For every natural number k such that $k \geqslant 1$ holds $\mathrm{Catalan}(k+1) = \frac{(2 \cdot k)!}{k! \cdot (k+1)!}$.

3. Basic Properties of Catalan Numbers

We now state several propositions:

(15) For every natural number n such that $n > 1$ holds $\mathrm{Catalan}(n) < \mathrm{Catalan}(n+1)$.

(16) For every natural number n holds $\mathrm{Catalan}(n) \leqslant \mathrm{Catalan}(n+1)$.

(17) For every natural number n holds $\mathrm{Catalan}(n) \geqslant 0$.

(18) For every natural number n holds Catalan(n) is a natural number.

(19) For every natural number n such that $n > 0$ holds $\mathrm{Catalan}(n+1) = 2 \cdot (2 - \frac{3}{n+1}) \cdot \mathrm{Catalan}(n)$.

Let n be a natural number. Note that Catalan(n) is natural.

Next we state the proposition

(20) For every natural number n such that $n > 0$ holds $\mathrm{Catalan}(n) > 0$.

Let n be a non empty natural number. One can verify that Catalan(n) is non empty.

One can prove the following proposition

(21) For every natural number n such that $n > 0$ holds $\mathrm{Catalan}(n+1) < 4 \cdot \mathrm{Catalan}(n)$.

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Axiomatization of Boolean Algebras Based on Sheffer Stroke

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Summary. We formalized another axiomatization of Boolean algebras. The classical one is introduced in [9], "the fourth set of postulates" due to Huntington [3] ([2] in Mizar) and the single axiom in terms of disjunction and negation is codified recently in [7]. In this article, we aimed at the description of Boolean algebras using Sheffer stroke according to [6], namely by the following three axioms:

$$(x|x)|(x|x) = x$$

$$x|(y|(y|y)) = x|x$$

$$(x|(y|z))|(x|(y|z)) = ((y|y)|x)|((z|z)|x)$$

($\upharpoonright$ is used instead of $|$ in the translation of our Mizar article). Since Sheffer in his original paper proved its equivalence and Huntington's "first set of postulates", we have also introduced this axiomatization of BAs.

MML Identifier: `SHEFFER1`.

The terminology and notation used here are introduced in the following articles: [8], [9], [5], [1], [4], and [2].

1. Preliminaries

The following two propositions are true:

(1) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. Then $(a+b)^{\rm c} = a^{\rm c} * b^{\rm c}$.

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(2) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. Then $(a * b)^{\mathrm{c}} = a^{\mathrm{c}} + b^{\mathrm{c}}$.

2. Huntington's First Axiomatization of Boolean Algebras

Let I_1 be a non empty lattice structure. We say that I_1 is upper-bounded' if and only if:

(Def. 1) There exists an element c of I_1 such that for every element a of I_1 holds $c \sqcap a = a$ and $a \sqcap c = a$.

Let L be a non empty lattice structure. Let us assume that L is upper-bounded'. The functor $\top'_L$ yields an element of L and is defined by:

(Def. 2) For every element a of L holds $\top'_L \sqcap a = a$ and $a \sqcap \top'_L = a$.

Let I_1 be a non empty lattice structure. We say that I_1 is lower-bounded' if and only if:

(Def. 3) There exists an element c of I_1 such that for every element a of I_1 holds $c \sqcup a = a$ and $a \sqcup c = a$.

Let L be a non empty lattice structure. Let us assume that L is lower-bounded'. The functor $\bot'_L$ yields an element of L and is defined as follows:

(Def. 4) For every element a of L holds $\bot'_L \sqcup a = a$ and $a \sqcup \bot'_L = a$.

Let I_1 be a non empty lattice structure. We say that I_1 is distributive' if and only if:

(Def. 5) For all elements a, b, c of I_1 holds $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$.

Let L be a non empty lattice structure and let a, b be elements of L. We say that a is a complement' of b if and only if:

(Def. 6) $b \sqcup a = \top'_L$ and $a \sqcup b = \top'_L$ and $b \sqcap a = \bot'_L$ and $a \sqcap b = \bot'_L$.

Let I_1 be a non empty lattice structure. We say that I_1 is complemented' if and only if:

(Def. 7) For every element b of I_1 holds there exists an element of I_1 which is a complement' of b.

Let L be a non empty lattice structure and let x be an element of L. Let us assume that L is complemented', distributive, upper-bounded', and meet-commutative. The functor $x^{\mathrm{c}'}$ yields an element of L and is defined as follows:

(Def. 8) $x^{\mathrm{c}'}$ is a complement' of x.

Let us mention that there exists a non empty lattice structure which is Boolean, join-idempotent, upper-bounded', complemented', distributive', lower-bounded', and lattice-like.

Next we state several propositions:

(3) Let L be a complemented' join-commutative meet-commutative distributive upper-bounded' distributive' non empty lattice structure and x be an element of L. Then $x \sqcup x^{c'} = \top'_L$.

(4) Let L be a complemented' join-commutative meet-commutative distributive upper-bounded' distributive' non empty lattice structure and x be an element of L. Then $x \sqcap x^{c'} = \bot'_L$.

(5) Let L be a complemented' join-commutative meet-commutative join-idempotent distributive upper-bounded' distributive' non empty lattice structure and x be an element of L. Then $x \sqcup \top'_L = \top'_L$.

(6) Let L be a complemented' join-commutative meet-commutative join-idempotent distributive upper-bounded' lower-bounded' distributive' non empty lattice structure and x be an element of L. Then $x \sqcap \bot'_L = \bot'_L$.

(7) Let L be a join-commutative meet-absorbing meet-commutative join-absorbing join-idempotent distributive non empty lattice structure and x, y, z be elements of L. Then $(x \sqcup y \sqcup z) \sqcap x = x$.

(8) Let L be a join-commutative meet-absorbing meet-commutative join-absorbing join-idempotent distributive' non empty lattice structure and x, y, z be elements of L. Then $(x \sqcap y \sqcap z) \sqcup x = x$.

Let G be a non empty $\sqcap$-semi lattice structure. We say that G is meet-idempotent if and only if:

(Def. 9) For every element x of G holds $x \sqcap x = x$.

Next we state a number of propositions:

(9) Every complemented' join-commutative meet-commutative distributive upper-bounded' lower-bounded' distributive' non empty lattice structure is meet-idempotent.

(10) Every complemented' join-commutative meet-commutative distributive upper-bounded' lower-bounded' distributive' non empty lattice structure is join-idempotent.

(11) Every complemented' join-commutative meet-commutative join-idempotent distributive upper-bounded' distributive' non empty lattice structure is meet-absorbing.

(12) Every complemented' join-commutative upper-bounded' meet-commutative join-idempotent distributive distributive' lower-bounded' non empty lattice structure is join-absorbing.

(13) Every complemented' join-commutative meet-commutative upper-bounded' lower-bounded' join-idempotent distributive distributive' non empty lattice structure is upper-bounded.

(14) Every Boolean lattice-like non empty lattice structure is upper-bounded'.

(15) Every complemented' join-commutative meet-commutative upper-bounded' lower-bounded' join-idempotent distributive distributive' non

empty lattice structure is lower-bounded.

(16) Every Boolean lattice-like non empty lattice structure is lower-bounded'.

(17) Every join-commutative meet-commutative meet-absorbing join-absorbing join-idempotent distributive non empty lattice structure is join-associative.

(18) Every join-commutative meet-commutative meet-absorbing join-absorbing join-idempotent distributive' non empty lattice structure is meet-associative.

(19) Let L be a complemented' join-commutative meet-commutative lower-bounded' upper-bounded' join-idempotent distributive distributive' non empty lattice structure. Then $\top_L = \top'_L$.

(20) Let L be a complemented' join-commutative meet-commutative lower-bounded' upper-bounded' join-idempotent distributive distributive' non empty lattice structure. Then $\bot_L = \bot'_L$.

(21) For every Boolean distributive' lattice-like non empty lattice structure L holds $\top_L = \top'_L$.

(22) Let L be a Boolean complemented lower-bounded upper-bounded distributive distributive' lattice-like non empty lattice structure. Then $\bot_L = \bot'_L$.

(23) Let L be a complemented' lower-bounded' upper-bounded' join-commutative meet-commutative join-idempotent distributive distributive' non empty lattice structure and x, y be elements of L. Then x is a complement' of y if and only if x is a complement of y.

(24) Every complemented' join-commutative meet-commutative lower-bounded' upper-bounded' join-idempotent distributive distributive' non empty lattice structure is complemented.

(25) Every Boolean lower-bounded' upper-bounded' distributive' lattice-like non empty lattice structure is complemented'.

(26) Let L be a non empty lattice structure. Then L is a Boolean lattice if and only if L is lower-bounded', upper-bounded', join-commutative, meet-commutative, distributive, distributive', and complemented'.

Let us note that every non empty lattice structure which is Boolean and lattice-like is also lower-bounded', upper-bounded', complemented', join-commutative, meet-commutative, distributive, and distributive' and every non empty lattice structure which is lower-bounded', upper-bounded', complemented', join-commutative, meet-commutative, distributive, and distributive' is also Boolean and lattice-like.

3. Axiomatization Based on Sheffer Stroke

We introduce Sheffer structures which are extensions of 1-sorted structure and are systems

⟨ a carrier, a Sheffer stroke ⟩,

where the carrier is a set and the Sheffer stroke is a binary operation on the carrier.

We consider Sheffer lattice structures as extensions of Sheffer structure and lattice structure as systems

⟨ a carrier, a join operation, a meet operation, a Sheffer stroke ⟩,

where the carrier is a set, the join operation is a binary operation on the carrier, the meet operation is a binary operation on the carrier, and the Sheffer stroke is a binary operation on the carrier.

We consider Sheffer ortholattice structures as extensions of Sheffer structure and ortholattice structure as systems

⟨ a carrier, a join operation, a meet operation, a complement operation, a Sheffer stroke ⟩,

where the carrier is a set, the join operation is a binary operation on the carrier, the meet operation is a binary operation on the carrier, the complement operation is a unary operation on the carrier, and the Sheffer stroke is a binary operation on the carrier.

The Sheffer ortholattice structure TrivShefferOrthoLattStr is defined by:

(Def. 10) $\mathrm{TrivShefferOrthoLattStr} = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_2, \mathrm{op}_1, \mathrm{op}_2 \rangle$.

One can verify the following observations:

* there exists a Sheffer structure which is non empty,
* there exists a Sheffer lattice structure which is non empty, and
* there exists a Sheffer ortholattice structure which is non empty.

Let L be a non empty Sheffer structure and let x, y be elements of L. The functor $x \upharpoonleft y$ yields an element of L and is defined as follows:

(Def. 11) $x \upharpoonleft y = (\text{the Sheffer stroke of } L)(x, y)$.

Let L be a non empty Sheffer ortholattice structure. We say that L is properly defined if and only if the conditions (Def. 12) are satisfied.

(Def. 12)(i) For every element x of L holds $x \upharpoonleft x = x^{\mathrm{c}}$,

(ii) for all elements x, y of L holds $x \sqcup y = x \upharpoonleft x \upharpoonleft (y \upharpoonleft y)$,

(iii) for all elements x, y of L holds $x \sqcap y = x \upharpoonleft y \upharpoonleft (x \upharpoonleft y)$, and

(iv) for all elements x, y of L holds $x \upharpoonleft y = x^{\mathrm{c}} + y^{\mathrm{c}}$.

Let L be a non empty Sheffer structure. We say that L satisfies (Sheffer$_1$) if and only if:

(Def. 13) For every element x of L holds $x \upharpoonleft x \upharpoonleft (x \upharpoonleft x) = x$.

We say that L satisfies (Sheffer$_2$) if and only if:

(Def. 14) For all elements x, y of L holds $x{\restriction}(y{\restriction}(y{\restriction}y)) = x{\restriction}x$.

We say that L satisfies (Sheffer$_3$) if and only if:

(Def. 15) For all elements x, y, z of L holds $(x{\restriction}(y{\restriction}z)){\restriction}(x{\restriction}(y{\restriction}z)) = y{\restriction}y{\restriction}x{\restriction}(z{\restriction}z{\restriction}x)$.

Let us note that every non empty Sheffer structure which is trivial satisfies also (Sheffer$_1$), (Sheffer$_2$), and (Sheffer$_3$).

One can verify that every non empty $\sqcup$-semi lattice structure which is trivial is also join-commutative and join-associative and every non empty $\sqcap$-semi lattice structure which is trivial is also meet-commutative and meet-associative.

Let us note that every non empty lattice structure which is trivial is also join-absorbing, meet-absorbing, and Boolean.

One can check the following observations:

* TrivShefferOrthoLattStr is non empty,
* TrivShefferOrthoLattStr is trivial, and
* TrivShefferOrthoLattStr is properly defined and well-complemented.

Let us mention that there exists a non empty Sheffer ortholattice structure which is properly defined, Boolean, well-complemented, and lattice-like and satisfies (Sheffer$_1$), (Sheffer$_2$), and (Sheffer$_3$).

Next we state three propositions:

(27) Every properly defined Boolean well-complemented lattice-like non empty Sheffer ortholattice structure satisfies (Sheffer$_1$).

(28) Every properly defined Boolean well-complemented lattice-like non empty Sheffer ortholattice structure satisfies (Sheffer$_2$).

(29) Every properly defined Boolean well-complemented lattice-like non empty Sheffer ortholattice structure satisfies (Sheffer$_3$).

Let L be a non empty Sheffer structure and let a be an element of L. The functor a^{-1} yielding an element of L is defined as follows:

(Def. 16) $a^{-1} = a{\restriction}a$.

One can prove the following propositions:

(30) Let L be a non empty Sheffer ortholattice structure satisfying (Sheffer$_3$) and x, y, z be elements of L. Then $(x{\restriction}(y{\restriction}z))^{-1} = y^{-1}{\restriction}x{\restriction}(z^{-1}{\restriction}x)$.

(31) For every non empty Sheffer ortholattice structure L satisfying (Sheffer$_1$) and for every element x of L holds $x = (x^{-1})^{-1}$.

(32) Let L be a properly defined non empty Sheffer ortholattice structure satisfying (Sheffer$_1$), (Sheffer$_2$), and (Sheffer$_3$) and x, y be elements of L. Then $x{\restriction}y = y{\restriction}x$.

(33) Let L be a properly defined non empty Sheffer ortholattice structure satisfying (Sheffer$_1$), (Sheffer$_2$), and (Sheffer$_3$) and x, y be elements of L. Then $x{\restriction}(x{\restriction}x) = y{\restriction}(y{\restriction}y)$.

(34) Every properly defined non empty Sheffer ortholattice structure satisfying (Sheffer_1), (Sheffer_2), and (Sheffer_3) is join-commutative.

(35) Every properly defined non empty Sheffer ortholattice structure satisfying (Sheffer_1), (Sheffer_2), and (Sheffer_3) is meet-commutative.

(36) Every properly defined non empty Sheffer ortholattice structure satisfying (Sheffer_1), (Sheffer_2), and (Sheffer_3) is distributive.

(37) Every properly defined non empty Sheffer ortholattice structure satisfying (Sheffer_1), (Sheffer_2), and (Sheffer_3) is distributive'.

(38) Every properly defined non empty Sheffer ortholattice structure satisfying (Sheffer_1), (Sheffer_2), and (Sheffer_3) is a Boolean lattice.

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Short Sheffer Stroke-Based Single Axiom for Boolean Algebras

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Summary. We continue the description of Boolean algebras in terms of the Sheffer stroke as defined in [2]. The single axiomatization for BAs in terms of disjunction and negation was shown in [3]. As was checked automatically with the help of automated theorem prover Otter, single axiom of the form

$$(x|((y|x)|x))|(y|(z|x)) = y \quad \text{(Sh1)}$$

is enough to axiomatize the class of all Boolean algebras ($\upharpoonleft$ is used instead of $|$ in translation of our Mizar article). Many theorems in Section 2 were automatically translated from the Otter proof object.

MML Identifier: `SHEFFER2`.

The terminology and notation used in this paper are introduced in the following papers: [4], [1], and [2].

1. First Implication

Let L be a non empty Sheffer structure. We say that L satisfies (Sh_1) if and only if:

(Def. 1) For all elements x, y, z of L holds $x \upharpoonleft (y \upharpoonleft x \upharpoonleft x) \upharpoonleft (y \upharpoonleft (z \upharpoonleft x)) = y$.

Let us observe that every non empty Sheffer structure which is trivial satisfies also (Sh_1).

Let us observe that there exists a non empty Sheffer structure which satisfies (Sh_1), $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, and $(\mathrm{Sheffer}_3)$.

In the sequel L is a non empty Sheffer structure satisfying (Sh_1).

One can prove the following propositions:

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(1) For all elements x, y, z, u of L holds $(x\restriction(y\restriction z)\restriction(x\restriction(x\restriction(y\restriction z))))\restriction(z\restriction(x\restriction z\restriction z)\restriction(u\restriction(x\restriction(y\restriction z)))) = z\restriction(x\restriction z\restriction z)$.

(2) For all elements x, y, z of L holds $(x\restriction y\restriction(y\restriction(z\restriction y\restriction y)\restriction(x\restriction y)\restriction(x\restriction y)))\restriction z = y\restriction(z\restriction y\restriction y)$.

(3) For all elements x, y, z of L holds $x\restriction(y\restriction x\restriction x)\restriction(y\restriction(z\restriction(x\restriction z\restriction z))) = y$.

(4) For all elements x, y of L holds $x\restriction(x\restriction(x\restriction x\restriction x)\restriction(y\restriction(x\restriction(x\restriction x\restriction x)))) = x\restriction(x\restriction x\restriction x)$.

(5) For every element x of L holds $x\restriction(x\restriction x\restriction x) = x\restriction x$.

(6) For every element x of L holds $x\restriction(x\restriction x\restriction x)\restriction(x\restriction x) = x$.

(7) For all elements x, y, z of L holds $x\restriction x\restriction(x\restriction(y\restriction x)) = x$.

(8) For all elements x, y of L holds $x\restriction(y\restriction y\restriction x\restriction x)\restriction y = y\restriction y$.

(9) For all elements x, y of L holds $(x\restriction y\restriction(x\restriction y\restriction(x\restriction y)\restriction(x\restriction y)))\restriction(x\restriction y\restriction(x\restriction y)) = y\restriction(x\restriction y\restriction(x\restriction y)\restriction y\restriction y)$.

(10) For all elements x, y of L holds $x\restriction(y\restriction x\restriction(y\restriction x)\restriction x\restriction x) = y\restriction x$.

(11) For all elements x, y of L holds $x\restriction x\restriction(y\restriction x) = x$.

(12) For all elements x, y of L holds $x\restriction(y\restriction(x\restriction x)) = x\restriction x$.

(13) For all elements x, y of L holds $x\restriction y\restriction(x\restriction y)\restriction y = x\restriction y$.

(14) For all elements x, y of L holds $x\restriction(y\restriction x\restriction x) = y\restriction x$.

(15) For all elements x, y, z of L holds $x\restriction y\restriction(x\restriction(z\restriction y)) = x$.

(16) For all elements x, y, z of L holds $x\restriction(y\restriction z)\restriction(x\restriction z) = x$.

(17) For all elements x, y, z of L holds $x\restriction(x\restriction y\restriction(z\restriction y)) = x\restriction y$.

(18) For all elements x, y, z of L holds $(x\restriction(y\restriction z)\restriction z)\restriction x = x\restriction(y\restriction z)$.

(19) For all elements x, y of L holds $x\restriction(y\restriction x\restriction x) = x\restriction y$.

(20) For all elements x, y of L holds $x\restriction y = y\restriction x$.

(21) For all elements x, y of L holds $x\restriction y\restriction(x\restriction x) = x$.

(22) For all elements x, y, z of L holds $x\restriction y\restriction(y\restriction(z\restriction x)) = y$.

(23) For all elements x, y, z of L holds $x\restriction(y\restriction z)\restriction(z\restriction x) = x$.

(24) For all elements x, y, z of L holds $x\restriction y\restriction(y\restriction(x\restriction z)) = y$.

(25) For all elements x, y, z of L holds $x\restriction(y\restriction z)\restriction(y\restriction x) = x$.

(26) For all elements x, y, z of L holds $x\restriction y\restriction(x\restriction z)\restriction z = x\restriction z$.

(27) For all elements x, y, z of L holds $x\restriction(y\restriction(x\restriction(y\restriction z))) = x\restriction(y\restriction z)$.

(28) For all elements x, y, z of L holds $(x\restriction(y\restriction(x\restriction z)))\restriction y = y\restriction(x\restriction z)$.

(29) For all elements x, y, z, u of L holds $(x\restriction(y\restriction z))\restriction(x\restriction(u\restriction(y\restriction x))) = x\restriction(y\restriction z)\restriction(y\restriction x)$.

(30) For all elements x, y, z of L holds $(x\restriction(y\restriction(x\restriction z)))\restriction y = y\restriction(z\restriction x)$.

(31) For all elements x, y, z, u of L holds $x\restriction(y\restriction z)\restriction(x\restriction(u\restriction(y\restriction x))) = x$.

(32) For all elements x, y of L holds $x\restriction(y\restriction(x\restriction y)) = x\restriction x$.

(33) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft z) = x\upharpoonleft(z\upharpoonleft y)$.

(34) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft(x\upharpoonleft(z\upharpoonleft(y\upharpoonleft x)))) = x\upharpoonleft x$.

(35) For all elements x, y, z of L holds $(x\upharpoonleft(y\upharpoonleft z))\upharpoonleft(y\upharpoonleft x\upharpoonleft x) = x\upharpoonleft(y\upharpoonleft z)\upharpoonleft(x\upharpoonleft(y\upharpoonleft z))$.

(36) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft x)\upharpoonleft y = y\upharpoonleft y$.

(37) For all elements x, y, z of L holds $(x\upharpoonleft y)\upharpoonleft z = z\upharpoonleft(y\upharpoonleft x)$.

(38) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(x\upharpoonleft y))) = x\upharpoonleft(y\upharpoonleft y)$.

(39) For all elements x, y, z of L holds $(x\upharpoonleft y\upharpoonleft y)\upharpoonleft(y\upharpoonleft(z\upharpoonleft x)) = y\upharpoonleft(z\upharpoonleft x)\upharpoonleft(y\upharpoonleft(z\upharpoonleft x))$.

(40) For all elements x, y, z, u of L holds $(x\upharpoonleft y)\upharpoonleft(z\upharpoonleft u) = u\upharpoonleft z\upharpoonleft(y\upharpoonleft x)$.

(41) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft(y\upharpoonleft x\upharpoonleft z)) = x\upharpoonleft(y\upharpoonleft y)$.

(42) For all elements x, y of L holds $x\upharpoonleft(y\upharpoonleft x) = x\upharpoonleft(y\upharpoonleft y)$.

(43) For all elements x, y of L holds $(x\upharpoonleft y)\upharpoonleft y = y\upharpoonleft(x\upharpoonleft x)$.

(44) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft y) = x\upharpoonleft(x\upharpoonleft y)$.

(45) For all elements x, y, z of L holds $(x\upharpoonleft(y\upharpoonleft y))\upharpoonleft(x\upharpoonleft(z\upharpoonleft y)) = x\upharpoonleft(z\upharpoonleft y)\upharpoonleft(x\upharpoonleft(z\upharpoonleft y))$.

(46) For all elements x, y, z of L holds $(x\upharpoonleft(y\upharpoonleft z))\upharpoonleft(x\upharpoonleft(y\upharpoonleft y)) = x\upharpoonleft(y\upharpoonleft z)\upharpoonleft(x\upharpoonleft(y\upharpoonleft z))$.

(47) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft y\upharpoonleft(z\upharpoonleft(x\upharpoonleft(x\upharpoonleft y)))) = x\upharpoonleft(y\upharpoonleft y\upharpoonleft(y\upharpoonleft y))$.

(48) For all elements x, y, z of L holds $(x\upharpoonleft(y\upharpoonleft z)\upharpoonleft(x\upharpoonleft(y\upharpoonleft z)))\upharpoonleft(y\upharpoonleft y) = x\upharpoonleft(y\upharpoonleft y)$.

(49) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft y\upharpoonleft(z\upharpoonleft(x\upharpoonleft(x\upharpoonleft y)))) = x\upharpoonleft y$.

(50) For all elements x, y, z of L holds $(x\upharpoonleft y\upharpoonleft(x\upharpoonleft y)\upharpoonleft(z\upharpoonleft(x\upharpoonleft y\upharpoonleft z)\upharpoonleft(x\upharpoonleft y)))\upharpoonleft(x\upharpoonleft x) = z\upharpoonleft(x\upharpoonleft y\upharpoonleft z)\upharpoonleft(x\upharpoonleft x)$.

(51) For all elements x, y, z of L holds $(x\upharpoonleft(y\upharpoonleft z\upharpoonleft x))\upharpoonleft(y\upharpoonleft y) = y\upharpoonleft z\upharpoonleft(y\upharpoonleft y)$.

(52) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft z\upharpoonleft x)\upharpoonleft(y\upharpoonleft y) = y$.

(53) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft(x\upharpoonleft z\upharpoonleft y)\upharpoonleft x) = y\upharpoonleft(x\upharpoonleft z\upharpoonleft y)$.

(54) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft(y\upharpoonleft(z\upharpoonleft x))\upharpoonleft x) = y\upharpoonleft(x\upharpoonleft(y\upharpoonleft(x\upharpoonleft z))\upharpoonleft y)$.

(55) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft(y\upharpoonleft(z\upharpoonleft x))\upharpoonleft x) = y\upharpoonleft(y\upharpoonleft(z\upharpoonleft x))$.

(56) For all elements x, y, z, u of L holds $x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft(u\upharpoonleft(y\upharpoonleft x))))) = x\upharpoonleft(y\upharpoonleft y)$.

(57) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft(y\upharpoonleft(z\upharpoonleft(x\upharpoonleft y)))) = x\upharpoonleft(y\upharpoonleft(x\upharpoonleft x))$.

(58) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft(y\upharpoonleft(z\upharpoonleft(x\upharpoonleft y)))) = x\upharpoonleft x$.

(59) For all elements x, y of L holds $x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y)) = x\upharpoonleft x$.

(60) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft(z\upharpoonleft x)\upharpoonleft(y\upharpoonleft(z\upharpoonleft x))\upharpoonleft(z\upharpoonleft z)) = x\upharpoonleft(y\upharpoonleft(z\upharpoonleft x))$.

(61) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft(z\upharpoonleft z)) = x\upharpoonleft(y\upharpoonleft(z\upharpoonleft x))$.

(62) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft(z\upharpoonleft z\upharpoonleft x)) = x\upharpoonleft(y\upharpoonleft z)$.

(63) For all elements x, y, z of L holds $(x\upharpoonleft(y\upharpoonleft y))\upharpoonleft(x\upharpoonleft(z\upharpoonleft(y\upharpoonleft y\upharpoonleft x))) = x\upharpoonleft(z\upharpoonleft y)\upharpoonleft(x\upharpoonleft(z\upharpoonleft y))$.

(64) For all elements x, y, z of L holds $(x \restriction (y \restriction y)) \restriction (x \restriction (z \restriction (x \restriction (y \restriction y)))) = x \restriction (z \restriction y) \restriction (x \restriction (z \restriction y))$.

(65) For all elements x, y, z of L holds $(x \restriction (y \restriction y)) \restriction (x \restriction (z \restriction z)) = x \restriction (z \restriction y) \restriction (x \restriction (z \restriction y))$.

(66) For all elements x, y, z of L holds $(x \restriction x \restriction y) \restriction (z \restriction z \restriction y) = y \restriction (x \restriction z) \restriction (y \restriction (x \restriction z))$.

(67) For every non empty Sheffer structure L such that L satisfies (Sh_1) holds L satisfies $(\mathrm{Sheffer}_1)$.

(68) For every non empty Sheffer structure L such that L satisfies (Sh_1) holds L satisfies $(\mathrm{Sheffer}_2)$.

(69) For every non empty Sheffer structure L such that L satisfies (Sh_1) holds L satisfies $(\mathrm{Sheffer}_3)$.

Let us mention that there exists a non empty Sheffer ortholattice structure which is properly defined, Boolean, well-complemented, lattice-like, and de Morgan and satisfies $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, $(\mathrm{Sheffer}_3)$, and (Sh_1).

Let us mention that every non empty Sheffer ortholattice structure which is properly defined satisfies $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, and $(\mathrm{Sheffer}_3)$ is also Boolean and lattice-like and every non empty Sheffer ortholattice structure which is Boolean, lattice-like, well-complemented, and properly defined satisfies also $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, and $(\mathrm{Sheffer}_3)$.

2. Second Implication

We adopt the following rules: L denotes a non empty Sheffer structure satisfying $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, and $(\mathrm{Sheffer}_3)$ and v, q, p, w, z, y, x denote elements of L.

One can prove the following propositions:

(70) For all x, w holds $w \restriction (x \restriction x \restriction x) = w \restriction w$.

(71) For all p, x holds $x = x \restriction x \restriction (p \restriction (p \restriction p))$.

(72) For all y, w holds $w \restriction w \restriction (w \restriction (y \restriction (y \restriction y))) = w$.

(73) For all q, p, y, w holds $(w \restriction (y \restriction (y \restriction y)) \restriction p) \restriction (q \restriction q \restriction p) = p \restriction (w \restriction q) \restriction (p \restriction (w \restriction q))$.

(74) For all q, p, x holds $(x \restriction p) \restriction (q \restriction q \restriction p) = p \restriction (x \restriction x \restriction q) \restriction (p \restriction (x \restriction x \restriction q))$.

(75) For all w, p, y, q holds $(w \restriction w \restriction p) \restriction (q \restriction (y \restriction (y \restriction y)) \restriction p) = p \restriction (w \restriction q) \restriction (p \restriction (w \restriction q))$.

(76) For all p, x holds $x = x \restriction x \restriction (p \restriction p \restriction p)$.

(77) For all y, w holds $w \restriction w \restriction (w \restriction (y \restriction y \restriction y)) = w$.

(78) For all y, w holds $w \restriction (y \restriction y \restriction y) \restriction (w \restriction w) = w$.

(79) For all p, y, w holds $w \restriction (y \restriction y \restriction y) \restriction (p \restriction (p \restriction p)) = w$.

(80) For all p, x, y holds $y \restriction (x \restriction x) \restriction (y \restriction (x \restriction x)) \restriction (p \restriction (p \restriction p)) = (x \restriction x) \restriction y$.

(81) For all x, y holds $y \restriction (x \restriction x) = (x \restriction x) \restriction y$.

(82) For all y, w holds $w \restriction y = y \restriction y \restriction (y \restriction y) \restriction w$.

(83) For all y, w holds $w|y = y|w$.

(84) For all x, p holds $(p|x)|(p|(x|x|(x|x))) = x|x|(x|x)|p|(x|x|(x|x)|p)$.

(85) For all x, p holds $(p|x)|(p|x) = x|x|(x|x)|p|(x|x|(x|x)|p)$.

(86) For all x, p holds $(p|x)|(p|x) = x|p|(x|x|(x|x)|p)$.

(87) For all x, p holds $(p|x)|(p|x) = x|p|(x|p)$.

(88) For all y, q, w holds $(w|q|(y|y|y))|(w|q|(w|q)) = w|w|(w|q)|(q|q|(w|q))$.

(89) For all q, w holds $w|q = w|w|(w|q)|(q|q|(w|q))$.

(90) For all q, p holds $(p|p)|(p|(q|q|q)) = q|q|(q|q)|p|(q|q|p)$.

(91) For all p, q holds $p = q|q|(q|q)|p|(q|q|p)$.

(92) For all p, q holds $p = q|p|(q|q|p)$.

(93) For all z, w, x holds $(x|x|w|(z|z|w))|(w|(x|z)|(w|(x|z))) = w|w|(w|(x|z))|(x|z|(x|z)|(w|(x|z)))$.

(94) For all z, w, x holds $(x|x|w|(z|z|w))|(w|(x|z)|(w|(x|z))) = w|(x|z)$.

(95) For all w, p holds $(p|p)|(p|(w|(w|w))) = w|w|p|(w|w|(w|w)|p)$.

(96) For all p, w holds $p = w|w|p|(w|w|(w|w)|p)$.

(97) For all p, w holds $p = w|w|p|(w|p)$.

(98) For all z, q, x holds $(x|x|q|(z|z|q))|(q|(x|z)|(q|(x|z))) = z|z|(z|z)|(x|x|q)|(q|q|(x|x|q))$.

(99) For all q, z, x holds $q|(x|z) = (z|z|(z|z)|(x|x|q))|(q|q|(x|x|q))$.

(100) For all q, z, x holds $q|(x|z) = (z|(x|x|q))|(q|q|(x|x|q))$.

(101) For all w, y holds $w|w = y|y|y|w$.

(102) For all w, p holds $p|w|(w|w|p) = p$.

(103) For all y, w holds $w|w|(w|w|(y|y|y)) = (y|y)|y$.

(104) For all y, w holds $w|w|w = y|y|y$.

(105) For all p, w holds $w|p|(p|(w|w)) = p$.

(106) For all w, p holds $p|(w|w)|(w|p) = p$.

(107) For all p, w holds $w|p|(w|(p|p)) = w$.

(108) For all x, y holds $y|(y|(x|x)|(y|(x|x))|(x|y)) = x|y$.

(109) For all p, w holds $w|(p|p)|(w|p) = w$.

(110) For all p, w, q, y holds $(y|y|y|q)|(w|w|q) = q|(p|(p|p)|(p|(p|p))|w)|(q|(p|(p|p)|(p|(p|p))|w))$.

(111) For all q, w, p holds $(q|q)|(w|w|q) = q|(p|(p|p)|(p|(p|p))|w)|(q|(p|(p|p)|(p|(p|p))|w))$.

(112) For all w, y, p holds $w|p|(w|(p|(y|(y|y)))) = w$.

(113) For all w, y, p holds $w|(p|(y|(y|y)))|(w|p) = w$.

(114) For all q, p, y holds $(y|y|y|p)|(q|q|p) = p|(p|p|q)|(p|(p|p|q))$.

(115) For all q, z, x holds $(q\upharpoonleft(x\upharpoonleft x\upharpoonleft z)\upharpoonleft(q\upharpoonleft(x\upharpoonleft x\upharpoonleft z)))\upharpoonleft(x\upharpoonleft q\upharpoonleft(z\upharpoonleft z\upharpoonleft q)) = z\upharpoonleft z\upharpoonleft(z\upharpoonleft z)\upharpoonleft(x\upharpoonleft q)\upharpoonleft(q\upharpoonleft q\upharpoonleft(x\upharpoonleft q))$.

(116) For all q, z, x holds $(q\upharpoonleft(x\upharpoonleft x\upharpoonleft z)\upharpoonleft(q\upharpoonleft(x\upharpoonleft x\upharpoonleft z)))\upharpoonleft(x\upharpoonleft q\upharpoonleft(z\upharpoonleft z\upharpoonleft q)) = z\upharpoonleft(x\upharpoonleft q)\upharpoonleft(q\upharpoonleft q\upharpoonleft(x\upharpoonleft q))$.

(117) For all w, q, z holds $(w\upharpoonleft w\upharpoonleft(z\upharpoonleft z\upharpoonleft q))\upharpoonleft(q\upharpoonleft(q\upharpoonleft q\upharpoonleft z)\upharpoonleft(q\upharpoonleft(q\upharpoonleft q\upharpoonleft z))) = z\upharpoonleft z\upharpoonleft q\upharpoonleft(w\upharpoonleft q)\upharpoonleft(z\upharpoonleft z\upharpoonleft q\upharpoonleft(w\upharpoonleft q))$.

(118) For all q, p, x holds $p\upharpoonleft(x\upharpoonleft p)\upharpoonleft(p\upharpoonleft(x\upharpoonleft p))\upharpoonleft(q\upharpoonleft(q\upharpoonleft q)) = (x\upharpoonleft x)\upharpoonleft p$.

(119) For all p, x holds $p\upharpoonleft(x\upharpoonleft p) = (x\upharpoonleft x)\upharpoonleft p$.

(120) For all p, y holds $(y\upharpoonleft p)\upharpoonleft(y\upharpoonleft y\upharpoonleft p) = p\upharpoonleft p\upharpoonleft(y\upharpoonleft p)$.

(121) For all x, y holds $x = x\upharpoonleft x\upharpoonleft(y\upharpoonleft x)$.

(122) For all x, y holds $(y\upharpoonleft x)\upharpoonleft x = x\upharpoonleft(y\upharpoonleft y)\upharpoonleft(x\upharpoonleft(y\upharpoonleft y))\upharpoonleft(y\upharpoonleft x)$.

(123) For all x, z, y holds $x\upharpoonleft(y\upharpoonleft y\upharpoonleft z)\upharpoonleft(x\upharpoonleft(y\upharpoonleft y\upharpoonleft z))\upharpoonleft(y\upharpoonleft x\upharpoonleft(z\upharpoonleft z\upharpoonleft x)) = (z\upharpoonleft(y\upharpoonleft x))\upharpoonleft x$.

(124) For all x, y, z holds $x\upharpoonleft(z\upharpoonleft(z\upharpoonleft z)\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))\upharpoonleft y)\upharpoonleft(x\upharpoonleft(z\upharpoonleft(z\upharpoonleft z)\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))\upharpoonleft y)) = x$.

(125) For all x, z, y holds $(x\upharpoonleft(y\upharpoonleft y\upharpoonleft z))\upharpoonleft z = z\upharpoonleft(y\upharpoonleft x)$.

(126) For all x, y holds $x\upharpoonleft(y\upharpoonleft x\upharpoonleft x) = y\upharpoonleft x$.

(127) For all z, y, x holds $y = x\upharpoonleft x\upharpoonleft x\upharpoonleft y\upharpoonleft(z\upharpoonleft z\upharpoonleft y)$.

(128) For all z, y holds $y\upharpoonleft(y\upharpoonleft y\upharpoonleft z)\upharpoonleft(y\upharpoonleft(y\upharpoonleft y\upharpoonleft z)) = y$.

(129) For all x, z, y holds $y\upharpoonleft y\upharpoonleft z\upharpoonleft(x\upharpoonleft z)\upharpoonleft(y\upharpoonleft y\upharpoonleft z\upharpoonleft(x\upharpoonleft z)) = (x\upharpoonleft x\upharpoonleft(y\upharpoonleft y\upharpoonleft z))\upharpoonleft z$.

(130) For all x, z, y holds $(y\upharpoonleft y\upharpoonleft z\upharpoonleft(x\upharpoonleft z))\upharpoonleft(y\upharpoonleft y\upharpoonleft z\upharpoonleft(x\upharpoonleft z)) = z\upharpoonleft(y\upharpoonleft(x\upharpoonleft x))$.

(131) For all y, x holds $x\upharpoonleft y\upharpoonleft(x\upharpoonleft y)\upharpoonleft x = x\upharpoonleft y$.

(132) For all p, w holds $w\upharpoonleft w\upharpoonleft(w\upharpoonleft p) = w$.

(133) For all w, p holds $p\upharpoonleft w\upharpoonleft(w\upharpoonleft w) = w$.

(134) For all p, y, w holds $w\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft(w\upharpoonleft p) = w$.

(135) For all p, w holds $w\upharpoonleft p\upharpoonleft(w\upharpoonleft w) = w$.

(136) For all y, p, w holds $w\upharpoonleft p\upharpoonleft(w\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))) = w$.

(137) For all p, q, w, y, x holds $(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft w\upharpoonleft(q\upharpoonleft q\upharpoonleft w))\upharpoonleft(w\upharpoonleft(x\upharpoonleft q)\upharpoonleft(w\upharpoonleft(x\upharpoonleft q))) = w\upharpoonleft(p\upharpoonleft(p\upharpoonleft p))\upharpoonleft(w\upharpoonleft(x\upharpoonleft q))\upharpoonleft(x\upharpoonleft q\upharpoonleft(x\upharpoonleft q)\upharpoonleft(w\upharpoonleft(x\upharpoonleft q)))$.

(138) For all q, w, y, x holds $(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft w\upharpoonleft(q\upharpoonleft q\upharpoonleft w))\upharpoonleft(w\upharpoonleft(x\upharpoonleft q)\upharpoonleft(w\upharpoonleft(x\upharpoonleft q))) = w\upharpoonleft(x\upharpoonleft q\upharpoonleft(x\upharpoonleft q)\upharpoonleft(w\upharpoonleft(x\upharpoonleft q)))$.

(139) For all q, w, y, x holds $(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft w\upharpoonleft(q\upharpoonleft q\upharpoonleft w))\upharpoonleft(w\upharpoonleft(x\upharpoonleft q)\upharpoonleft(w\upharpoonleft(x\upharpoonleft q))) = w\upharpoonleft(x\upharpoonleft q)$.

(140) For all z, p, q, y, x holds $(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q\upharpoonleft(z\upharpoonleft z\upharpoonleft q))\upharpoonleft(q\upharpoonleft(x\upharpoonleft z)\upharpoonleft(q\upharpoonleft(x\upharpoonleft z))) = z\upharpoonleft z\upharpoonleft(p\upharpoonleft(p\upharpoonleft p))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q)\upharpoonleft(q\upharpoonleft q\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q))$.

(141) For all z, p, q, y, x holds $q\upharpoonleft(x\upharpoonleft z) = (z\upharpoonleft z\upharpoonleft(p\upharpoonleft(p\upharpoonleft p))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q))\upharpoonleft(q\upharpoonleft q\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q))$.

(142) For all z, q, y, x holds $q\upharpoonleft(x\upharpoonleft z) = (z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q))\upharpoonleft(q\upharpoonleft q\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q))$.

(143) For all v, p, y, x holds $p\upharpoonleft(x\upharpoonleft v) = (v\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft p))\upharpoonleft p$.

(144) For all y, w, z, v, x holds $(w\upharpoonleft(z\upharpoonleft(x\upharpoonleft v)))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z\upharpoonleft(v\upharpoonleft v\upharpoonleft z)) = z\upharpoonleft(x\upharpoonleft v)$.

(145) For all y, z, x holds $(y\upharpoonleft(x\upharpoonleft x\upharpoonleft z)\upharpoonleft(y\upharpoonleft(x\upharpoonleft x\upharpoonleft z)))\upharpoonleft(x\upharpoonleft y\upharpoonleft(z\upharpoonleft z\upharpoonleft y)) = y\upharpoonleft(x\upharpoonleft x\upharpoonleft z)$.

(146) For all z, y, x holds $(z\upharpoonleft(x\upharpoonleft y))\upharpoonleft y = y\upharpoonleft(x\upharpoonleft x\upharpoonleft z)$.

(147) For all x, w, y, z holds $(x\upharpoonleft x\upharpoonleft w\upharpoonleft(z\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft w))\upharpoonleft w = w\upharpoonleft(x\upharpoonleft z)$.

(148) For all z, w, x holds $w\upharpoonleft(z\upharpoonleft(x\upharpoonleft x\upharpoonleft w)) = w\upharpoonleft(x\upharpoonleft z)$.

(149) For all p, z, y, x holds $(z\upharpoonleft(x\upharpoonleft p)\upharpoonleft(z\upharpoonleft(x\upharpoonleft p)))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z\upharpoonleft(p\upharpoonleft p\upharpoonleft z)) = p\upharpoonleft p\upharpoonleft z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z)\upharpoonleft(p\upharpoonleft p\upharpoonleft z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z))$.

(150) For all p, z, y, x holds $z\upharpoonleft(x\upharpoonleft p) = (p\upharpoonleft p\upharpoonleft z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z))\upharpoonleft(p\upharpoonleft p\upharpoonleft z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z))$.

(151) For all z, p, y, x holds $z\upharpoonleft(x\upharpoonleft p) = z\upharpoonleft(p\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y)))))$.

(152) For all z, p, x holds $z\upharpoonleft(x\upharpoonleft p) = z\upharpoonleft(p\upharpoonleft x)$.

(153) For all w, q, p holds $(p\upharpoonleft q)\upharpoonleft w = w\upharpoonleft(q\upharpoonleft p)$.

(154) For all w, p, q holds $(q\upharpoonleft p\upharpoonleft w)\upharpoonleft q = q\upharpoonleft(p\upharpoonleft p\upharpoonleft w)$.

(155) For all z, w, y, x holds $w\upharpoonleft x = w\upharpoonleft(x\upharpoonleft z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y)))\upharpoonleft w))$.

(156) For all w, z, x holds $w\upharpoonleft x = w\upharpoonleft(x\upharpoonleft z\upharpoonleft(x\upharpoonleft w))$.

(157) For all q, x, z, y holds $(x\upharpoonleft y)\upharpoonleft(x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z)))\upharpoonleft q\upharpoonleft x) = x\upharpoonleft y\upharpoonleft(x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))))$.

(158) For all x, q, z, y holds $(x\upharpoonleft y)\upharpoonleft(x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z)))\upharpoonleft q)) = x\upharpoonleft y\upharpoonleft(x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))))$.

(159) For all z, x, q, y holds $(x\upharpoonleft y)\upharpoonleft(x\upharpoonleft(y\upharpoonleft q)) = x\upharpoonleft y\upharpoonleft(x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))))$.

(160) For all x, q, y holds $x\upharpoonleft y\upharpoonleft(x\upharpoonleft(y\upharpoonleft q)) = x$.

(161) L satisfies (Sh_1).

Let us mention that every non empty Sheffer structure which satisfies $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, and $(\mathrm{Sheffer}_3)$ satisfies also (Sh_1) and every non empty Sheffer structure which satisfies (Sh_1) satisfies also $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, and $(\mathrm{Sheffer}_3)$.

Let us observe that every non empty Sheffer ortholattice structure which is properly defined satisfies (Sh_1) is also Boolean and lattice-like and every non empty Sheffer ortholattice structure which is Boolean, lattice-like, well-complemented, and properly defined satisfies also (Sh_1).

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Differentiable Functions on Normed Linear Spaces. Part II

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Summary. A continuation of [7], the basic properties of the differentiable functions on normed linear spaces are described.

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The terminology and notation used in this paper have been introduced in the following articles: [16], [3], [19], [5], [4], [1], [15], [6], [17], [18], [9], [8], [2], [20], [12], [14], [10], [13], [7], and [11].

For simplicity, we adopt the following rules: S, T denote non trivial real normed spaces, x_0 denotes a point of S, f denotes a partial function from S to T, h denotes a convergent to 0 sequence of S, and c denotes a constant sequence of S.

Let X, Y, Z be real normed spaces, let f be an element of $\mathrm{BdLinOps}(X,Y)$, and let g be an element of $\mathrm{BdLinOps}(Y,Z)$. The functor $g \cdot f$ yielding an element of $\mathrm{BdLinOps}(X,Z)$ is defined by:

(Def. 1) $g \cdot f = \mathrm{modetrans}(g, Y, Z) \cdot \mathrm{modetrans}(f, X, Y)$.

Let X, Y, Z be real normed spaces, let f be a point of $\mathrm{RNormSpaceOfBoundedLinearOperators}(X,Y)$, and let g be a point of $\mathrm{RNormSpaceOfBoundedLinearOperators}(Y,Z)$. The functor $g \cdot f$ yields a point of $\mathrm{RNormSpaceOfBoundedLinearOperators}(X,Z)$ and is defined by:

(Def. 2) $g \cdot f = \mathrm{modetrans}(g, Y, Z) \cdot \mathrm{modetrans}(f, X, Y)$.

Next we state three propositions:

(1) Let x_0 be a point of S. Suppose f is differentiable in x_0. Then there exists a neighbourhood N of x_0 such that

(i) $N \subseteq \mathrm{dom}\, f$, and

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(ii) for every point z of S and for every convergent to 0 sequence h of real numbers and for every c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h \cdot z + c) \subseteq N$ holds $h^{-1}(f \cdot (h \cdot z + c) - f \cdot c)$ is convergent and $f'(x_0)(z) = \lim(h^{-1}(f \cdot (h \cdot z + c) - f \cdot c))$.

(2) Let x_0 be a point of S. Suppose f is differentiable in x_0. Let z be a point of S, h be a convergent to 0 sequence of real numbers, and given c. Suppose $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h \cdot z + c) \subseteq \operatorname{dom} f$. Then $h^{-1}(f \cdot (h \cdot z + c) - f \cdot c)$ is convergent and $f'(x_0)(z) = \lim(h^{-1}(f \cdot (h \cdot z + c) - f \cdot c))$.

(3) Let x_0 be a point of S and N be a neighbourhood of x_0. Suppose $N \subseteq \operatorname{dom} f$. Let z be a point of S and d_1 be a point of T. Then the following statements are equivalent

(i) for every convergent to 0 sequence h of real numbers and for every c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h \cdot z + c) \subseteq N$ holds $h^{-1}(f \cdot (h \cdot z + c) - f \cdot c)$ is convergent and $d_1 = \lim(h^{-1}(f \cdot (h \cdot z + c) - f \cdot c))$,

(ii) for every real number e such that $e > 0$ there exists a real number d such that $d > 0$ and for every real number h such that $|h| < d$ and $h \neq 0$ and $h \cdot z + x_0 \in N$ holds $\|h^{-1} \cdot (f_{h \cdot z + x_0} - f_{x_0}) - d_1\| < e$.

Let us consider S, T, let us consider f, let x_0 be a point of S, and let z be a point of S. We say that f is Gateaux differentiable in x_0, z if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exists a neighbourhood N of x_0 such that

(i) $N \subseteq \operatorname{dom} f$, and

(ii) there exists a point d_1 of T such that for every real number e such that $e > 0$ there exists a real number d such that $d > 0$ and for every real number h such that $|h| < d$ and $h \neq 0$ and $h \cdot z + x_0 \in N$ holds $\|h^{-1} \cdot (f_{h \cdot z + x_0} - f_{x_0}) - d_1\| < e$.

One can prove the following proposition

(4) For every real normed space X and for all points x, y of X holds $\|x - y\| > 0$ iff $x \neq y$ and for every real normed space X and for all points x, y of X holds $\|x - y\| = \|y - x\|$ and for every real normed space X and for all points x, y of X holds $\|x - y\| = 0$ iff $x = y$ and for every real normed space X and for all points x, y of X holds $\|x - y\| \neq 0$ iff $x \neq y$ and for every real normed space X and for all points x, y, z of X and for every real number e such that $e > 0$ holds if $\|x - z\| < \frac{e}{2}$ and $\|z - y\| < \frac{e}{2}$, then $\|x - y\| < e$ and for every real normed space X and for all points x, y, z of X and for every real number e such that $e > 0$ holds if $\|x - z\| < \frac{e}{2}$ and $\|y - z\| < \frac{e}{2}$, then $\|x - y\| < e$ and for every real normed space X and for every point x of X such that for every real number e such that $e > 0$ holds $\|x\| < e$ holds $x = 0_X$ and for every real normed space X and for all points x, y of X such that for every real number e such that $e > 0$ holds $\|x - y\| < e$ holds $x = y$.

Let us consider S, T, let us consider f, let x_0 be a point of S, and let z be a point of S. Let us assume that f is Gateaux differentiable in x_0, z. The functor $\mathrm{GateauxDiff}_z(f, x_0)$ yields a point of T and is defined by the condition (Def. 4).

(Def. 4) There exists a neighbourhood N of x_0 such that

(i) $N \subseteq \mathrm{dom}\, f$, and

(ii) for every real number e such that $e > 0$ there exists a real number d such that $d > 0$ and for every real number h such that $|h| < d$ and $h \neq 0$ and $h \cdot z + x_0 \in N$ holds $\|h^{-1} \cdot (f_{h \cdot z + x_0} - f_{x_0}) - \mathrm{GateauxDiff}_z(f, x_0)\| < e$.

We now state two propositions:

(5) Let x_0 be a point of S and z be a point of S. Then f is Gateaux differentiable in x_0, z if and only if there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and there exists a point d_1 of T such that for every convergent to 0 sequence h of real numbers and for every c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h \cdot z + c) \subseteq N$ holds $h^{-1}\, (f \cdot (h \cdot z + c) - f \cdot c)$ is convergent and $d_1 = \lim(h^{-1}\, (f \cdot (h \cdot z + c) - f \cdot c))$.

(6) Let x_0 be a point of S. Suppose f is differentiable in x_0. Let z be a point of S. Then

(i) f is Gateaux differentiable in x_0, z,

(ii) $\mathrm{GateauxDiff}_z(f, x_0) = f'(x_0)(z)$, and

(iii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and for every convergent to 0 sequence h of real numbers and for every c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h \cdot z + c) \subseteq N$ holds $h^{-1}\, (f \cdot (h \cdot z + c) - f \cdot c)$ is convergent and $\mathrm{GateauxDiff}_z(f, x_0) = \lim(h^{-1}\, (f \cdot (h \cdot z + c) - f \cdot c))$.

In the sequel U is a non trivial real normed space.

Next we state several propositions:

(7) Let R be a rest of S, T. Suppose $R_{0_S} = 0_T$. Let e be a real number. Suppose $e > 0$. Then there exists a real number d such that $d > 0$ and for every point h of S such that $\|h\| < d$ holds $\|R_h\| \leqslant e \cdot \|h\|$.

(8) Let R be a rest of T, U. Suppose $R_{0_T} = 0_U$. Let L be a bounded linear operator from S into T. Then $R \cdot L$ is a rest of S, U.

(9) For every rest R of S, T and for every bounded linear operator L from T into U holds $L \cdot R$ is a rest of S, U.

(10) Let R_1 be a rest of S, T. Suppose $(R_1)_{0_S} = 0_T$. Let R_2 be a rest of T, U. If $(R_2)_{0_T} = 0_U$, then $R_2 \cdot R_1$ is a rest of S, U.

(11) Let R_1 be a rest of S, T. Suppose $(R_1)_{0_S} = 0_T$. Let R_2 be a rest of T, U. Suppose $(R_2)_{0_T} = 0_U$. Let L be a bounded linear operator from S into T. Then $R_2 \cdot (L + R_1)$ is a rest of S, U.

(12) Let R_1 be a rest of S, T. Suppose $(R_1)_{0_S} = 0_T$. Let R_2 be a rest of T, U. Suppose $(R_2)_{0_T} = 0_U$. Let L_1 be a bounded linear operator from S into T and L_2 be a bounded linear operator from T into U. Then

$L_2 \cdot R_1 + R_2 \cdot (L_1 + R_1)$ is a rest of S, U.

(13) Let f_1 be a partial function from S to T. Suppose f_1 is differentiable in x_0. Let f_2 be a partial function from T to U. Suppose f_2 is differentiable in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is differentiable in x_0 and $(f_2 \cdot f_1)'(x_0) = f_2{}'((f_1)_{x_0}) \cdot f_1{}'(x_0)$.

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Logical Correctness of Vector Calculation Programs

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Summary. In C-program, vectors of n-dimension are sometimes represented by arrays, where the dimension n is saved in the 0-th element of each array. If we write the program in non-overwriting type, we can give Logical-Model to each program. Here, we give a program calculating inner product of 2 vectors, as an example of such a type, and its Logical-Model. If the Logical-Model is well defined, and theorems tying the model with previous definitions are given, we can say that the program is logically correct. In case the program is given as implicit function form (i.e., the result of calculation is given by a variable of one of arguments of a function), its Logical-Model is given by a definition of a new predicate form. Logical correctness of such a program is shown by theorems following the definition. As examples of such programs, we presented vector calculation of add, sub, minus and scalar product.

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The articles [16], [18], [14], [20], [8], [4], [5], [11], [3], [10], [2], [6], [19], [17], [12], [9], [13], [1], [15], and [7] provide the terminology and notation for this paper.

In this paper m, n, i are natural numbers and D is a set.

The following proposition is true

(1) For all n, m holds $n \in m$ iff $n < m$.

Let D be a non empty set. One can check that there exists a finite 0-sequence of D which is non empty.

The following proposition is true

(2) For every non empty set D and for every non empty finite 0-sequence f of D holds $\operatorname{len} f > 0$.

Let D be a set and let q be a finite sequence of elements of D. The functor FS2XFS(q) yields a finite 0-sequence of D and is defined by:

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(Def. 1) $\operatorname{len} \mathrm{FS2XFS}(q) = \operatorname{len} q$ and for every i such that $i < \operatorname{len} q$ holds $q(i+1) = (\mathrm{FS2XFS}(q))(i)$.

Let D be a set and let q be a finite 0-sequence of D. The functor $\mathrm{XFS2FS}(q)$ yielding a finite sequence of elements of D is defined as follows:

(Def. 2) $\operatorname{len} \mathrm{XFS2FS}(q) = \operatorname{len} q$ and for every i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} q$ holds $q(i -' 1) = (\mathrm{XFS2FS}(q))(i)$.

One can prove the following two propositions:

(3) For every natural number k and for every set a holds $k \longmapsto a$ is a finite 0-sequence.

(4) Let D be a set, n be a natural number, and r be a set. Suppose $r \in D$. Then $n \longmapsto r$ is a finite 0-sequence of D and for every finite 0-sequence q_2 such that $q_2 = n \longmapsto r$ holds $\operatorname{len} q_2 = n$.

Let D be a non empty set, let q be a finite sequence of elements of D, and let n be a natural number. Let us assume that $n > \operatorname{len} q$ and $\mathbb{N} \subseteq D$. The functor $\mathrm{FS2XFS}^\star(q, n)$ yields a non empty finite 0-sequence of D and is defined by the conditions (Def. 3).

(Def. 3)(i) $\operatorname{len} q = (\mathrm{FS2XFS}^\star(q, n))(0)$,

(ii) $\operatorname{len} \mathrm{FS2XFS}^\star(q, n) = n$,

(iii) for every i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} q$ holds $(\mathrm{FS2XFS}^\star(q, n))(i) = q(i)$, and

(iv) for every natural number j such that $\operatorname{len} q < j$ and $j < n$ holds $(\mathrm{FS2XFS}^\star(q, n))(j) = 0$.

Let D be a non empty set and let p be a non empty finite 0-sequence of D. Let us assume that $\mathbb{N} \subseteq D$ and $p(0)$ is a natural number and $p(0) \in \operatorname{len} p$. The functor $\mathrm{XFS2FS}^\star(p)$ yielding a finite sequence of elements of D is defined by:

(Def. 4) For every m such that $m = p(0)$ holds $\operatorname{len} \mathrm{XFS2FS}^\star(p) = m$ and for every i such that $1 \leqslant i$ and $i \leqslant m$ holds $(\mathrm{XFS2FS}^\star(p))(i) = p(i)$.

The following proposition is true

(5) For every non empty set D and for every non empty finite 0-sequence p of D such that $\mathbb{N} \subseteq D$ and $p(0) = 0$ and $0 < \operatorname{len} p$ holds $\mathrm{XFS2FS}^\star(p) = \emptyset$.

Let D be a non empty set, let p be a finite 0-sequence of D, and let q be a finite sequence of elements of D. We say that p is an xrep of q if and only if:

(Def. 5) $\mathbb{N} \subseteq D$ and $p(0) = \operatorname{len} q$ and $\operatorname{len} q < \operatorname{len} p$ and for every i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} q$ holds $p(i) = q(i)$.

The following proposition is true

(6) Let D be a non empty set and p be a non empty finite 0-sequence of D. Suppose $\mathbb{N} \subseteq D$ and $p(0)$ is a natural number and $p(0) \in \operatorname{len} p$. Then p is an xrep of $\mathrm{XFS2FS}^\star(p)$.

Let x, y, a, b, c be sets. The functor $\mathrm{IFLGT}(x, y, a, b, c)$ yielding a set is defined by:

(Def. 6) $$\mathrm{IFLGT}(x, y, a, b, c) = \begin{cases} a, & \text{if } x \in y, \\ b, & \text{if } x = y, \\ c, & \text{otherwise.} \end{cases}$$

Next we state the proposition

(7) Let D be a non empty set, q be a finite sequence of elements of D, and n be a natural number. Suppose $\mathbb{N} \subseteq D$ and $n > \mathrm{len}\, q$. Then there exists a finite 0-sequence p of D such that $\mathrm{len}\, p = n$ and p is an xrep of q.

Let b be a finite 0-sequence of $\mathbb{R}$ and let n be a natural number. Then $b(n)$ is a real number.

Let a, b be finite 0-sequences of $\mathbb{R}$. Let us assume that $b(0)$ is a natural number and $0 \leqslant b(0)$ and $b(0) < \mathrm{len}\, a$. The functor $\mathrm{InnerPrdPrg}(a, b)$ yielding a real number is defined by the condition (Def. 7).

(Def. 7) There exists a finite 0-sequence s of $\mathbb{R}$ and there exists an integer n such that

- (i) $\mathrm{len}\, s = \mathrm{len}\, a$,
- (ii) $s(0) = 0$,
- (iii) $n = b(0)$,
- (iv) if $n \neq 0$, then for every natural number i such that $i < n$ holds $s(i+1) = s(i) + a(i+1) \cdot b(i+1)$, and
- (v) $\mathrm{InnerPrdPrg}(a, b) = s(n)$.

The following propositions are true:

(8) Let a be a finite sequence of elements of $\mathbb{R}$ and s be a finite 0-sequence of $\mathbb{R}$. Suppose $\mathrm{len}\, s > \mathrm{len}\, a$ and $s(0) = 0$ and for every i such that $i < \mathrm{len}\, a$ holds $s(i+1) = s(i) + a(i+1)$. Then $\sum a = s(\mathrm{len}\, a)$.

(9) Let a be a finite sequence of elements of $\mathbb{R}$. Then there exists a finite 0-sequence s of $\mathbb{R}$ such that $\mathrm{len}\, s = \mathrm{len}\, a + 1$ and $s(0) = 0$ and for every i such that $i < \mathrm{len}\, a$ holds $s(i+1) = s(i) + a(i+1)$ and $\sum a = s(\mathrm{len}\, a)$.

(10) Let a, b be finite sequences of elements of $\mathbb{R}$ and n be a natural number. If $\mathrm{len}\, a = \mathrm{len}\, b$ and $n > \mathrm{len}\, a$, then $|(a, b)| = \mathrm{InnerPrdPrg}(\mathrm{FS2XFS}^{\star}(a, n), \mathrm{FS2XFS}^{\star}(b, n))$.

Let b, c be finite 0-sequences of $\mathbb{R}$, let a be a real number, and let m be an integer. We say that m scalar prd prg of c, a, b if and only if the conditions (Def. 8) are satisfied.

(Def. 8)(i) $\mathrm{len}\, c = m$,

- (ii) $\mathrm{len}\, b = m$, and
- (iii) there exists an integer n such that $c(0) = b(0)$ and $n = b(0)$ and if $n \neq 0$, then for every natural number i such that $1 \leqslant i$ and $i \leqslant n$ holds $c(i) = a \cdot b(i)$.

We now state the proposition

(11) Let b be a non empty finite 0-sequence of $\mathbb{R}$, a be a real number, and m be a natural number. Suppose $b(0)$ is a natural number and $\operatorname{len} b = m$ and $0 \leqslant b(0)$ and $b(0) < m$. Then

(i) there exists a finite 0-sequence c of $\mathbb{R}$ such that m scalar prd prg of c, a, b, and

(ii) for every non empty finite 0-sequence c of $\mathbb{R}$ such that m scalar prd prg of c, a, b holds $\text{XFS2FS}^\star(c) = a \cdot \text{XFS2FS}^\star(b)$.

Let b, c be finite 0-sequences of $\mathbb{R}$ and let m be an integer. We say that m vector minus prg of c, b if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) $\operatorname{len} c = m$,

(ii) $\operatorname{len} b = m$, and

(iii) there exists an integer n such that $c(0) = b(0)$ and $n = b(0)$ and if $n \neq 0$, then for every natural number i such that $1 \leqslant i$ and $i \leqslant n$ holds $c(i) = -b(i)$.

The following proposition is true

(12) Let b be a non empty finite 0-sequence of $\mathbb{R}$ and m be a natural number. Suppose $b(0)$ is a natural number and $\operatorname{len} b = m$ and $0 \leqslant b(0)$ and $b(0) < m$. Then

(i) there exists a finite 0-sequence c of $\mathbb{R}$ such that m vector minus prg of c, b, and

(ii) for every non empty finite 0-sequence c of $\mathbb{R}$ such that m vector minus prg of c, b holds $\text{XFS2FS}^\star(c) = -\text{XFS2FS}^\star(b)$.

Let a, b, c be finite 0-sequences of $\mathbb{R}$ and let m be an integer. We say that m vector add prg of c, a, b if and only if the conditions (Def. 10) are satisfied.

(Def. 10)(i) $\operatorname{len} c = m$,

(ii) $\operatorname{len} a = m$,

(iii) $\operatorname{len} b = m$, and

(iv) there exists an integer n such that $c(0) = b(0)$ and $n = b(0)$ and if $n \neq 0$, then for every natural number i such that $1 \leqslant i$ and $i \leqslant n$ holds $c(i) = a(i) + b(i)$.

Next we state the proposition

(13) Let a, b be non empty finite 0-sequences of $\mathbb{R}$ and m be a natural number. Suppose $b(0)$ is a natural number and $\operatorname{len} a = m$ and $\operatorname{len} b = m$ and $a(0) = b(0)$ and $0 \leqslant b(0)$ and $b(0) < m$. Then

(i) there exists a finite 0-sequence c of $\mathbb{R}$ such that m vector add prg of c, a, b, and

(ii) for every non empty finite 0-sequence c of $\mathbb{R}$ such that m vector add prg of c, a, b holds $\text{XFS2FS}^\star(c) = \text{XFS2FS}^\star(a) + \text{XFS2FS}^\star(b)$.

Let a, b, c be finite 0-sequences of $\mathbb{R}$ and let m be an integer. We say that m vector sub prg of c, a, b if and only if the conditions (Def. 11) are satisfied.

(Def. 11)(i) $\operatorname{len} c = m$,

(ii) $\operatorname{len} a = m$,

(iii) $\operatorname{len} b = m$, and

(iv) there exists an integer n such that $c(0) = b(0)$ and $n = b(0)$ and if $n \neq 0$, then for every natural number i such that $1 \leqslant i$ and $i \leqslant n$ holds $c(i) = a(i) - b(i)$.

One can prove the following proposition

(14) Let a, b be non empty finite 0-sequences of $\mathbb{R}$ and m be a natural number. Suppose $b(0)$ is a natural number and $\operatorname{len} a = m$ and $\operatorname{len} b = m$ and $a(0) = b(0)$ and $0 \leqslant b(0)$ and $b(0) < m$. Then

(i) there exists a finite 0-sequence c of $\mathbb{R}$ such that m vector sub prg of c, a, b, and

(ii) for every non empty finite 0-sequence c of $\mathbb{R}$ such that m vector sub prg of c, a, b holds $\mathrm{XFS2FS}^\star(c) = \mathrm{XFS2FS}^\star(a) - \mathrm{XFS2FS}^\star(b)$.

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Continuous Mappings between Finite and One-Dimensional Finite Topological Spaces

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Summary. We showed relations between separateness and inflation operation. We also gave some relations between separateness and connectedness defined before. For two finite topological spaces, we defined a continuous function from one to another. Some topological concepts are preserved by such continuous functions. We gave one-dimensional concrete models of finite topological space.

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The notation and terminology used here are introduced in the following articles: [12], [5], [13], [1], [14], [3], [4], [2], [6], [10], [9], [11], [7], and [8].

Let F_1 be a non empty finite topology space and let A, B be subsets of F_1. We say that A and B are separated if and only if:

(Def. 1) A^b misses B and A misses B^b.

Next we state a number of propositions:

(1) Let F_1 be a filled non empty finite topology space, A be a subset of F_1, and n, m be natural numbers. If $n \leqslant m$, then $\mathrm{Finf}(A, n) \subseteq \mathrm{Finf}(A, m)$.

(2) Let F_1 be a filled non empty finite topology space, A be a subset of F_1, and n, m be natural numbers. If $n \leqslant m$, then $\mathrm{Fcl}(A, n) \subseteq \mathrm{Fcl}(A, m)$.

(3) Let F_1 be a filled non empty finite topology space, A be a subset of F_1, and n, m be natural numbers. If $n \leqslant m$, then $\mathrm{Fdfl}(A, m) \subseteq \mathrm{Fdfl}(A, n)$.

(4) Let F_1 be a filled non empty finite topology space, A be a subset of F_1, and n, m be natural numbers. If $n \leqslant m$, then $\mathrm{Fint}(A, m) \subseteq \mathrm{Fint}(A, n)$.

(5) Let F_1 be a non empty finite topology space and A, B be subsets of F_1. If A and B are separated, then B and A are separated.

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(6) Let F_1 be a filled non empty finite topology space and A, B be subsets of F_1. If A and B are separated, then A misses B.

(7) Let F_1 be a non empty finite topology space and A, B be subsets of F_1. Suppose F_1 is symmetric. Then A and B are separated if and only if A^f misses B and A misses B^f.

(8) Let F_1 be a filled non empty finite topology space and A, B be subsets of F_1. If F_1 is symmetric and A^b misses B, then A misses B^b.

(9) Let F_1 be a filled non empty finite topology space and A, B be subsets of F_1. If F_1 is symmetric and A misses B^b, then A^b misses B.

(10) Let F_1 be a filled non empty finite topology space and A, B be subsets of F_1. Suppose F_1 is symmetric. Then A and B are separated if and only if A^b misses B.

(11) Let F_1 be a filled non empty finite topology space and A, B be subsets of F_1. Suppose F_1 is symmetric. Then A and B are separated if and only if A misses B^b.

(12) Let F_1 be a filled non empty finite topology space and I_1 be a subset of F_1. Suppose F_1 is symmetric. Then I_1 is connected if and only if for all subsets A, B of F_1 such that $I_1 = A \cup B$ and A and B are separated holds $A = I_1$ or $B = I_1$.

(13) Let F_1 be a filled non empty finite topology space and B be a subset of F_1. Suppose F_1 is symmetric. Then B is connected if and only if it is not true that there exists a subset C of F_1 such that $C \neq \emptyset$ and $B \setminus C \neq \emptyset$ and $C \subseteq B$ and C^b misses $B \setminus C$.

Let F_2, F_3 be non empty finite topology spaces, let f be a function from the carrier of F_2 into the carrier of F_3, and let n be a natural number. We say that f is continuous n if and only if:

(Def. 2) For every element x of F_2 and for every element y of F_3 such that $x \in$ the carrier of F_2 and $y = f(x)$ holds $f^\circ U(x, 0) \subseteq U(y, n)$.

Next we state four propositions:

(14) Let F_2 be a non empty finite topology space, F_3 be a filled non empty finite topology space, n be a natural number, and f be a function from the carrier of F_2 into the carrier of F_3. If f is continuous 0, then f is continuous n.

(15) Let F_2 be a non empty finite topology space, F_3 be a filled non empty finite topology space, n_0, n be natural numbers, and f be a function from the carrier of F_2 into the carrier of F_3. If f is continuous n_0 and $n_0 \leqslant n$, then f is continuous n.

(16) Let F_2, F_3 be non empty finite topology spaces, A be a subset of F_2, B be a subset of F_3, and f be a function from the carrier of F_2 into the carrier of F_3. If f is continuous 0 and $B = f^\circ A$, then $f^\circ A^b \subseteq B^b$.

(17) Let F_2, F_3 be non empty finite topology spaces, A be a subset of F_2, B be a subset of F_3, and f be a function from the carrier of F_2 into the carrier of F_3. Suppose A is connected and f is continuous 0 and $B = f^\circ A$. Then B is connected.

Let n be a natural number. The functor $\mathrm{Nbdl1}(n)$ yielding a function from $\mathrm{Seg}\,n$ into $2^{\mathrm{Seg}\,n}$ is defined as follows:

(Def. 3) $\mathrm{dom}\,\mathrm{Nbdl1}(n) = \mathrm{Seg}\,n$ and for every natural number i such that $i \in \mathrm{Seg}\,n$ holds $(\mathrm{Nbdl1}(n))(i) = \{i, \max(i -' 1, 1), \min(i + 1, n)\}$.

Let n be a natural number. Let us assume that $n > 0$. The functor $\mathrm{FTSL1}(n)$ yielding a non empty finite topology space is defined as follows:

(Def. 4) $\mathrm{FTSL1}(n) = \langle \mathrm{Seg}\,n, \mathrm{Nbdl1}(n) \rangle$.

We now state two propositions:

(18) For every natural number n such that $n > 0$ holds $\mathrm{FTSL1}(n)$ is filled.

(19) For every natural number n such that $n > 0$ holds $\mathrm{FTSL1}(n)$ is symmetric.

Let n be a natural number. The functor $\mathrm{Nbdc1}(n)$ yielding a function from $\mathrm{Seg}\,n$ into $2^{\mathrm{Seg}\,n}$ is defined by the conditions (Def. 5).

(Def. 5)(i) $\mathrm{dom}\,\mathrm{Nbdc1}(n) = \mathrm{Seg}\,n$, and

(ii) for every natural number i such that $i \in \mathrm{Seg}\,n$ holds if $1 < i$ and $i < n$, then $(\mathrm{Nbdc1}(n))(i) = \{i, i -' 1, i + 1\}$ and if $i = 1$ and $i < n$, then $(\mathrm{Nbdc1}(n))(i) = \{i, n, i + 1\}$ and if $1 < i$ and $i = n$, then $(\mathrm{Nbdc1}(n))(i) = \{i, i -' 1, 1\}$ and if $i = 1$ and $i = n$, then $(\mathrm{Nbdc1}(n))(i) = \{i\}$.

Let n be a natural number. Let us assume that $n > 0$. The functor $\mathrm{FTSC1}(n)$ yielding a non empty finite topology space is defined as follows:

(Def. 6) $\mathrm{FTSC1}(n) = \langle \mathrm{Seg}\,n, \mathrm{Nbdc1}(n) \rangle$.

We now state two propositions:

(20) For every natural number n such that $n > 0$ holds $\mathrm{FTSC1}(n)$ is filled.

(21) For every natural number n such that $n > 0$ holds $\mathrm{FTSC1}(n)$ is symmetric.

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The Nagata-Smirnov Theorem. Part II[1]

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Summary. In this paper we show some auxiliary facts for sequence function to be pseudo-metric. Next we prove the Nagata-Smirnov theorem that every topological space is metrizable if and only if it has σ-locally finite basis. We attach also the proof of the Bing's theorem that every topological space is metrizable if and only if its basis is σ-discrete.

MML Identifier: `NAGATA_2`.

The terminology and notation used in this paper have been introduced in the following articles: [9], [27], [28], [32], [20], [5], [12], [8], [21], [15], [2], [17], [14], [18], [19], [6], [10], [11], [24], [23], [4], [33], [1], [3], [25], [16], [26], [7], [13], [29], [31], [34], [30], and [22].

For simplicity, we adopt the following convention: i, k, m, n denote natural numbers, r, s denote real numbers, X denotes a set, T, T_1, T_2 denote non empty topological spaces, p denotes a point of T, A denotes a subset of T, A' denotes a non empty subset of T, p_1 denotes an element of [: the carrier of T, the carrier of T :], p_2 denotes a function from [: the carrier of T, the carrier of T :] into $\mathbb{R}$, p'_1 denotes a real map of [: T, T :], f denotes a real map of T, F_2 denotes a sequence of partial functions from [: the carrier of T, the carrier of T :] into $\mathbb{R}$, and s_1 denotes a sequence of real numbers.

The following proposition is true

(1) For every i such that $i > 0$ there exist n, m such that $i = 2^n \cdot (2 \cdot m + 1)$.

The function PairFunc from [: $\mathbb{N}$, $\mathbb{N}$:] into $\mathbb{N}$ is defined by:

(Def. 1) For all n, m holds $\mathrm{PairFunc}(\langle n, m\rangle) = 2^n \cdot (2 \cdot m + 1) - 1$.

We now state the proposition

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(2) PairFunc is bijective.

Let X be a set, let f be a function from $[\!:X, X:\!]$ into $\mathbb{R}$, and let x be an element of X. The functor $\rho(f, x)$ yielding a function from X into $\mathbb{R}$ is defined as follows:

(Def. 2) For every element y of X holds $(\rho(f, x))(y) = f(x, y)$.

The following two propositions are true:

(3) Let D be a subset of $[\!:T_1, T_2:\!]$. Suppose D is open. Let x_1 be a point of T_1, x_2 be a point of T_2, X_1 be a subset of T_1, and X_2 be a subset of T_2. Then

(i) if $X_1 = \pi_1(($the carrier of $T_1) \times$ the carrier of $T_2)^\circ(D \cap [\!:$the carrier of $T_1, \{x_2\}:\!])$, then X_1 is open, and

(ii) if $X_2 = \pi_2(($the carrier of $T_1) \times$ the carrier of $T_2)^\circ(D \cap [\!:\{x_1\},$ the carrier of $T_2:\!])$, then X_2 is open.

(4) For every p_2 such that for every p'_1 such that $p_2 = p'_1$ holds p'_1 is continuous and for every point x of T holds $\rho(p_2, x)$ is continuous.

Let X be a non empty set, let f be a function from $[\!:X, X:\!]$ into $\mathbb{R}$, and let A be a subset of X. The functor $\inf(f, A)$ yielding a function from X into $\mathbb{R}$ is defined by:

(Def. 3) For every element x of X holds $(\inf(f, A))(x) = \inf((\rho(f, x))^\circ A)$.

One can prove the following propositions:

(5) Let X be a non empty set and f be a function from $[\!:X, X:\!]$ into $\mathbb{R}$. Suppose f is a pseudometric of. Let A be a non empty subset of X and x be an element of X. Then $(\inf(f, A))(x) \geqslant 0$.

(6) Let X be a non empty set and f be a function from $[\!:X, X:\!]$ into $\mathbb{R}$. Suppose f is a pseudometric of. Let A be a subset of X and x be an element of X. If $x \in A$, then $(\inf(f, A))(x) = 0$.

(7) Let given p_2. Suppose p_2 is a pseudometric of. Let x, y be points of T and A be a non empty subset of T. Then $|(\inf(p_2, A))(x) - (\inf(p_2, A))(y)| \leqslant p_2(x, y)$.

(8) Let given p_2. Suppose p_2 is a pseudometric of and for every p holds $\rho(p_2, p)$ is continuous. Let A be a non empty subset of T. Then $\inf(p_2, A)$ is continuous.

(9) For every function f from $[\!:X, X:\!]$ into $\mathbb{R}$ such that f is a metric of X holds f is a pseudometric of.

(10) Let given p_2. Suppose p_2 is a metric of the carrier of T and for every non empty subset A of T holds $\overline{A} = \{p; p$ ranges over points of T: $(\inf(p_2, A))(p) = 0\}$. Then T is metrizable.

(11) Let given F_2. Suppose for every n there exists p_2 such that $F_2(n) = p_2$ and p_2 is a pseudometric of and for every p_1 holds $F_2 \# p_1$ is summable.

Let given p_2. If for every p_1 holds $p_2(p_1) = \sum(F_2 \# p_1)$, then p_2 is a pseudometric of.

(12) For all n, s_1 such that for every m such that $m \leqslant n$ holds $s_1(m) \leqslant r$ and for every m such that $m \leqslant n$ holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) \leqslant r \cdot (m+1)$.

(13) For every k holds $|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(k)| \leqslant (\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(14) Let F_1 be a sequence of partial functions from the carrier of T into $\mathbb{R}$. Suppose that

(i) for every n there exists f such that $F_1(n) = f$ and f is continuous and for every p holds $f(p) \geqslant 0$, and

(ii) there exists s_1 such that s_1 is summable and for all n, p holds $(F_1 \# p)(n) \leqslant s_1(n)$.

Let given f. If for every p holds $f(p) = \sum(F_1 \# p)$, then f is continuous.

(15) Let given s, F_2. Suppose that for every n there exists p_2 such that $F_2(n) = p_2$ and p_2 is a pseudometric of and for every p_1 holds $p_2(p_1) \leqslant s$ and for every p'_1 such that $p_2 = p'_1$ holds p'_1 is continuous. Let given p_2. Suppose that for every p_1 holds $p_2(p_1) = \sum(((\frac{1}{2})^{\kappa})_{\kappa\in\mathbb{N}} (F_2 \# p_1))$. Then p_2 is a pseudometric of and for every p'_1 such that $p_2 = p'_1$ holds p'_1 is continuous.

(16) Let given p_2. Suppose p_2 is a pseudometric of and for every p'_1 such that $p_2 = p'_1$ holds p'_1 is continuous. Let A be a non empty subset of T and given p. If $p \in \overline{A}$, then $(\inf(p_2, A))(p) = 0$.

(17) Let given T. Suppose T is a T_1 space. Let given s, F_2. Suppose that

(i) for every n there exists p_2 such that $F_2(n) = p_2$ and p_2 is a pseudometric of and for every p_1 holds $p_2(p_1) \leqslant s$ and for every p'_1 such that $p_2 = p'_1$ holds p'_1 is continuous, and

(ii) for all p, A' such that $p \notin A'$ and A' is closed there exists n such that for every p_2 such that $F_2(n) = p_2$ holds $(\inf(p_2, A'))(p) > 0$.

Then there exists p_2 such that p_2 is a metric of the carrier of T and for every p_1 holds $p_2(p_1) = \sum(((\frac{1}{2})^{\kappa})_{\kappa\in\mathbb{N}} (F_2 \# p_1))$ and T is metrizable.

(18) Let D be a non empty set, p, q be finite sequences of elements of D, and B be a binary operation on D. Suppose that

(i) p is one-to-one,

(ii) q is one-to-one,

(iii) $\operatorname{rng} q \subseteq \operatorname{rng} p$,

(iv) B is commutative and associative, and

(v) B has a unity or $\operatorname{len} q \geqslant 1$ and $\operatorname{len} p > \operatorname{len} q$.

Then there exists a finite sequence r of elements of D such that r is one-to-one and $\operatorname{rng} r = \operatorname{rng} p \setminus \operatorname{rng} q$ and $B \odot p = B(B \odot q, B \odot r)$.

(19) Let given T. Then T is a T_3 space and a T_1 space and there exists a family sequence of T which is Basis-sigma-locally finite if and only if T is metrizable.

(20) Suppose T is metrizable. Let F_3 be a family of subsets of T. Suppose F_3 is a cover of T and open. Then there exists a family sequence U_1 of T such that $\bigcup U_1$ is open and $\bigcup U_1$ is a cover of T and $\bigcup U_1$ is finer than F_3 and U_1 is sigma-discrete.

(21) For every T such that T is metrizable holds there exists a family sequence of T which is Basis-sigma-discrete.

(22) For every T holds T is a T_3 space and a T_1 space and there exists a family sequence of T which is Basis-sigma-discrete iff T is metrizable.

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On the Isomorphism of Fundamental Groups[1]

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The terminology and notation used here have been introduced in the following articles: [24], [7], [27], [28], [22], [4], [29], [5], [2], [18], [23], [3], [6], [21], [19], [26], [25], [9], [8], [20], [16], [11], [10], [1], [13], [14], [12], [15], and [17].

1. Preliminaries

One can prove the following propositions:

(1) Let A, B, a, b be sets and f be a function from A into B. If $a \in A$ and $b \in B$, then $f{+\cdot}(a \longmapsto b)$ is a function from A into B.

(2) For every function f and for all sets X, x such that $f{\restriction}X$ is one-to-one and $x \in \operatorname{rng}(f{\restriction}X)$ holds $(f \cdot (f{\restriction}X)^{-1})(x) = x$.

(3) Let x, y, X, Y, Z be sets, f be a function from $[\!: X, Y :\!]$ into Z, and g be a function. If $Z \neq \emptyset$ and $x \in X$ and $y \in Y$, then $(g \cdot f)(x, y) = g(f(x, y))$.

(4) For all sets X, a, b and for every function f from X into $\{a, b\}$ holds $X = f^{-1}(\{a\}) \cup f^{-1}(\{b\})$.

(5) For all non empty 1-sorted structures S, T and for every point s of S and for every point t of T holds $(S \longmapsto t)(s) = t$.

(6) Let T be a non empty topological structure, t be a point of T, and A be a subset of T. If $A = \{t\}$, then $\operatorname{Sspace}(t) = T{\restriction}A$.

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(7) Let T be a topological space, A, B be subsets of T, and C, D be subsets of the topological structure of T. Suppose $A = C$ and $B = D$. Then A and B are separated if and only if C and D are separated.

(8) For every non empty topological space T holds T is connected iff there exists no map from T into $\{\{0,1\}\}_{\rm top}$ which is continuous and onto.

One can verify that every topological structure which is empty is also connected.

We now state the proposition

(9) For every topological space T such that the topological structure of T is connected holds T is connected.

Let T be a connected topological space. One can check that the topological structure of T is connected.

One can prove the following proposition

(10) Let S, T be non empty topological spaces. Suppose S and T are homeomorphic and S is arcwise connected. Then T is arcwise connected.

One can verify that every non empty topological space which is trivial is also arcwise connected.

One can prove the following propositions:

(11) For every subspace T of $\mathcal{E}^2_{\rm T}$ such that the carrier of T is a simple closed curve holds T is arcwise connected.

(12) Let T be a topological space. Then there exists a family F of subsets of T such that $F = \{$the carrier of $T\}$ and F is a cover of T and open.

Let T be a topological space. Note that there exists a family of subsets of T which is non empty, mutually-disjoint, open, and closed.

The following proposition is true

(13) Let T be a topological space, D be a mutually-disjoint open family of subsets of T, A be a subset of T, and X be a set. If A is connected and $X \in D$ and X meets A and D is a cover of A, then $A \subseteq X$.

2. On the Product of Topologies

One can prove the following three propositions:

(14) Let S, T be topological spaces. Then the topological structure of $[\!:S, T:\!] = [\!:$the topological structure of S, the topological structure of $T:\!]$.

(15) For all topological spaces S, T and for every subset A of S and for every subset B of T holds $\overline{[\!:A, B:\!]} = [\!:\overline{A}, \overline{B}:\!]$.

(16) Let S, T be topological spaces, A be a closed subset of S, and B be a closed subset of T. Then $[\!:A, B:\!]$ is closed.

Let A, B be connected topological spaces. One can check that $[: A, B :]$ is connected.

One can prove the following propositions:

(17) Let S, T be topological spaces, A be a subset of S, and B be a subset of T. If A is connected and B is connected, then $[: A, B :]$ is connected.

(18) Let S, T be topological spaces, Y be a non empty topological space, A be a subset of S, f be a map from $[: S, T :]$ into Y, and g be a map from $[: S \restriction A, T :]$ into Y. If $g = f \restriction [: A$, the carrier of $T :]$ and f is continuous, then g is continuous.

(19) Let S, T be topological spaces, Y be a non empty topological space, A be a subset of T, f be a map from $[: S, T :]$ into Y, and g be a map from $[: S, T \restriction A :]$ into Y. If $g = f \restriction [:$ the carrier of S, $A :]$ and f is continuous, then g is continuous.

(20) Let S, T, T_1, T_2, Y be non empty topological spaces, f be a map from $[: Y, T_1 :]$ into S, g be a map from $[: Y, T_2 :]$ into S, and F_1, F_2 be closed subsets of T. Suppose that T_1 is a subspace of T and T_2 is a subspace of T and $F_1 = \Omega_{(T_1)}$ and $F_2 = \Omega_{(T_2)}$ and $\Omega_{(T_1)} \cup \Omega_{(T_2)} = \Omega_T$ and f is continuous and g is continuous and for every set p such that $p \in \Omega_{[: Y, T_1 :]} \cap \Omega_{[: Y, T_2 :]}$ holds $f(p) = g(p)$. Then there exists a map h from $[: Y, T :]$ into S such that $h = f {+\cdot} g$ and h is continuous.

(21) Let S, T, T_1, T_2, Y be non empty topological spaces, f be a map from $[: T_1, Y :]$ into S, g be a map from $[: T_2, Y :]$ into S, and F_1, F_2 be closed subsets of T. Suppose that T_1 is a subspace of T and T_2 is a subspace of T and $F_1 = \Omega_{(T_1)}$ and $F_2 = \Omega_{(T_2)}$ and $\Omega_{(T_1)} \cup \Omega_{(T_2)} = \Omega_T$ and f is continuous and g is continuous and for every set p such that $p \in \Omega_{[: T_1, Y :]} \cap \Omega_{[: T_2, Y :]}$ holds $f(p) = g(p)$. Then there exists a map h from $[: T, Y :]$ into S such that $h = f {+\cdot} g$ and h is continuous.

3. On the Fundamental Groups

Let T be a non empty topological space and let t be a point of T. Observe that every loop of t is continuous.

We now state a number of propositions:

(22) Let T be a non empty topological space, t be a point of T, x be a point of $\mathbb{I}$, and P be a constant loop of t. Then $P(x) = t$.

(23) For every non empty topological space T and for every point t of T and for every loop P of t holds $P(0) = t$ and $P(1) = t$.

(24) Let S, T be non empty topological spaces, f be a continuous map from S into T, and a, b be points of S. If a, b are connected, then $f(a)$, $f(b)$ are connected.

(25) Let S, T be non empty topological spaces, f be a continuous map from S into T, a, b be points of S, and P be a path from a to b. If a, b are connected, then $f \cdot P$ is a path from $f(a)$ to $f(b)$.

(26) Let S be a non empty arcwise connected topological space, T be a non empty topological space, f be a continuous map from S into T, a, b be points of S, and P be a path from a to b. Then $f \cdot P$ is a path from $f(a)$ to $f(b)$.

(27) Let S, T be non empty topological spaces, f be a continuous map from S into T, a be a point of S, and P be a loop of a. Then $f \cdot P$ is a loop of $f(a)$.

(28) Let S, T be non empty topological spaces, f be a continuous map from S into T, a, b be points of S, P, Q be paths from a to b, and P_1, Q_1 be paths from $f(a)$ to $f(b)$. Suppose P, Q are homotopic and $P_1 = f \cdot P$ and $Q_1 = f \cdot Q$. Then P_1, Q_1 are homotopic.

(29) Let S, T be non empty topological spaces, f be a continuous map from S into T, a, b be points of S, P, Q be paths from a to b, P_1, Q_1 be paths from $f(a)$ to $f(b)$, and F be a homotopy between P and Q. Suppose P, Q are homotopic and $P_1 = f \cdot P$ and $Q_1 = f \cdot Q$. Then $f \cdot F$ is a homotopy between P_1 and Q_1.

(30) Let S, T be non empty topological spaces, f be a continuous map from S into T, a, b, c be points of S, P be a path from a to b, Q be a path from b to c, P_1 be a path from $f(a)$ to $f(b)$, and Q_1 be a path from $f(b)$ to $f(c)$. Suppose a, b are connected and b, c are connected and $P_1 = f \cdot P$ and $Q_1 = f \cdot Q$. Then $P_1 + Q_1 = f \cdot (P + Q)$.

(31) Let S be a non empty topological space, s be a point of S, x, y be elements of $\pi_1(S, s)$, and P, Q be loops of s. If $x = [P]_{\mathrm{EqRel}(S,s)}$ and $y = [Q]_{\mathrm{EqRel}(S,s)}$, then $x \cdot y = [P + Q]_{\mathrm{EqRel}(S,s)}$.

Let S, T be non empty topological spaces, let s be a point of S, and let f be a map from S into T. Let us assume that f is continuous. The functor $\mathrm{FundGrIso}(f, s)$ yielding a map from $\pi_1(S, s)$ into $\pi_1(T, f(s))$ is defined by the condition (Def. 1).

(Def. 1) Let x be an element of $\pi_1(S, s)$. Then there exists a loop l_1 of s and there exists a loop l_2 of $f(s)$ such that $x = [l_1]_{\mathrm{EqRel}(S,s)}$ and $l_2 = f \cdot l_1$ and $(\mathrm{FundGrIso}(f, s))(x) = [l_2]_{\mathrm{EqRel}(T,f(s))}$.

The following proposition is true

(32) Let S, T be non empty topological spaces, s be a point of S, f be a continuous map from S into T, x be an element of $\pi_1(S, s)$, l_1 be a loop of s, and l_2 be a loop of $f(s)$. If $x = [l_1]_{\mathrm{EqRel}(S,s)}$ and $l_2 = f \cdot l_1$, then $(\mathrm{FundGrIso}(f, s))(x) = [l_2]_{\mathrm{EqRel}(T,f(s))}$.

Let S, T be non empty topological spaces, let s be a point of S, and let f

be a continuous map from S into T. Then FundGrIso(f, s) is a homomorphism from $\pi_1(S, s)$ to $\pi_1(T, f(s))$.

We now state three propositions:

(33) Let S, T be non empty topological spaces, s be a point of S, and f be a continuous map from S into T. If f is a homeomorphism, then FundGrIso(f, s) is an isomorphism.

(34) Let S, T be non empty topological spaces, s be a point of S, t be a point of T, f be a continuous map from S into T, P be a path from t to $f(s)$, and h be a homomorphism from $\pi_1(S, s)$ to $\pi_1(T, t)$. Suppose f is a homeomorphism and $f(s)$, t are connected and $h = \pi_1\text{-iso}(P) \cdot$ FundGrIso(f, s). Then h is an isomorphism.

(35) Let S be a non empty topological space, T be a non empty arcwise connected topological space, s be a point of S, and t be a point of T. If S and T are homeomorphic, then $\pi_1(S, s)$ and $\pi_1(T, t)$ are isomorphic.

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Algebra of Complex Vector Valued Functions

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Summary. This article is an extension of [17].

MML Identifier: VFUNCT_2.

The notation and terminology used here have been introduced in the following papers: [12], [15], [2], [11], [4], [16], [5], [7], [14], [9], [8], [3], [1], [13], [10], and [6].

For simplicity, we follow the rules: M denotes a non empty set, V denotes a complex normed space, f, f_1, f_2, f_3 denote partial functions from M to the carrier of V, and z, z_1, z_2 denote complex numbers.

Let M be a non empty set, let V be a complex normed space, and let f_1, f_2 be partial functions from M to the carrier of V. The functor $f_1 + f_2$ yields a partial function from M to the carrier of V and is defined by:

(Def. 1) $\mathrm{dom}(f_1 + f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every element c of M such that $c \in \mathrm{dom}(f_1 + f_2)$ holds $(f_1 + f_2)_c = (f_1)_c + (f_2)_c$.

The functor $f_1 - f_2$ yields a partial function from M to the carrier of V and is defined as follows:

(Def. 2) $\mathrm{dom}(f_1 - f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every element c of M such that $c \in \mathrm{dom}(f_1 - f_2)$ holds $(f_1 - f_2)_c = (f_1)_c - (f_2)_c$.

Let M be a non empty set, let V be a complex normed space, let f_1 be a partial function from M to $\mathbb{C}$, and let f_2 be a partial function from M to the carrier of V. The functor $f_1\, f_2$ yielding a partial function from M to the carrier of V is defined by:

(Def. 3) $\mathrm{dom}(f_1\, f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every element c of M such that $c \in \mathrm{dom}(f_1\, f_2)$ holds $(f_1\, f_2)_c = (f_1)_c \cdot (f_2)_c$.

Let X be a non empty set, let V be a complex normed space, let f be a partial function from X to the carrier of V, and let z be a complex number. The

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functor $z\,f$ yields a partial function from X to the carrier of V and is defined as follows:

(Def. 4) $\mathrm{dom}(z\,f) = \mathrm{dom}\,f$ and for every element x of X such that $x \in \mathrm{dom}(z\,f)$ holds $(z\,f)_x = z \cdot f_x$.

Let X be a non empty set, let V be a complex normed space, and let f be a partial function from X to the carrier of V. The functor $\|f\|$ yielding a partial function from X to $\mathbb{R}$ is defined as follows:

(Def. 5) $\mathrm{dom}\|f\| = \mathrm{dom}\,f$ and for every element x of X such that $x \in \mathrm{dom}\|f\|$ holds $\|f\|(x) = \|f_x\|$.

The functor $-f$ yields a partial function from X to the carrier of V and is defined by:

(Def. 6) $\mathrm{dom}(-f) = \mathrm{dom}\,f$ and for every element x of X such that $x \in \mathrm{dom}(-f)$ holds $(-f)_x = -f_x$.

The following propositions are true:

(1) Let f_1 be a partial function from M to $\mathbb{C}$ and f_2 be a partial function from M to the carrier of V. Then $\mathrm{dom}(f_1\,f_2) \setminus (f_1\,f_2)^{-1}(\{0_V\}) = (\mathrm{dom}\,f_1 \setminus {f_1}^{-1}(\{0\})) \cap (\mathrm{dom}\,f_2 \setminus {f_2}^{-1}(\{0_V\}))$.

(2) $\|f\|^{-1}(\{0\}) = f^{-1}(\{0_V\})$ and $(-f)^{-1}(\{0_V\}) = f^{-1}(\{0_V\})$.

(3) If $z \neq 0_{\mathbb{C}}$, then $(z\,f)^{-1}(\{0_V\}) = f^{-1}(\{0_V\})$.

(4) $f_1 + f_2 = f_2 + f_1$.

(5) $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(6) Let f_1, f_2 be partial functions from M to $\mathbb{C}$ and f_3 be a partial function from M to the carrier of V. Then $(f_1\,f_2)\,f_3 = f_1\,(f_2\,f_3)$.

(7) For all partial functions f_1, f_2 from M to $\mathbb{C}$ holds $(f_1 + f_2)\,f_3 = f_1\,f_3 + f_2\,f_3$.

(8) For every partial function f_3 from M to $\mathbb{C}$ holds $f_3\,(f_1 + f_2) = f_3\,f_1 + f_3\,f_2$.

(9) For every partial function f_1 from M to $\mathbb{C}$ holds $z\,(f_1\,f_2) = (z\,f_1)\,f_2$.

(10) For every partial function f_1 from M to $\mathbb{C}$ holds $z\,(f_1\,f_2) = f_1\,(z\,f_2)$.

(11) For all partial functions f_1, f_2 from M to $\mathbb{C}$ holds $(f_1 - f_2)\,f_3 = f_1\,f_3 - f_2\,f_3$.

(12) For every partial function f_3 from M to $\mathbb{C}$ holds $f_3\,f_1 - f_3\,f_2 = f_3\,(f_1 - f_2)$.

(13) $z\,(f_1 + f_2) = z\,f_1 + z\,f_2$.

(14) $(z_1 \cdot z_2)\,f = z_1\,(z_2\,f)$.

(15) $z\,(f_1 - f_2) = z\,f_1 - z\,f_2$.

(16) $f_1 - f_2 = (-1_{\mathbb{C}})\,(f_2 - f_1)$.

(17) $f_1 - (f_2 + f_3) = f_1 - f_2 - f_3$.

(18) $1_{\mathbb{C}} f = f$.

(19) $f_1 - (f_2 - f_3) = (f_1 - f_2) + f_3$.

(20) $f_1 + (f_2 - f_3) = (f_1 + f_2) - f_3$.

(21) For every partial function f_1 from M to $\mathbb{C}$ holds $\|f_1\, f_2\| = |f_1|\, \|f_2\|$.

(22) $\|z\, f\| = |z|\, \|f\|$.

(23) $-f = (-1_{\mathbb{C}})\, f$.

(24) $--f = f$.

(25) $f_1 - f_2 = f_1 + -f_2$.

(26) $f_1 - -f_2 = f_1 + f_2$.

In the sequel X, Y denote sets.

We now state a number of propositions:

(27) $(f_1+f_2){\restriction}X = f_1{\restriction}X+f_2{\restriction}X$ and $(f_1+f_2){\restriction}X = f_1{\restriction}X+f_2$ and $(f_1+f_2){\restriction}X = f_1 + f_2{\restriction}X$.

(28) For every partial function f_1 from M to $\mathbb{C}$ holds $(f_1\, f_2){\restriction}X = (f_1{\restriction}X)\,(f_2{\restriction}X)$ and $(f_1\, f_2){\restriction}X = (f_1{\restriction}X)\, f_2$ and $(f_1\, f_2){\restriction}X = f_1\,(f_2{\restriction}X)$.

(29) $(-f){\restriction}X = -f{\restriction}X$ and $\|f\|{\restriction}X = \|f{\restriction}X\|$.

(30) $(f_1-f_2){\restriction}X = f_1{\restriction}X-f_2{\restriction}X$ and $(f_1-f_2){\restriction}X = f_1{\restriction}X-f_2$ and $(f_1-f_2){\restriction}X = f_1 - f_2{\restriction}X$.

(31) $(z\, f){\restriction}X = z\,(f{\restriction}X)$.

(32) f_1 is total and f_2 is total iff $f_1 + f_2$ is total and f_1 is total and f_2 is total iff $f_1 - f_2$ is total.

(33) For every partial function f_1 from M to $\mathbb{C}$ holds f_1 is total and f_2 is total iff $f_1\, f_2$ is total.

(34) f is total iff $z\, f$ is total.

(35) f is total iff $-f$ is total.

(36) f is total iff $\|f\|$ is total.

(37) For every element x of M such that f_1 is total and f_2 is total holds $(f_1 + f_2)_x = (f_1)_x + (f_2)_x$ and $(f_1 - f_2)_x = (f_1)_x - (f_2)_x$.

(38) Let f_1 be a partial function from M to $\mathbb{C}$ and x be an element of M. If f_1 is total and f_2 is total, then $(f_1\, f_2)_x = (f_1)_x \cdot (f_2)_x$.

(39) For every element x of M such that f is total holds $(z\, f)_x = z \cdot f_x$.

(40) For every element x of M such that f is total holds $(-f)_x = -f_x$ and $\|f\|(x) = \|f_x\|$.

Let us consider M, let us consider V, and let us consider f, Y. We say that f is bounded on Y if and only if:

(Def. 7) There exists a real number r such that for every element x of M such that $x \in Y \cap \operatorname{dom} f$ holds $\|f_x\| \leqslant r$.

One can prove the following propositions:

(41) If $Y \subseteq X$ and f is bounded on X, then f is bounded on Y.

(42) If X misses $\operatorname{dom} f$, then f is bounded on X.

(43) $0_{\mathbb{C}}\, f$ is bounded on Y.

(44) If f is bounded on Y, then $z\, f$ is bounded on Y.

(45) If f is bounded on Y, then $\|f\|$ is bounded on Y and $-f$ is bounded on Y.

(46) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(47) For every partial function f_1 from M to $\mathbb{C}$ such that f_1 is bounded on X and f_2 is bounded on Y holds $f_1\, f_2$ is bounded on $X \cap Y$.

(48) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 - f_2$ is bounded on $X \cap Y$.

(49) If f is bounded on X and bounded on Y, then f is bounded on $X \cup Y$.

(50) If f_1 is a constant on X and f_2 is a constant on Y, then $f_1 + f_2$ is a constant on $X \cap Y$ and $f_1 - f_2$ is a constant on $X \cap Y$.

(51) Let f_1 be a partial function from M to $\mathbb{C}$. Suppose f_1 is a constant on X and f_2 is a constant on Y. Then $f_1\, f_2$ is a constant on $X \cap Y$.

(52) If f is a constant on Y, then $z\, f$ is a constant on Y.

(53) If f is a constant on Y, then $\|f\|$ is a constant on Y and $-f$ is a constant on Y.

(54) If f is a constant on Y, then f is bounded on Y.

(55) If f is a constant on Y, then for every z holds $z\, f$ is bounded on Y and $-f$ is bounded on Y and $\|f\|$ is bounded on Y.

(56) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(57) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is bounded on $X \cap Y$ and $f_2 - f_1$ is bounded on $X \cap Y$.

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Continuous Functions on Real and Complex Normed Linear Spaces

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Summary. This article is an extension of [18].

MML Identifier: `NCFCONT1`.

The notation and terminology used here are introduced in the following papers: [25], [28], [29], [4], [30], [6], [14], [5], [2], [24], [10], [26], [27], [19], [15], [12], [13], [11], [31], [20], [3], [1], [16], [21], [17], [23], [7], [8], [22], [18], and [9].

For simplicity, we use the following convention: n denotes a natural number, r, s denote real numbers, z denotes a complex number, C_1, C_2, C_3 denote complex normed spaces, and R_1 denotes a real normed space.

Let C_4 be a complex linear space and let s_1 be a sequence of C_4. The functor $-s_1$ yields a sequence of C_4 and is defined by:

(Def. 1) For every n holds $(-s_1)(n) = -s_1(n)$.

The following propositions are true:

(1) For all sequences s_2, s_3 of C_1 holds $s_2 - s_3 = s_2 + -s_3$.

(2) For every sequence s_1 of C_1 holds $-s_1 = (-1_{\mathbb{C}}) \cdot s_1$.

Let us consider C_2, C_3 and let f be a partial function from C_2 to C_3. The functor $\|f\|$ yielding a partial function from the carrier of C_2 to $\mathbb{R}$ is defined by:

(Def. 2) $\mathrm{dom}\|f\| = \mathrm{dom}\, f$ and for every point c of C_2 such that $c \in \mathrm{dom}\|f\|$ holds $\|f\|(c) = \|f_c\|$.

Let us consider C_1, R_1 and let f be a partial function from C_1 to R_1. The functor $\|f\|$ yielding a partial function from the carrier of C_1 to $\mathbb{R}$ is defined as follows:

(Def. 3) $\mathrm{dom}\|f\| = \mathrm{dom}\, f$ and for every point c of C_1 such that $c \in \mathrm{dom}\|f\|$ holds $\|f\|(c) = \|f_c\|$.

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Let us consider R_1, C_1 and let f be a partial function from R_1 to C_1. The functor $\|f\|$ yielding a partial function from the carrier of R_1 to $\mathbb{R}$ is defined by:

(Def. 4) $\operatorname{dom}\|f\| = \operatorname{dom} f$ and for every point c of R_1 such that $c \in \operatorname{dom}\|f\|$ holds $\|f\|(c) = \|f_c\|$.

Let us consider C_1 and let x_0 be a point of C_1. A subset of C_1 is called a neighbourhood of x_0 if:

(Def. 5) There exists a real number g such that $0 < g$ and $\{y; y$ ranges over points of C_1: $\|y - x_0\| < g\} \subseteq$ it.

Next we state two propositions:

(3) Let x_0 be a point of C_1 and g be a real number. If $0 < g$, then $\{y; y$ ranges over points of C_1: $\|y - x_0\| < g\}$ is a neighbourhood of x_0.

(4) For every point x_0 of C_1 and for every neighbourhood N of x_0 holds $x_0 \in N$.

Let us consider C_1 and let X be a subset of C_1. We say that X is compact if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let s_4 be a sequence of C_1. Suppose $\operatorname{rng} s_4 \subseteq X$. Then there exists a sequence s_5 of C_1 such that s_5 is a subsequence of s_4 and convergent and $\lim s_5 \in X$.

Let us consider C_1 and let X be a subset of C_1. We say that X is closed if and only if:

(Def. 7) For every sequence s_4 of C_1 such that $\operatorname{rng} s_4 \subseteq X$ and s_4 is convergent holds $\lim s_4 \in X$.

Let us consider C_1 and let X be a subset of C_1. We say that X is open if and only if:

(Def. 8) X^{c} is closed.

Let us consider C_2, C_3, let f be a partial function from C_2 to C_3, and let s_1 be a sequence of C_2. Let us assume that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$. The functor $f \cdot s_1$ yields a sequence of C_3 and is defined by:

(Def. 9) $f \cdot s_1 = (f$ **qua** function$) \cdot (s_1)$.

Let us consider C_1, R_1, let f be a partial function from C_1 to R_1, and let s_1 be a sequence of C_1. Let us assume that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$. The functor $f \cdot s_1$ yielding a sequence of R_1 is defined by:

(Def. 10) $f \cdot s_1 = (f$ **qua** function$) \cdot (s_1)$.

Let us consider C_1, R_1, let f be a partial function from R_1 to C_1, and let s_1 be a sequence of R_1. Let us assume that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$. The functor $f \cdot s_1$ yields a sequence of C_1 and is defined by:

(Def. 11) $f \cdot s_1 = (f$ **qua** function$) \cdot (s_1)$.

Let us consider C_1, let f be a partial function from the carrier of C_1 to $\mathbb{C}$, and let s_1 be a sequence of C_1. Let us assume that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$. The functor

$f \cdot s_1$ yields a complex sequence and is defined as follows:

(Def. 12) $f \cdot s_1 = (f$ **qua** function$)\ \cdot(s_1)$.

Let us consider R_1, let f be a partial function from the carrier of R_1 to $\mathbb{C}$, and let s_1 be a sequence of R_1. Let us assume that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$. The functor $f \cdot s_1$ yielding a complex sequence is defined by:

(Def. 13) $f \cdot s_1 = (f$ **qua** function$)\ \cdot(s_1)$.

Let us consider C_1, let f be a partial function from the carrier of C_1 to $\mathbb{R}$, and let s_1 be a sequence of C_1. Let us assume that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$. The functor $f \cdot s_1$ yielding a sequence of real numbers is defined as follows:

(Def. 14) $f \cdot s_1 = (f$ **qua** function$)\ \cdot(s_1)$.

Let us consider C_2, C_3, let f be a partial function from C_2 to C_3, and let x_0 be a point of C_2. We say that f is continuous in x_0 if and only if the conditions (Def. 15) are satisfied.

(Def. 15)(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_1 of C_2 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider C_1, R_1, let f be a partial function from C_1 to R_1, and let x_0 be a point of C_1. We say that f is continuous in x_0 if and only if the conditions (Def. 16) are satisfied.

(Def. 16)(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_1 of C_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider R_1, let us consider C_1, let f be a partial function from R_1 to C_1, and let x_0 be a point of R_1. We say that f is continuous in x_0 if and only if the conditions (Def. 17) are satisfied.

(Def. 17)(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_1 of R_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider C_1, let f be a partial function from the carrier of C_1 to $\mathbb{C}$, and let x_0 be a point of C_1. We say that f is continuous in x_0 if and only if the conditions (Def. 18) are satisfied.

(Def. 18)(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_1 of C_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider C_1, let f be a partial function from the carrier of C_1 to $\mathbb{R}$, and let x_0 be a point of C_1. We say that f is continuous in x_0 if and only if the conditions (Def. 19) are satisfied.

(Def. 19)(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_1 of C_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider R_1, let f be a partial function from the carrier of R_1 to $\mathbb{C}$, and let x_0 be a point of R_1. We say that f is continuous in x_0 if and only if the conditions (Def. 20) are satisfied.

(Def. 20)(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_1 of R_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

The following propositions are true:

(5) For every sequence s_1 of C_2 and for every partial function h from C_2 to C_3 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$ holds $s_1(n) \in \operatorname{dom} h$.

(6) For every sequence s_1 of C_1 and for every partial function h from C_1 to R_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$ holds $s_1(n) \in \operatorname{dom} h$.

(7) For every sequence s_1 of R_1 and for every partial function h from R_1 to C_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$ holds $s_1(n) \in \operatorname{dom} h$.

(8) For every sequence s_1 of C_1 and for every set x holds $x \in \operatorname{rng} s_1$ iff there exists n such that $x = s_1(n)$.

(9) For all sequences s_1, s_2 of C_1 such that s_2 is a subsequence of s_1 holds $\operatorname{rng} s_2 \subseteq \operatorname{rng} s_1$.

(10) Let f be a partial function from C_2 to C_3 and C_5 be a sequence of C_2. If $\operatorname{rng} C_5 \subseteq \operatorname{dom} f$, then for every n holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(11) Let f be a partial function from C_1 to R_1 and C_5 be a sequence of C_1. If $\operatorname{rng} C_5 \subseteq \operatorname{dom} f$, then for every n holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(12) Let f be a partial function from R_1 to C_1 and R_2 be a sequence of R_1. If $\operatorname{rng} R_2 \subseteq \operatorname{dom} f$, then for every n holds $(f \cdot R_2)(n) = f_{R_2(n)}$.

(13) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$ and C_5 be a sequence of C_1. If $\operatorname{rng} C_5 \subseteq \operatorname{dom} f$, then for every n holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(14) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$ and C_5 be a sequence of C_1. If $\operatorname{rng} C_5 \subseteq \operatorname{dom} f$, then for every n holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(15) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$ and R_2 be a sequence of R_1. If $\operatorname{rng} R_2 \subseteq \operatorname{dom} f$, then for every n holds $(f \cdot R_2)(n) = f_{R_2(n)}$.

(16) Let h be a partial function from C_2 to C_3, C_5 be a sequence of C_2, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} C_5 \subseteq \operatorname{dom} h$, then $(h \cdot C_5) \cdot N_1 = h \cdot (C_5 \cdot N_1)$.

(17) Let h be a partial function from C_1 to R_1, C_6 be a sequence of C_1, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} C_6 \subseteq \operatorname{dom} h$, then $(h \cdot C_6) \cdot N_1 = h \cdot (C_6 \cdot N_1)$.

(18) Let h be a partial function from R_1 to C_1, R_3 be a sequence of R_1,

and N_1 be an increasing sequence of naturals. If $\operatorname{rng} R_3 \subseteq \operatorname{dom} h$, then $(h \cdot R_3) \cdot N_1 = h \cdot (R_3 \cdot N_1)$.

(19) Let h be a partial function from the carrier of C_1 to $\mathbb{C}$, C_6 be a sequence of C_1, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} C_6 \subseteq \operatorname{dom} h$, then $(h \cdot C_6) \cdot N_1 = h \cdot (C_6 \cdot N_1)$.

(20) Let h be a partial function from the carrier of C_1 to $\mathbb{R}$, C_6 be a sequence of C_1, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} C_6 \subseteq \operatorname{dom} h$, then $(h \cdot C_6) \cdot N_1 = h \cdot (C_6 \cdot N_1)$.

(21) Let h be a partial function from the carrier of R_1 to $\mathbb{C}$, R_3 be a sequence of R_1, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} R_3 \subseteq \operatorname{dom} h$, then $(h \cdot R_3) \cdot N_1 = h \cdot (R_3 \cdot N_1)$.

(22) Let h be a partial function from C_2 to C_3 and C_7, C_8 be sequences of C_2. If $\operatorname{rng} C_7 \subseteq \operatorname{dom} h$ and C_8 is a subsequence of C_7, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(23) Let h be a partial function from C_1 to R_1 and C_7, C_8 be sequences of C_1. If $\operatorname{rng} C_7 \subseteq \operatorname{dom} h$ and C_8 is a subsequence of C_7, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(24) Let h be a partial function from R_1 to C_1 and R_4, R_5 be sequences of R_1. If $\operatorname{rng} R_4 \subseteq \operatorname{dom} h$ and R_5 is a subsequence of R_4, then $h \cdot R_5$ is a subsequence of $h \cdot R_4$.

(25) Let s_1 be a complex sequence, n be a natural number, and N_2 be an increasing sequence of naturals. Then $(s_1 \cdot N_2)(n) = s_1(N_2(n))$.

(26) Let h be a partial function from the carrier of C_1 to $\mathbb{C}$ and C_7, C_8 be sequences of C_1. If $\operatorname{rng} C_7 \subseteq \operatorname{dom} h$ and C_8 is a subsequence of C_7, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(27) Let h be a partial function from the carrier of C_1 to $\mathbb{R}$ and C_7, C_8 be sequences of C_1. If $\operatorname{rng} C_7 \subseteq \operatorname{dom} h$ and C_8 is a subsequence of C_7, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(28) Let h be a partial function from the carrier of R_1 to $\mathbb{C}$ and R_4, R_5 be sequences of R_1. If $\operatorname{rng} R_4 \subseteq \operatorname{dom} h$ and R_5 is a subsequence of R_4, then $h \cdot R_5$ is a subsequence of $h \cdot R_4$.

(29) Let f be a partial function from C_2 to C_3 and x_0 be a point of C_2. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_2 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(30) Let f be a partial function from C_1 to R_1 and x_0 be a point of C_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(31) Let f be a partial function from R_1 to C_1 and x_0 be a point of R_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:
(i) $x_0 \in \operatorname{dom} f$, and
(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of R_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(32) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$ and x_0 be a point of C_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:
(i) $x_0 \in \operatorname{dom} f$, and
(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(33) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$ and x_0 be a point of C_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:
(i) $x_0 \in \operatorname{dom} f$, and
(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(34) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$ and x_0 be a point of R_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:
(i) $x_0 \in \operatorname{dom} f$, and
(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of R_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(35) Let f be a partial function from C_2 to C_3 and x_0 be a point of C_2. Then f is continuous in x_0 if and only if the following conditions are satisfied:
(i) $x_0 \in \operatorname{dom} f$, and
(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that for every point x_1 of C_2 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_3$.

(36) Let f be a partial function from C_1 to R_1 and x_0 be a point of C_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:
(i) $x_0 \in \operatorname{dom} f$, and
(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that for every point x_1 of C_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_3$.

(37) Let f be a partial function from R_1 to C_1 and x_0 be a point of R_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that for every point x_1 of R_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_3$.

(38) Let f be a partial function from C_2 to C_3 and x_0 be a point of C_2. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_3$.

(39) Let f be a partial function from C_1 to R_1 and x_0 be a point of C_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_3$.

(40) Let f be a partial function from R_1 to C_1 and x_0 be a point of R_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_3$.

(41) Let f be a partial function from C_2 to C_3 and x_0 be a point of C_2. Suppose $x_0 \in \operatorname{dom} f$ and there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$. Then f is continuous in x_0.

(42) Let f be a partial function from C_1 to R_1 and x_0 be a point of C_1. Suppose $x_0 \in \operatorname{dom} f$ and there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$. Then f is continuous in x_0.

(43) Let f be a partial function from R_1 to C_1 and x_0 be a point of R_1. Suppose $x_0 \in \operatorname{dom} f$ and there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$. Then f is continuous in x_0.

(44) Let h_1, h_2 be partial functions from C_2 to C_3 and s_1 be a sequence of C_2. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1$ and $(h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1$.

(45) Let h_1, h_2 be partial functions from C_1 to R_1 and s_1 be a sequence of C_1. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1$ and $(h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1$.

(46) Let h_1, h_2 be partial functions from R_1 to C_1 and s_1 be a sequence of R_1. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1$ and $(h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1$.

(47) Let h be a partial function from C_2 to C_3, s_1 be a sequence of C_2, and z be a complex number. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $(z\,h) \cdot s_1 = z \cdot (h \cdot s_1)$.

(48) Let h be a partial function from C_1 to R_1, s_1 be a sequence of C_1, and r be a real number. If $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, h$, then $(r\, h) \cdot s_1 = r \cdot (h \cdot s_1)$.

(49) Let h be a partial function from R_1 to C_1, s_1 be a sequence of R_1, and z be a complex number. If $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, h$, then $(z\, h) \cdot s_1 = z \cdot (h \cdot s_1)$.

(50) Let h be a partial function from C_2 to C_3 and s_1 be a sequence of C_2. If $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, h$, then $\|h \cdot s_1\| = \|h\| \cdot s_1$ and $-h \cdot s_1 = (-h) \cdot s_1$.

(51) Let h be a partial function from C_1 to R_1 and s_1 be a sequence of C_1. If $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, h$, then $\|h \cdot s_1\| = \|h\| \cdot s_1$ and $-h \cdot s_1 = (-h) \cdot s_1$.

(52) Let h be a partial function from R_1 to C_1 and s_1 be a sequence of R_1. If $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, h$, then $\|h \cdot s_1\| = \|h\| \cdot s_1$ and $-h \cdot s_1 = (-h) \cdot s_1$.

(53) Let f_1, f_2 be partial functions from C_2 to C_3 and x_0 be a point of C_2. Suppose f_1 is continuous in x_0 and f_2 is continuous in x_0. Then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0.

(54) Let f_1, f_2 be partial functions from C_1 to R_1 and x_0 be a point of C_1. Suppose f_1 is continuous in x_0 and f_2 is continuous in x_0. Then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0.

(55) Let f_1, f_2 be partial functions from R_1 to C_1 and x_0 be a point of R_1. Suppose f_1 is continuous in x_0 and f_2 is continuous in x_0. Then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0.

(56) Let f be a partial function from C_2 to C_3, x_0 be a point of C_2, and z be a complex number. If f is continuous in x_0, then $z\, f$ is continuous in x_0.

(57) Let f be a partial function from C_1 to R_1, x_0 be a point of C_1, and r be a real number. If f is continuous in x_0, then $r\, f$ is continuous in x_0.

(58) Let f be a partial function from R_1 to C_1, x_0 be a point of R_1, and z be a complex number. If f is continuous in x_0, then $z\, f$ is continuous in x_0.

(59) Let f be a partial function from C_2 to C_3 and x_0 be a point of C_2. If f is continuous in x_0, then $\|f\|$ is continuous in x_0 and $-f$ is continuous in x_0.

(60) Let f be a partial function from C_1 to R_1 and x_0 be a point of C_1. If f is continuous in x_0, then $\|f\|$ is continuous in x_0 and $-f$ is continuous in x_0.

(61) Let f be a partial function from R_1 to C_1 and x_0 be a point of R_1. If f is continuous in x_0, then $\|f\|$ is continuous in x_0 and $-f$ is continuous in x_0.

Let C_2, C_3 be complex normed spaces, let f be a partial function from C_2 to C_3, and let X be a set. We say that f is continuous on X if and only if:

(Def. 21) $X \subseteq \mathrm{dom}\, f$ and for every point x_0 of C_2 such that $x_0 \in X$ holds $f{\restriction}X$ is continuous in x_0.

Let C_1 be a complex normed space, let R_1 be a real normed space, let f be a partial function from C_1 to R_1, and let X be a set. We say that f is continuous on X if and only if:

(Def. 22) $X \subseteq \operatorname{dom} f$ and for every point x_0 of C_1 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

Let R_1 be a real normed space, let C_1 be a complex normed space, let g be a partial function from R_1 to C_1, and let X be a set. We say that g is continuous on X if and only if:

(Def. 23) $X \subseteq \operatorname{dom} g$ and for every point x_0 of R_1 such that $x_0 \in X$ holds $g \restriction X$ is continuous in x_0.

Let C_1 be a complex normed space, let f be a partial function from the carrier of C_1 to $\mathbb{C}$, and let X be a set. We say that f is continuous on X if and only if:

(Def. 24) $X \subseteq \operatorname{dom} f$ and for every point x_0 of C_1 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

Let C_1 be a complex normed space, let f be a partial function from the carrier of C_1 to $\mathbb{R}$, and let X be a set. We say that f is continuous on X if and only if:

(Def. 25) $X \subseteq \operatorname{dom} f$ and for every point x_0 of C_1 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

Let R_1 be a real normed space, let f be a partial function from the carrier of R_1 to $\mathbb{C}$, and let X be a set. We say that f is continuous on X if and only if:

(Def. 26) $X \subseteq \operatorname{dom} f$ and for every point x_0 of R_1 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

In the sequel X, X_1 denote sets.

The following propositions are true:

(62) Let f be a partial function from C_2 to C_3. Then f is continuous on X if and only if the following conditions are satisfied:
  (i) $X \subseteq \operatorname{dom} f$, and
  (ii) for every sequence s_4 of C_2 such that $\operatorname{rng} s_4 \subseteq X$ and s_4 is convergent and $\lim s_4 \in X$ holds $f \cdot s_4$ is convergent and $f_{\lim s_4} = \lim(f \cdot s_4)$.

(63) Let f be a partial function from C_1 to R_1. Then f is continuous on X if and only if the following conditions are satisfied:
  (i) $X \subseteq \operatorname{dom} f$, and
  (ii) for every sequence s_4 of C_1 such that $\operatorname{rng} s_4 \subseteq X$ and s_4 is convergent and $\lim s_4 \in X$ holds $f \cdot s_4$ is convergent and $f_{\lim s_4} = \lim(f \cdot s_4)$.

(64) Let f be a partial function from R_1 to C_1. Then f is continuous on X if and only if the following conditions are satisfied:
  (i) $X \subseteq \operatorname{dom} f$, and
  (ii) for every sequence s_4 of R_1 such that $\operatorname{rng} s_4 \subseteq X$ and s_4 is convergent and $\lim s_4 \in X$ holds $f \cdot s_4$ is convergent and $f_{\lim s_4} = \lim(f \cdot s_4)$.

(65) Let f be a partial function from C_2 to C_3. Then f is continuous on X if and only if the following conditions are satisfied:
 (i) $X \subseteq \operatorname{dom} f$, and
 (ii) for every point x_0 of C_2 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_2 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(66) Let f be a partial function from C_1 to R_1. Then f is continuous on X if and only if the following conditions are satisfied:
 (i) $X \subseteq \operatorname{dom} f$, and
 (ii) for every point x_0 of C_1 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(67) Let f be a partial function from R_1 to C_1. Then f is continuous on X if and only if the following conditions are satisfied:
 (i) $X \subseteq \operatorname{dom} f$, and
 (ii) for every point x_0 of R_1 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of R_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(68) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$. Then f is continuous on X if and only if the following conditions are satisfied:
 (i) $X \subseteq \operatorname{dom} f$, and
 (ii) for every point x_0 of C_1 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(69) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. Then f is continuous on X if and only if the following conditions are satisfied:
 (i) $X \subseteq \operatorname{dom} f$, and
 (ii) for every point x_0 of C_1 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(70) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$. Then f is continuous on X if and only if the following conditions are satisfied:
 (i) $X \subseteq \operatorname{dom} f$, and
 (ii) for every point x_0 of R_1 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of R_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(71) For every partial function f from C_2 to C_3 holds f is continuous on X iff $f{\restriction}X$ is continuous on X.

(72) For every partial function f from C_1 to R_1 holds f is continuous on X iff $f{\restriction}X$ is continuous on X.

(73) For every partial function f from R_1 to C_1 holds f is continuous on X iff $f{\restriction}X$ is continuous on X.

(74) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$. Then f is continuous on X if and only if $f{\restriction}X$ is continuous on X.

(75) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. Then f is continuous on X if and only if $f{\restriction}X$ is continuous on X.

(76) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$. Then f is continuous on X if and only if $f{\restriction}X$ is continuous on X.

(77) For every partial function f from C_2 to C_3 such that f is continuous on X and $X_1 \subseteq X$ holds f is continuous on X_1.

(78) For every partial function f from C_1 to R_1 such that f is continuous on X and $X_1 \subseteq X$ holds f is continuous on X_1.

(79) For every partial function f from R_1 to C_1 such that f is continuous on X and $X_1 \subseteq X$ holds f is continuous on X_1.

(80) For every partial function f from C_2 to C_3 and for every point x_0 of C_2 such that $x_0 \in \operatorname{dom} f$ holds f is continuous on $\{x_0\}$.

(81) For every partial function f from C_1 to R_1 and for every point x_0 of C_1 such that $x_0 \in \operatorname{dom} f$ holds f is continuous on $\{x_0\}$.

(82) For every partial function f from R_1 to C_1 and for every point x_0 of R_1 such that $x_0 \in \operatorname{dom} f$ holds f is continuous on $\{x_0\}$.

(83) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is continuous on X and f_2 is continuous on X. Then $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X.

(84) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is continuous on X and f_2 is continuous on X. Then $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X.

(85) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is continuous on X and f_2 is continuous on X. Then $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X.

(86) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is continuous on X and f_2 is continuous on X_1. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

(87) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is continuous on X and f_2 is continuous on X_1. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

(88) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is continuous on X and f_2 is continuous on X_1. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

(89) For every partial function f from C_2 to C_3 such that f is continuous on

X holds $z\,f$ is continuous on X.

(90) For every partial function f from C_1 to R_1 such that f is continuous on X holds $r\,f$ is continuous on X.

(91) For every partial function f from R_1 to C_1 such that f is continuous on X holds $z\,f$ is continuous on X.

(92) Let f be a partial function from C_2 to C_3. If f is continuous on X, then $\|f\|$ is continuous on X and $-f$ is continuous on X.

(93) Let f be a partial function from C_1 to R_1. If f is continuous on X, then $\|f\|$ is continuous on X and $-f$ is continuous on X.

(94) Let f be a partial function from R_1 to C_1. If f is continuous on X, then $\|f\|$ is continuous on X and $-f$ is continuous on X.

(95) Let f be a partial function from C_2 to C_3. Suppose f is total and for all points x_1, x_2 of C_2 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists a point x_0 of C_2 such that f is continuous in x_0. Then f is continuous on the carrier of C_2.

(96) Let f be a partial function from C_1 to R_1. Suppose f is total and for all points x_1, x_2 of C_1 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists a point x_0 of C_1 such that f is continuous in x_0. Then f is continuous on the carrier of C_1.

(97) Let f be a partial function from R_1 to C_1. Suppose f is total and for all points x_1, x_2 of R_1 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists a point x_0 of R_1 such that f is continuous in x_0. Then f is continuous on the carrier of R_1.

(98) For every partial function f from C_2 to C_3 such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds $\operatorname{rng} f$ is compact.

(99) For every partial function f from C_1 to R_1 such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds $\operatorname{rng} f$ is compact.

(100) For every partial function f from R_1 to C_1 such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds $\operatorname{rng} f$ is compact.

(101) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$. If $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$, then $\operatorname{rng} f$ is compact.

(102) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. If $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$, then $\operatorname{rng} f$ is compact.

(103) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$. If $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$, then $\operatorname{rng} f$ is compact.

(104) Let Y be a subset of C_2 and f be a partial function from C_2 to C_3. If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y, then $f^\circ Y$ is compact.

(105) Let Y be a subset of C_1 and f be a partial function from C_1 to R_1.

If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y, then $f^\circ Y$ is compact.

(106) Let Y be a subset of R_1 and f be a partial function from R_1 to C_1. If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y, then $f^\circ Y$ is compact.

(107) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. Suppose $\operatorname{dom} f \neq \emptyset$ and $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$. Then there exist points x_1, x_2 of C_1 such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $f_{x_1} = \sup \operatorname{rng} f$ and $f_{x_2} = \inf \operatorname{rng} f$.

(108) Let f be a partial function from C_2 to C_3. Suppose $\operatorname{dom} f \neq \emptyset$ and $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$. Then there exist points x_1, x_2 of C_2 such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $\|f\|_{x_1} = \sup \operatorname{rng}\|f\|$ and $\|f\|_{x_2} = \inf \operatorname{rng}\|f\|$.

(109) Let f be a partial function from C_1 to R_1. Suppose $\operatorname{dom} f \neq \emptyset$ and $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$. Then there exist points x_1, x_2 of C_1 such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $\|f\|_{x_1} = \sup \operatorname{rng}\|f\|$ and $\|f\|_{x_2} = \inf \operatorname{rng}\|f\|$.

(110) Let f be a partial function from R_1 to C_1. Suppose $\operatorname{dom} f \neq \emptyset$ and $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$. Then there exist points x_1, x_2 of R_1 such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $\|f\|_{x_1} = \sup \operatorname{rng}\|f\|$ and $\|f\|_{x_2} = \inf \operatorname{rng}\|f\|$.

(111) For every partial function f from C_2 to C_3 holds $\|f\| \restriction X = \|f \restriction X\|$.

(112) For every partial function f from C_1 to R_1 holds $\|f\| \restriction X = \|f \restriction X\|$.

(113) For every partial function f from R_1 to C_1 holds $\|f\| \restriction X = \|f \restriction X\|$.

(114) Let f be a partial function from C_2 to C_3 and Y be a subset of C_2. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y. Then there exist points x_1, x_2 of C_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $\|f\|_{x_1} = \sup(\|f\|^\circ Y)$ and $\|f\|_{x_2} = \inf(\|f\|^\circ Y)$.

(115) Let f be a partial function from C_1 to R_1 and Y be a subset of C_1. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y. Then there exist points x_1, x_2 of C_1 such that $x_1 \in Y$ and $x_2 \in Y$ and $\|f\|_{x_1} = \sup(\|f\|^\circ Y)$ and $\|f\|_{x_2} = \inf(\|f\|^\circ Y)$.

(116) Let f be a partial function from R_1 to C_1 and Y be a subset of R_1. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y. Then there exist points x_1, x_2 of R_1 such that $x_1 \in Y$ and $x_2 \in Y$ and $\|f\|_{x_1} = \sup(\|f\|^\circ Y)$ and $\|f\|_{x_2} = \inf(\|f\|^\circ Y)$.

(117) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$ and Y be a subset of C_1. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y. Then there exist points x_1, x_2 of C_1 such that $x_1 \in Y$ and $x_2 \in Y$ and $f_{x_1} = \sup(f^\circ Y)$ and $f_{x_2} = \inf(f^\circ Y)$.

Let C_2, C_3 be complex normed spaces, let X be a set, and let f be a partial function from C_2 to C_3. We say that f is Lipschitzian on X if and only if:

(Def. 27) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of C_2 such that $x_1 \in X$ and $x_2 \in X$ holds $\|f_{x_1} - f_{x_2}\| \leqslant r \cdot \|x_1 - x_2\|$.

Let C_1 be a complex normed space, let R_1 be a real normed space, let X be a set, and let f be a partial function from C_1 to R_1. We say that f is Lipschitzian on X if and only if:

(Def. 28) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ holds $\|f_{x_1} - f_{x_2}\| \leqslant r \cdot \|x_1 - x_2\|$.

Let R_1 be a real normed space, let C_1 be a complex normed space, let X be a set, and let f be a partial function from R_1 to C_1. We say that f is Lipschitzian on X if and only if:

(Def. 29) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of R_1 such that $x_1 \in X$ and $x_2 \in X$ holds $\|f_{x_1} - f_{x_2}\| \leqslant r \cdot \|x_1 - x_2\|$.

Let C_1 be a complex normed space, let X be a set, and let f be a partial function from the carrier of C_1 to $\mathbb{C}$. We say that f is Lipschitzian on X if and only if:

(Def. 30) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ holds $|f_{x_1} - f_{x_2}| \leqslant r \cdot \|x_1 - x_2\|$.

Let C_1 be a complex normed space, let X be a set, and let f be a partial function from the carrier of C_1 to $\mathbb{R}$. We say that f is Lipschitzian on X if and only if:

(Def. 31) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ holds $|f_{x_1} - f_{x_2}| \leqslant r \cdot \|x_1 - x_2\|$.

Let R_1 be a real normed space, let X be a set, and let f be a partial function from the carrier of R_1 to $\mathbb{C}$. We say that f is Lipschitzian on X if and only if:

(Def. 32) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of R_1 such that $x_1 \in X$ and $x_2 \in X$ holds $|f_{x_1} - f_{x_2}| \leqslant r \cdot \|x_1 - x_2\|$.

Next we state a number of propositions:

(118) For every partial function f from C_2 to C_3 such that f is Lipschitzian on X and $X_1 \subseteq X$ holds f is Lipschitzian on X_1.

(119) For every partial function f from C_1 to R_1 such that f is Lipschitzian on X and $X_1 \subseteq X$ holds f is Lipschitzian on X_1.

(120) For every partial function f from R_1 to C_1 such that f is Lipschitzian on X and $X_1 \subseteq X$ holds f is Lipschitzian on X_1.

(121) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(122) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(123) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(124) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(125) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(126) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(127) For every partial function f from C_2 to C_3 such that f is Lipschitzian on X holds $z\,f$ is Lipschitzian on X.

(128) For every partial function f from C_1 to R_1 such that f is Lipschitzian on X holds $r\,f$ is Lipschitzian on X.

(129) For every partial function f from R_1 to C_1 such that f is Lipschitzian on X holds $z\,f$ is Lipschitzian on X.

(130) Let f be a partial function from C_2 to C_3. Suppose f is Lipschitzian on X. Then $-f$ is Lipschitzian on X and $\|f\|$ is Lipschitzian on X.

(131) Let f be a partial function from C_1 to R_1. Suppose f is Lipschitzian on X. Then $-f$ is Lipschitzian on X and $\|f\|$ is Lipschitzian on X.

(132) Let f be a partial function from R_1 to C_1. Suppose f is Lipschitzian on X. Then $-f$ is Lipschitzian on X and $\|f\|$ is Lipschitzian on X.

(133) Let X be a set and f be a partial function from C_2 to C_3. If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is Lipschitzian on X.

(134) Let X be a set and f be a partial function from C_1 to R_1. If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is Lipschitzian on X.

(135) Let X be a set and f be a partial function from R_1 to C_1. If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is Lipschitzian on X.

(136) For every subset Y of C_1 holds id_Y is Lipschitzian on Y.

(137) For every partial function f from C_2 to C_3 such that f is Lipschitzian on X holds f is continuous on X.

(138) For every partial function f from C_1 to R_1 such that f is Lipschitzian on X holds f is continuous on X.

(139) For every partial function f from R_1 to C_1 such that f is Lipschitzian on X holds f is continuous on X.

(140) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$. If f is Lipschitzian on X, then f is continuous on X.

(141) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. If f is Lipschitzian on X, then f is continuous on X.

(142) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$. If f is Lipschitzian on X, then f is continuous on X.

(143) For every partial function f from C_2 to C_3 such that there exists a point r of C_3 such that $\operatorname{rng} f = \{r\}$ holds f is continuous on $\operatorname{dom} f$.

(144) For every partial function f from C_1 to R_1 such that there exists a point r of R_1 such that $\operatorname{rng} f = \{r\}$ holds f is continuous on $\operatorname{dom} f$.

(145) For every partial function f from R_1 to C_1 such that there exists a point r of C_1 such that $\operatorname{rng} f = \{r\}$ holds f is continuous on $\operatorname{dom} f$.

(146) For every partial function f from C_2 to C_3 such that $X \subseteq \operatorname{dom} f$ and f is a constant on X holds f is continuous on X.

(147) For every partial function f from C_1 to R_1 such that $X \subseteq \operatorname{dom} f$ and f is a constant on X holds f is continuous on X.

(148) For every partial function f from R_1 to C_1 such that $X \subseteq \operatorname{dom} f$ and f is a constant on X holds f is continuous on X.

(149) Let f be a partial function from C_1 to C_1. Suppose that for every point x_0 of C_1 such that $x_0 \in \operatorname{dom} f$ holds $f_{x_0} = x_0$. Then f is continuous on $\operatorname{dom} f$.

(150) For every partial function f from C_1 to C_1 such that $f = \mathrm{id}_{\operatorname{dom} f}$ holds f is continuous on $\operatorname{dom} f$.

(151) Let f be a partial function from C_1 to C_1 and Y be a subset of C_1. If $Y \subseteq \operatorname{dom} f$ and $f \restriction Y = \mathrm{id}_Y$, then f is continuous on Y.

(152) Let f be a partial function from C_1 to C_1, z be a complex number, and p be a point of C_1. Suppose $X \subseteq \operatorname{dom} f$ and for every point x_0 of C_1 such that $x_0 \in X$ holds $f_{x_0} = z \cdot x_0 + p$. Then f is continuous on X.

(153) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. Suppose that for every point x_0 of C_1 such that $x_0 \in \operatorname{dom} f$ holds $f_{x_0} = \|x_0\|$. Then f is continuous on $\operatorname{dom} f$.

(154) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. Suppose $X \subseteq \operatorname{dom} f$ and for every point x_0 of C_1 such that $x_0 \in X$ holds $f_{x_0} = \|x_0\|$. Then f is continuous on X.

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On the Fundamental Groups of Products of Topological Spaces

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Summary. In the paper we show that fundamental group of the product of two topological spaces is isomorphic to the product of fundamental groups of the spaces.

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The articles [15], [7], [14], [19], [5], [20], [6], [3], [4], [1], [2], [12], [17], [18], [10], [13], [16], [8], [9], and [11] provide the terminology and notation for this paper.

1. On the Product of Groups

The following proposition is true

(1) Let G, H be non empty groupoids and x be an element of $\prod\langle G, H\rangle$. Then there exists an element g of G and there exists an element h of H such that $x = \langle g, h\rangle$.

Let G_1, G_2, H_1, H_2 be non empty groupoids, let f be a map from G_1 into H_1, and let g be a map from G_2 into H_2. The functor $\mathrm{Gr2Iso}(f, g)$ yields a map from $\prod\langle G_1, G_2\rangle$ into $\prod\langle H_1, H_2\rangle$ and is defined by the condition (Def. 1).

(Def. 1) Let x be an element of $\prod\langle G_1, G_2\rangle$. Then there exists an element x_1 of G_1 and there exists an element x_2 of G_2 such that $x = \langle x_1, x_2\rangle$ and $(\mathrm{Gr2Iso}(f, g))(x) = \langle f(x_1), g(x_2)\rangle$.

The following proposition is true

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(2) Let G_1, G_2, H_1, H_2 be non empty groupoids, f be a map from G_1 into H_1, g be a map from G_2 into H_2, x_1 be an element of G_1, and x_2 be an element of G_2. Then $(\mathrm{Gr2Iso}(f,g))(\langle x_1, x_2\rangle) = \langle f(x_1), g(x_2)\rangle$.

Let G_1, G_2, H_1, H_2 be groups, let f be a homomorphism from G_1 to H_1, and let g be a homomorphism from G_2 to H_2. Then $\mathrm{Gr2Iso}(f,g)$ is a homomorphism from $\prod\langle G_1, G_2\rangle$ to $\prod\langle H_1, H_2\rangle$.

One can prove the following four propositions:

(3) Let G_1, G_2, H_1, H_2 be non empty groupoids, f be a map from G_1 into H_1, and g be a map from G_2 into H_2. If f is one-to-one and g is one-to-one, then $\mathrm{Gr2Iso}(f,g)$ is one-to-one.

(4) Let G_1, G_2, H_1, H_2 be non empty groupoids, f be a map from G_1 into H_1, and g be a map from G_2 into H_2. If f is onto and g is onto, then $\mathrm{Gr2Iso}(f,g)$ is onto.

(5) Let G_1, G_2, H_1, H_2 be groups, f be a homomorphism from G_1 to H_1, and g be a homomorphism from G_2 to H_2. If f is an isomorphism and g is an isomorphism, then $\mathrm{Gr2Iso}(f,g)$ is an isomorphism.

(6) Let G_1, G_2, H_1, H_2 be groups. Suppose G_1 and H_1 are isomorphic and G_2 and H_2 are isomorphic. Then $\prod\langle G_1, G_2\rangle$ and $\prod\langle H_1, H_2\rangle$ are isomorphic.

2. On the Fundamental Groups of Products of Topological Spaces

For simplicity, we adopt the following rules: S, T, Y denote non empty topological spaces, s, s_1, s_2, s_3 denote points of S, t, t_1, t_2, t_3 denote points of T, l_1, l_2 denote paths from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$, and H denotes a homotopy between l_1 and l_2.

We now state two propositions:

(7) For all functions f, g such that $\mathrm{dom}\, f = \mathrm{dom}\, g$ holds $\mathrm{pr1}(\langle f, g\rangle) = f$.

(8) For all functions f, g such that $\mathrm{dom}\, f = \mathrm{dom}\, g$ holds $\mathrm{pr2}(\langle f, g\rangle) = g$.

Let us consider S, T, Y, let f be a map from Y into S, and let g be a map from Y into T. Then $\langle f, g\rangle$ is a map from Y into $[\!: S, T :\!]$.

Let us consider S, T, Y and let f be a map from Y into $[\!: S, T :\!]$. Then $\mathrm{pr1}(f)$ is a map from Y into S. Then $\mathrm{pr2}(f)$ is a map from Y into T.

The following propositions are true:

(9) For every continuous map f from Y into $[\!: S, T :\!]$ holds $\mathrm{pr1}(f)$ is continuous.

(10) For every continuous map f from Y into $[\!: S, T :\!]$ holds $\mathrm{pr2}(f)$ is continuous.

(11) If $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$ are connected, then s_1, s_2 are connected.

(12) If $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$ are connected, then t_1, t_2 are connected.

(13) If $\langle s_1, t_1 \rangle$, $\langle s_2, t_2 \rangle$ are connected, then for every path L from $\langle s_1, t_1 \rangle$ to $\langle s_2, t_2 \rangle$ holds $\mathrm{pr1}(L)$ is a path from s_1 to s_2.

(14) If $\langle s_1, t_1 \rangle$, $\langle s_2, t_2 \rangle$ are connected, then for every path L from $\langle s_1, t_1 \rangle$ to $\langle s_2, t_2 \rangle$ holds $\mathrm{pr2}(L)$ is a path from t_1 to t_2.

(15) If s_1, s_2 are connected and t_1, t_2 are connected, then $\langle s_1, t_1 \rangle$, $\langle s_2, t_2 \rangle$ are connected.

(16) Suppose s_1, s_2 are connected and t_1, t_2 are connected. Let L_1 be a path from s_1 to s_2 and L_2 be a path from t_1 to t_2. Then $\langle L_1, L_2 \rangle$ is a path from $\langle s_1, t_1 \rangle$ to $\langle s_2, t_2 \rangle$.

Let S, T be non empty arcwise connected topological spaces, let s_1, s_2 be points of S, let t_1, t_2 be points of T, let L_1 be a path from s_1 to s_2, and let L_2 be a path from t_1 to t_2. Then $\langle L_1, L_2 \rangle$ is a path from $\langle s_1, t_1 \rangle$ to $\langle s_2, t_2 \rangle$.

Let S, T be non empty topological spaces, let s be a point of S, let t be a point of T, let L_1 be a loop of s, and let L_2 be a loop of t. Then $\langle L_1, L_2 \rangle$ is a loop of $\langle s, t \rangle$.

Let S, T be non empty arcwise connected topological spaces. One can verify that $[\!: S, T :\!]$ is arcwise connected.

Let S, T be non empty arcwise connected topological spaces, let s_1, s_2 be points of S, let t_1, t_2 be points of T, and let L be a path from $\langle s_1, t_1 \rangle$ to $\langle s_2, t_2 \rangle$. Then $\mathrm{pr1}(L)$ is a path from s_1 to s_2. Then $\mathrm{pr2}(L)$ is a path from t_1 to t_2.

Let S, T be non empty topological spaces, let s be a point of S, let t be a point of T, and let L be a loop of $\langle s, t \rangle$. Then $\mathrm{pr1}(L)$ is a loop of s. Then $\mathrm{pr2}(L)$ is a loop of t.

Next we state a number of propositions:

(17) Let p, q be paths from s_1 to s_2. Suppose $p = \mathrm{pr1}(l_1)$ and $q = \mathrm{pr1}(l_2)$ and l_1, l_2 are homotopic. Then $\mathrm{pr1}(H)$ is a homotopy between p and q.

(18) Let p, q be paths from t_1 to t_2. Suppose $p = \mathrm{pr2}(l_1)$ and $q = \mathrm{pr2}(l_2)$ and l_1, l_2 are homotopic. Then $\mathrm{pr2}(H)$ is a homotopy between p and q.

(19) For all paths p, q from s_1 to s_2 such that $p = \mathrm{pr1}(l_1)$ and $q = \mathrm{pr1}(l_2)$ and l_1, l_2 are homotopic holds p, q are homotopic.

(20) For all paths p, q from t_1 to t_2 such that $p = \mathrm{pr2}(l_1)$ and $q = \mathrm{pr2}(l_2)$ and l_1, l_2 are homotopic holds p, q are homotopic.

(21) Let p, q be paths from s_1 to s_2, x, y be paths from t_1 to t_2, f be a homotopy between p and q, and g be a homotopy between x and y. Suppose $p = \mathrm{pr1}(l_1)$ and $q = \mathrm{pr1}(l_2)$ and $x = \mathrm{pr2}(l_1)$ and $y = \mathrm{pr2}(l_2)$ and p, q are homotopic and x, y are homotopic. Then $\langle f, g \rangle$ is a homotopy between l_1 and l_2.

(22) Let p, q be paths from s_1 to s_2 and x, y be paths from t_1 to t_2. Suppose $p = \mathrm{pr1}(l_1)$ and $q = \mathrm{pr1}(l_2)$ and $x = \mathrm{pr2}(l_1)$ and $y = \mathrm{pr2}(l_2)$ and p, q are homotopic and x, y are homotopic. Then l_1, l_2 are homotopic.

(23) Let l_1 be a path from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$, l_2 be a path from $\langle s_2, t_2\rangle$ to $\langle s_3, t_3\rangle$, p_1 be a path from s_1 to s_2, and p_2 be a path from s_2 to s_3. Suppose $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$ are connected and $\langle s_2, t_2\rangle$, $\langle s_3, t_3\rangle$ are connected and $p_1 = \mathrm{pr1}(l_1)$ and $p_2 = \mathrm{pr1}(l_2)$. Then $\mathrm{pr1}(l_1 + l_2) = p_1 + p_2$.

(24) Let S, T be non empty arcwise connected topological spaces, s_1, s_2, s_3 be points of S, t_1, t_2, t_3 be points of T, l_1 be a path from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$, and l_2 be a path from $\langle s_2, t_2\rangle$ to $\langle s_3, t_3\rangle$. Then $\mathrm{pr1}(l_1 + l_2) = \mathrm{pr1}(l_1) + \mathrm{pr1}(l_2)$.

(25) Let l_1 be a path from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$, l_2 be a path from $\langle s_2, t_2\rangle$ to $\langle s_3, t_3\rangle$, p_1 be a path from t_1 to t_2, and p_2 be a path from t_2 to t_3. Suppose $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$ are connected and $\langle s_2, t_2\rangle$, $\langle s_3, t_3\rangle$ are connected and $p_1 = \mathrm{pr2}(l_1)$ and $p_2 = \mathrm{pr2}(l_2)$. Then $\mathrm{pr2}(l_1 + l_2) = p_1 + p_2$.

(26) Let S, T be non empty arcwise connected topological spaces, s_1, s_2, s_3 be points of S, t_1, t_2, t_3 be points of T, l_1 be a path from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$, and l_2 be a path from $\langle s_2, t_2\rangle$ to $\langle s_3, t_3\rangle$. Then $\mathrm{pr2}(l_1 + l_2) = \mathrm{pr2}(l_1) + \mathrm{pr2}(l_2)$.

Let S, T be non empty topological spaces, let s be a point of S, and let t be a point of T. The functor $\mathrm{FGPrIso}(s, t)$ yielding a map from $\pi_1([\!:S, T:\!], \langle s, t\rangle)$ into $\prod\langle\pi_1(S, s), \pi_1(T, t)\rangle$ is defined as follows:

(Def. 2) For every point x of $\pi_1([\!:S, T:\!], \langle s, t\rangle)$ there exists a loop l of $\langle s, t\rangle$ such that $x = [l]_{\mathrm{EqRel}([\!:S, T:\!], \langle s, t\rangle)}$ and $(\mathrm{FGPrIso}(s, t))(x) = \langle[\mathrm{pr1}(l)]_{\mathrm{EqRel}(S,s)}, [\mathrm{pr2}(l)]_{\mathrm{EqRel}(T,t)}\rangle$.

The following propositions are true:

(27) For every point x of $\pi_1([\!:S, T:\!], \langle s, t\rangle)$ and for every loop l of $\langle s, t\rangle$ such that $x = [l]_{\mathrm{EqRel}([\!:S, T:\!], \langle s, t\rangle)}$ holds $(\mathrm{FGPrIso}(s, t))(x) = \langle[\mathrm{pr1}(l)]_{\mathrm{EqRel}(S,s)}, [\mathrm{pr2}(l)]_{\mathrm{EqRel}(T,t)}\rangle$.

(28) For every loop l of $\langle s, t\rangle$ holds $(\mathrm{FGPrIso}(s, t))([l]_{\mathrm{EqRel}([\!:S, T:\!], \langle s, t\rangle)}) = \langle[\mathrm{pr1}(l)]_{\mathrm{EqRel}(S,s)}, [\mathrm{pr2}(l)]_{\mathrm{EqRel}(T,t)}\rangle$.

Let S, T be non empty topological spaces, let s be a point of S, and let t be a point of T. Observe that $\mathrm{FGPrIso}(s, t)$ is one-to-one and onto.

Let S, T be non empty topological spaces, let s be a point of S, and let t be a point of T. Then $\mathrm{FGPrIso}(s, t)$ is a homomorphism from $\pi_1([\!:S, T:\!], \langle s, t\rangle)$ to $\prod\langle\pi_1(S, s), \pi_1(T, t)\rangle$.

The following propositions are true:

(29) $\mathrm{FGPrIso}(s, t)$ is an isomorphism.

(30) $\pi_1([\!:S, T:\!], \langle s, t\rangle)$ and $\prod\langle\pi_1(S, s), \pi_1(T, t)\rangle$ are isomorphic.

(31) Let f be a homomorphism from $\pi_1(S, s_1)$ to $\pi_1(S, s_2)$ and g be a homomorphism from $\pi_1(T, t_1)$ to $\pi_1(T, t_2)$. Suppose f is an isomorphism and g is an isomorphism. Then $\mathrm{Gr2Iso}(f, g) \cdot \mathrm{FGPrIso}(s_1, t_1)$ is an isomorphism.

(32) Let S, T be non empty arcwise connected topological spaces, s_1, s_2 be points of S, and t_1, t_2 be points of T. Then $\pi_1([\!:S, T\,:\!], \langle s_1, t_1 \rangle)$ and $\prod \langle \pi_1(S, s_2), \pi_1(T, t_2) \rangle$ are isomorphic.

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