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Sorting Operators for Finite Sequences

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Summary. Two kinds of sorting operators, descendent one and ascendent one are introduced for finite sequences of reals. They are also called rearrangement of finite sequences of reals. Maximum and minimum values of finite sequences of reals are also defined. We also discuss relations between these concepts.

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The articles [13], [12], [15], [4], [5], [2], [1], [9], [14], [10], [6], [7], [3], [11], and [8] provide the notation and terminology for this paper.

Let f be a finite sequence of elements of $\mathbb{R}$. The functor $\max_{\rm p} f$ yielding a natural number is defined by the conditions (Def. 1).

(Def. 1)(i) If $\operatorname{len} f = 0$, then $\max_{\rm p} f = 0$, and

(ii) if $\operatorname{len} f > 0$, then $\max_{\rm p} f \in \operatorname{dom} f$ and for every natural number i and for all real numbers r_1, r_2 such that $i \in \operatorname{dom} f$ and $r_1 = f(i)$ and $r_2 = f(\max_{\rm p} f)$ holds $r_1 \leqslant r_2$ and for every natural number j such that $j \in \operatorname{dom} f$ and $f(j) = f(\max_{\rm p} f)$ holds $\max_{\rm p} f \leqslant j$.

Let f be a finite sequence of elements of $\mathbb{R}$. The functor $\min_{\rm p} f$ yields a natural number and is defined by the conditions (Def. 2).

(Def. 2)(i) If $\operatorname{len} f = 0$, then $\min_{\rm p} f = 0$, and

(ii) if $\operatorname{len} f > 0$, then $\min_{\rm p} f \in \operatorname{dom} f$ and for every natural number i and for all real numbers r_1, r_2 such that $i \in \operatorname{dom} f$ and $r_1 = f(i)$ and $r_2 = f(\min_{\rm p} f)$ holds $r_1 \geqslant r_2$ and for every natural number j such that $j \in \operatorname{dom} f$ and $f(j) = f(\min_{\rm p} f)$ holds $\min_{\rm p} f \leqslant j$.

Let f be a finite sequence of elements of $\mathbb{R}$. The functor $\max f$ yields a real number and is defined by:

(Def. 3) $\max f = f(\max_{\rm p} f)$.

The functor $\min f$ yields a real number and is defined by:

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(Def. 4) $\min f = f(\min_{\rm p} f)$.

The following propositions are true:

(1) Let f be a finite sequence of elements of $\mathbb{R}$ and i be a natural number. If $1 \leqslant i$ and $i \leqslant \operatorname{len} f$, then $f(i) \leqslant f(\max_{\rm p} f)$ and $f(i) \leqslant \max f$.

(2) Let f be a finite sequence of elements of $\mathbb{R}$ and i be a natural number. If $1 \leqslant i$ and $i \leqslant \operatorname{len} f$, then $f(i) \geqslant f(\min_{\rm p} f)$ and $f(i) \geqslant \min f$.

(3) For every finite sequence f of elements of $\mathbb{R}$ and for every real number r such that $f = \langle r \rangle$ holds $\max_{\rm p} f = 1$ and $\max f = r$.

(4) For every finite sequence f of elements of $\mathbb{R}$ and for every real number r such that $f = \langle r \rangle$ holds $\min_{\rm p} f = 1$ and $\min f = r$.

(5) Let f be a finite sequence of elements of $\mathbb{R}$ and r_1, r_2 be real numbers. If $f = \langle r_1, r_2 \rangle$, then $\max f = \max(r_1, r_2)$ and $\max_{\rm p} f = (r_1 = \max(r_1, r_2) \rightarrow 1, 2)$.

(6) Let f be a finite sequence of elements of $\mathbb{R}$ and r_1, r_2 be real numbers. If $f = \langle r_1, r_2 \rangle$, then $\min f = \min(r_1, r_2)$ and $\min_{\rm p} f = (r_1 = \min(r_1, r_2) \rightarrow 1, 2)$.

(7) For all finite sequences f_1, f_2 of elements of $\mathbb{R}$ such that $\operatorname{len} f_1 = \operatorname{len} f_2$ and $\operatorname{len} f_1 > 0$ holds $\max(f_1 + f_2) \leqslant \max f_1 + \max f_2$.

(8) For all finite sequences f_1, f_2 of elements of $\mathbb{R}$ such that $\operatorname{len} f_1 = \operatorname{len} f_2$ and $\operatorname{len} f_1 > 0$ holds $\min(f_1 + f_2) \geqslant \min f_1 + \min f_2$.

(9) Let f be a finite sequence of elements of $\mathbb{R}$ and a be a real number. If $\operatorname{len} f > 0$ and $a > 0$, then $\max(a \cdot f) = a \cdot \max f$ and $\max_{\rm p}(a \cdot f) = \max_{\rm p} f$.

(10) Let f be a finite sequence of elements of $\mathbb{R}$ and a be a real number. If $\operatorname{len} f > 0$ and $a > 0$, then $\min(a \cdot f) = a \cdot \min f$ and $\min_{\rm p}(a \cdot f) = \min_{\rm p} f$.

(11) For every finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{len} f > 0$ holds $\max(-f) = -\min f$ and $\max_{\rm p}(-f) = \min_{\rm p} f$.

(12) For every finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{len} f > 0$ holds $\min(-f) = -\max f$ and $\min_{\rm p}(-f) = \max_{\rm p} f$.

(13) Let f be a finite sequence of elements of $\mathbb{R}$ and n be a natural number. If $1 \leqslant n$ and $n < \operatorname{len} f$, then $\max(f_{\restriction n}) \leqslant \max f$ and $\min(f_{\restriction n}) \geqslant \min f$.

(14) For all finite sequences f, g of elements of $\mathbb{R}$ such that f and g are fiberwise equipotent holds $\max f = \max g$.

(15) For all finite sequences f, g of elements of $\mathbb{R}$ such that f and g are fiberwise equipotent holds $\min f = \min g$.

Let f be a finite sequence of elements of $\mathbb{R}$. The functor $\operatorname{sort_d} f$ yields a non-increasing finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 5) f and $\operatorname{sort_d} f$ are fiberwise equipotent.

Next we state four propositions:

(16) For every finite sequence R of elements of $\mathbb{R}$ such that $\operatorname{len} R = 0$ or $\operatorname{len} R = 1$ holds R is non-decreasing.

(17) Let R be a finite sequence of elements of $\mathbb{R}$. Then R is non-decreasing if and only if for all natural numbers n, m such that $n \in \operatorname{dom} R$ and $m \in \operatorname{dom} R$ and $n < m$ holds $R(n) \leqslant R(m)$.

(18) Let R be a non-decreasing finite sequence of elements of $\mathbb{R}$ and n be a natural number. Then $R{\restriction}n$ is a non-decreasing finite sequence of elements of $\mathbb{R}$.

(19) Let R_1, R_2 be non-decreasing finite sequences of elements of $\mathbb{R}$. If R_1 and R_2 are fiberwise equipotent, then $R_1 = R_2$.

Let f be a finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{sort_a}\, f$ yields a non-decreasing finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 6) f and $\mathrm{sort_a}\, f$ are fiberwise equipotent.

Next we state a number of propositions:

(20) For every non-increasing finite sequence f of elements of $\mathbb{R}$ holds $\mathrm{sort_d}\, f = f$.

(21) For every non-decreasing finite sequence f of elements of $\mathbb{R}$ holds $\mathrm{sort_a}\, f = f$.

(22) For every finite sequence f of elements of $\mathbb{R}$ holds $\mathrm{sort_d}\,\mathrm{sort_d}\, f = \mathrm{sort_d}\, f$.

(23) For every finite sequence f of elements of $\mathbb{R}$ holds $\mathrm{sort_a}\,\mathrm{sort_a}\, f = \mathrm{sort_a}\, f$.

(24) For every finite sequence f of elements of $\mathbb{R}$ such that f is non-increasing holds $-f$ is non-decreasing.

(25) For every finite sequence f of elements of $\mathbb{R}$ such that f is non-decreasing holds $-f$ is non-increasing.

(26) Let f, g be finite sequences of elements of $\mathbb{R}$ and P be a permutation of $\operatorname{dom} g$. If $f = g \cdot P$ and $\operatorname{len} g \geqslant 1$, then $-f = (-g) \cdot P$.

(27) Let f, g be finite sequences of elements of $\mathbb{R}$. Suppose f and g are fiberwise equipotent. Then $-f$ and $-g$ are fiberwise equipotent.

(28) For every finite sequence f of elements of $\mathbb{R}$ holds $\mathrm{sort_d}(-f) = -\mathrm{sort_a}\, f$.

(29) For every finite sequence f of elements of $\mathbb{R}$ holds $\mathrm{sort_a}(-f) = -\mathrm{sort_d}\, f$.

(30) For every finite sequence f of elements of $\mathbb{R}$ holds $\operatorname{dom}\mathrm{sort_d}\, f = \operatorname{dom} f$ and $\operatorname{len}\mathrm{sort_d}\, f = \operatorname{len} f$.

(31) For every finite sequence f of elements of $\mathbb{R}$ holds $\operatorname{dom}\mathrm{sort_a}\, f = \operatorname{dom} f$ and $\operatorname{len}\mathrm{sort_a}\, f = \operatorname{len} f$.

(32) For every finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{len} f \geqslant 1$ holds $\max_{\mathrm{p}}\mathrm{sort_d}\, f = 1$ and $\min_{\mathrm{p}}\mathrm{sort_a}\, f = 1$ and $(\mathrm{sort_d}\, f)(1) = \max f$ and $(\mathrm{sort_a}\, f)(1) = \min f$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.

[7] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[8] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Scalar multiple of Riemann definite integral. *Formalized Mathematics*, 9(**1**):191–196, 2001.

[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[10] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[11] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(**2**):275–278, 1992.

[12] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[14] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[15] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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