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The Class of Series-Parallel Graphs. Part III

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Summary. This paper contains some facts and theorems relating to the following operations on graphs: union, sum, complement and "embeds". We also introduce connected graphs to prove that a finite irreflexive symmetric N-free graph is a finite series-parallel graph. This article continues the formalization of [22].

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The papers [25], [24], [28], [12], [29], [31], [30], [2], [13], [1], [27], [18], [17], [8], [14], [16], [20], [23], [7], [10], [26], [11], [4], [6], [19], [15], [5], [21], [3], and [9] provide the notation and terminology for this paper.

1. Preliminaries

In this paper A, B, a, b, c, d, e, f, g, h denote sets.

One can prove the following three propositions:

(1) $\mathrm{id}_A \restriction B = \mathrm{id}_A \cap [\!:B, B:\!]$.

(2) $\mathrm{id}_{\{a,b,c,d\}} = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle\}$.

(3) $[\!:\{a,b,c,d\}, \{e,f,g,h\}:\!] = \{\langle a, e\rangle, \langle a, f\rangle, \langle b, e\rangle, \langle b, f\rangle, \langle a, g\rangle, \langle a, h\rangle, \langle b, g\rangle, \langle b, h\rangle\} \cup \{\langle c, e\rangle, \langle c, f\rangle, \langle d, e\rangle, \langle d, f\rangle, \langle c, g\rangle, \langle c, h\rangle, \langle d, g\rangle, \langle d, h\rangle\}$.

Let X, Y be trivial sets. Observe that every relation between X and Y is trivial.

We now state the proposition

(4) For every trivial set X and for every binary relation R on X such that R is non empty there exists a set x such that $R = \{\langle x, x\rangle\}$.

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Let X be a trivial set. Observe that every binary relation on X is trivial, reflexive, symmetric, transitive, and strongly connected.

We now state the proposition

(5) For every non empty trivial set X holds every binary relation on X is symmetric in X.

One can verify that there exists a relational structure which is non empty, strict, finite, irreflexive, and symmetric.

Let L be an irreflexive relational structure. Observe that every full relational substructure of L is irreflexive.

Let L be a symmetric relational structure. Note that every full relational substructure of L is symmetric.

One can prove the following proposition

(6) Let R be an irreflexive symmetric relational structure. Suppose $\overline{\overline{\text{the carrier of } R}} = 2$. Then there exist sets a, b such that the carrier of $R = \{a, b\}$ but the internal relation of $R = \{\langle a, b\rangle, \langle b, a\rangle\}$ or the internal relation of $R = \emptyset$.

2. Some Facts about Operations "UnionOf" and "SumOf"

Let R be a non empty relational structure and let S be a relational structure. Note that $\mathrm{UnionOf}(R, S)$ is non empty and $\mathrm{SumOf}(R, S)$ is non empty.

Let R be a relational structure and let S be a non empty relational structure. Observe that $\mathrm{UnionOf}(R, S)$ is non empty and $\mathrm{SumOf}(R, S)$ is non empty.

Let R, S be finite relational structures. One can check that $\mathrm{UnionOf}(R, S)$ is finite and $\mathrm{SumOf}(R, S)$ is finite.

Let R, S be symmetric relational structures. One can check that $\mathrm{UnionOf}(R, S)$ is symmetric and $\mathrm{SumOf}(R, S)$ is symmetric.

Let R, S be irreflexive relational structures. Observe that $\mathrm{UnionOf}(R, S)$ is irreflexive.

The following four propositions are true:

(7) Let R, S be irreflexive relational structures. Suppose the carrier of R misses the carrier of S. Then $\mathrm{SumOf}(R, S)$ is irreflexive.

(8) For all relational structures R_1, R_2 holds $\mathrm{UnionOf}(R_1, R_2) = \mathrm{UnionOf}(R_2, R_1)$ and $\mathrm{SumOf}(R_1, R_2) = \mathrm{SumOf}(R_2, R_1)$.

(9) Let G be an irreflexive relational structure and G_1, G_2 be relational structures. If $G = \mathrm{UnionOf}(G_1, G_2)$ or $G = \mathrm{SumOf}(G_1, G_2)$, then G_1 is irreflexive and G_2 is irreflexive.

(10) Let G be a non empty relational structure and H_1, H_2 be relational structures. Suppose that

(i) the carrier of H_1 misses the carrier of H_2, and

(ii) the relational structure of $G = \mathrm{UnionOf}(H_1, H_2)$ or the relational structure of $G = \mathrm{SumOf}(H_1, H_2)$.
Then H_1 is a full relational substructure of G and H_2 is a full relational substructure of G.

3. Theorems Relating to the Complement of Relational Structure

One can prove the following proposition

(11) The internal relation of ComplRelStr Necklace 4 $= \{\langle 0, 2\rangle, \langle 2, 0\rangle, \langle 0, 3\rangle, \langle 3, 0\rangle, \langle 1, 3\rangle, \langle 3, 1\rangle\}$.

Let R be a relational structure. Note that ComplRelStr R is irreflexive.

Let R be a symmetric relational structure. Note that ComplRelStr R is symmetric.

Next we state several propositions:

(12) For every relational structure R holds the internal relation of R misses the internal relation of ComplRelStr R.

(13) For every relational structure R holds $\mathrm{id}_{\text{the carrier of } R}$ misses the internal relation of ComplRelStr R.

(14) Let G be a relational structure. Then $[\!:$ the carrier of G, the carrier of $G\,:\!] = \mathrm{id}_{\text{the carrier of } G} \cup$ the internal relation of $G \cup$ the internal relation of ComplRelStr G.

(15) For every strict irreflexive relational structure G such that G is trivial holds ComplRelStr $G = G$.

(16) For every strict irreflexive relational structure G holds ComplRelStr ComplRelStr $G = G$.

(17) For all relational structures G_1, G_2 such that the carrier of G_1 misses the carrier of G_2 holds ComplRelStr $\mathrm{UnionOf}(G_1, G_2) = \mathrm{SumOf}(\text{ComplRelStr } G_1, \text{ComplRelStr } G_2)$.

(18) For all relational structures G_1, G_2 such that the carrier of G_1 misses the carrier of G_2 holds ComplRelStr $\mathrm{SumOf}(G_1, G_2) = \mathrm{UnionOf}(\text{ComplRelStr } G_1, \text{ComplRelStr } G_2)$.

(19) Let G be a relational structure and H be a full relational substructure of G. Then the internal relation of ComplRelStr $H =$ (the internal relation of ComplRelStr G) $|^2$ (the carrier of ComplRelStr H).

(20) Let G be a non empty irreflexive relational structure, x be an element of the carrier of G, and x' be an element of the carrier of ComplRelStr G. If $x = x'$, then ComplRelStr $\mathrm{sub}(\Omega_G \setminus \{x\}) = \mathrm{sub}(\Omega_{\text{ComplRelStr } G} \setminus \{x'\})$.

4. Another Facts Relating to Operation "embeds"

Let us observe that every non empty relational structure which is trivial and strict is also N-free.

The following propositions are true:

(21) Let R be a reflexive antisymmetric relational structure and S be a relational structure. Then there exists a map f from R into S such that for all elements x, y of the carrier of R holds $\langle x, y\rangle \in$ the internal relation of R iff $\langle f(x), f(y)\rangle \in$ the internal relation of S if and only if S embeds R.

(22) Let G be a non empty relational structure and H be a non empty full relational substructure of G. Then G embeds H.

(23) Let G be a non empty relational structure and H be a non empty full relational substructure of G. If G is N-free, then H is N-free.

(24) For every non empty irreflexive relational structure G holds G embeds Necklace 4 iff ComplRelStr G embeds Necklace 4.

(25) For every non empty irreflexive relational structure G holds G is N-free iff ComplRelStr G is N-free.

5. Connected Graphs

Let R be a relational structure. A path of R is a reduction sequence w.r.t. the internal relation of R.

Let R be a relational structure. We say that R is path-connected if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let x, y be sets. Suppose $x \in$ the carrier of R and $y \in$ the carrier of R and $x \neq y$. Then the internal relation of R reduces x to y or the internal relation of R reduces y to x.

One can check that every relational structure which is empty is also path-connected.

One can check that every non empty relational structure which is connected is also path-connected.

We now state the proposition

(26) Let R be a non empty transitive reflexive relational structure and x, y be elements of R. Suppose the internal relation of R reduces x to y. Then $\langle x, y\rangle \in$ the internal relation of R.

One can check that every non empty transitive reflexive relational structure which is path-connected is also connected.

Next we state the proposition

(27) Let R be a symmetric relational structure and x, y be sets. Suppose $x \in$ the carrier of R and $y \in$ the carrier of R. Suppose the internal relation of R reduces x to y. Then the internal relation of R reduces y to x.

Let R be a symmetric relational structure. Let us observe that R is path-connected if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let x, y be sets. Suppose $x \in$ the carrier of R and $y \in$ the carrier of R and $x \neq y$. Then the internal relation of R reduces x to y.

Let R be a relational structure and let x be an element of R. The functor component(x) yielding a subset of R is defined as follows:

(Def. 3) $\text{component}(x) = [x]_{\text{EqCl(the internal relation of } R)}$.

Next we state the proposition

(28) For every non empty relational structure R and for every element x of R holds $x \in \text{component}(x)$.

Let R be a non empty relational structure and let x be an element of R. Note that component(x) is non empty.

Next we state a number of propositions:

(29) Let R be a relational structure, x be an element of R, and y be a set. If $y \in \text{component}(x)$, then $\langle x, y \rangle \in$ EqCl(the internal relation of R).

(30) Let R be a relational structure, x be an element of R, and A be a set. Then $A = \text{component}(x)$ if and only if for every set y holds $y \in A$ iff $\langle x, y \rangle \in$ EqCl(the internal relation of R).

(31) Let R be a non empty irreflexive symmetric relational structure. Suppose R is not path-connected. Then there exist non empty strict irreflexive symmetric relational structures G_1, G_2 such that the carrier of G_1 misses the carrier of G_2 and the relational structure of $R = \text{UnionOf}(G_1, G_2)$.

(32) Let R be a non empty irreflexive symmetric relational structure. Suppose ComplRelStr R is not path-connected. Then there exist non empty strict irreflexive symmetric relational structures G_1, G_2 such that the carrier of G_1 misses the carrier of G_2 and the relational structure of $R = \text{SumOf}(G_1, G_2)$.

(33) For every irreflexive relational structure G such that $G \in$ FinRelStrSp holds ComplRelStr $G \in$ FinRelStrSp.

(34) Let R be an irreflexive symmetric relational structure. Suppose $\overline{\overline{\text{the carrier of } R}} = 2$ and the carrier of $R \in \mathbf{U}_0$. Then the relational structure of $R \in$ FinRelStrSp.

(35) For every relational structure R such that $R \in$ FinRelStrSp holds R is symmetric.

(36) Let G be a relational structure, H_1, H_2 be non empty relational structures, x be an element of the carrier of H_1, and y be an element of the

carrier of H_2. Suppose $G = \text{UnionOf}(H_1, H_2)$ and the carrier of H_1 misses the carrier of H_2. Then $\langle x, y\rangle \notin$ the internal relation of G.

(37) Let G be a relational structure, H_1, H_2 be non empty relational structures, x be an element of the carrier of H_1, and y be an element of the carrier of H_2. If $G = \text{SumOf}(H_1, H_2)$, then $\langle x, y\rangle \notin$ the internal relation of ComplRelStr G.

(38) Let G be a non empty symmetric relational structure, x be an element of the carrier of G, and R_1, R_2 be non empty relational structures. Suppose the carrier of R_1 misses the carrier of R_2 and $\text{sub}(\Omega_G \setminus \{x\}) = \text{UnionOf}(R_1, R_2)$ and G is path-connected. Then there exists an element b of the carrier of R_1 such that $\langle b, x\rangle \in$ the internal relation of G.

(39) Let G be a non empty symmetric irreflexive relational structure, a, b, c, d be elements of the carrier of G, and Z be a subset of the carrier of G. Suppose that $Z = \{a, b, c, d\}$ and a, b, c, d are mutually different and $\langle a, b\rangle \in$ the internal relation of G and $\langle b, c\rangle \in$ the internal relation of G and $\langle c, d\rangle \in$ the internal relation of G and $\langle a, c\rangle \notin$ the internal relation of G and $\langle a, d\rangle \notin$ the internal relation of G and $\langle b, d\rangle \notin$ the internal relation of G. Then $\text{sub}(Z)$ embeds Necklace 4.

(40) Let G be a non empty irreflexive symmetric relational structure, x be an element of the carrier of G, and R_1, R_2 be non empty relational structures. Suppose that

(i) the carrier of R_1 misses the carrier of R_2,

(ii) $\text{sub}(\Omega_G \setminus \{x\}) = \text{UnionOf}(R_1, R_2)$,

(iii) G is non trivial and path-connected, and

(iv) ComplRelStr G is path-connected.

Then G embeds Necklace 4.

(41) Let G be a non empty strict finite irreflexive symmetric relational structure. Suppose G is N-free and the carrier of $G \in \mathbf{U}_0$. Then the relational structure of $G \in$ FinRelStrSp .

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