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Banach Space of Bounded Real Sequences

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Summary. We introduce the arithmetic addition and multiplication in the set of bounded real sequences and also introduce the norm. This set has the structure of the Banach space.

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The articles [23], [6], [27], [29], [28], [15], [21], [3], [1], [2], [20], [24], [9], [4], [5], [7], [26], [22], [16], [17], [14], [11], [12], [10], [25], [13], [8], [19], and [18] provide the notation and terminology for this paper.

1. The Banach Space of Bounded Real Sequences

The subset the set of bounded real sequences of the linear space of real sequences is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in$ the set of bounded real sequences if and only if $x \in$ the set of real sequences and $\mathrm{id}_{\mathrm{seq}}(x)$ is bounded.

Let us note that the set of bounded real sequences is non empty and the set of bounded real sequences is linearly closed.

One can prove the following proposition

(1) ⟨the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences)⟩ is a subspace of the linear space of real sequences.

One can verify that ⟨the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded

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real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences)⟩ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

The function linfty-norm from the set of bounded real sequences into $\mathbb{R}$ is defined by:

(Def. 2) For every set x such that $x \in$ the set of bounded real sequences holds linfty-norm$(x) = \sup \mathrm{rng}|\mathrm{id}_{\mathrm{seq}}(x)|$.

The following proposition is true

(2) Let r_1 be a sequence of real numbers. Then r_1 is bounded and $\sup \mathrm{rng}|r_1| = 0$ if and only if for every natural number n holds $r_1(n) = 0$.

Let us mention that ⟨the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences), linfty-norm⟩ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

The non empty normed structure linfty-Space is defined by the condition (Def. 3).

(Def. 3) linfty-Space = ⟨the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences), linfty-norm⟩.

We now state two propositions:

(3) The carrier of linfty-Space = the set of bounded real sequences and for every set x holds x is a vector of linfty-Space iff x is a sequence of real numbers and $\mathrm{id}_{\mathrm{seq}}(x)$ is bounded and $0_{\text{linfty-Space}} = \mathrm{Zeroseq}$ and for every vector u of linfty-Space holds $u = \mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of linfty-Space holds $u + v = \mathrm{id}_{\mathrm{seq}}(u) + \mathrm{id}_{\mathrm{seq}}(v)$ and for every real number r and for every vector u of linfty-Space holds $r \cdot u = r\ \mathrm{id}_{\mathrm{seq}}(u)$ and for every vector u of linfty-Space holds $-u = -\mathrm{id}_{\mathrm{seq}}(u)$ and $\mathrm{id}_{\mathrm{seq}}(-u) = -\mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of linfty-Space holds $u - v = \mathrm{id}_{\mathrm{seq}}(u) - \mathrm{id}_{\mathrm{seq}}(v)$ and for every vector v of linfty-Space holds $\mathrm{id}_{\mathrm{seq}}(v)$ is bounded and for every vector v of linfty-Space holds $\|v\| = \sup \mathrm{rng}|\mathrm{id}_{\mathrm{seq}}(v)|$.

(4) Let x, y be points of linfty-Space and a be a real number. Then $\|x\| = 0$ iff $x = 0_{\text{linfty-Space}}$ and $0 \leqslant \|x\|$ and $\|x+y\| \leqslant \|x\| + \|y\|$ and $\|a \cdot x\| = |a| \cdot \|x\|$.

Let us observe that linfty-Space is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

Next we state the proposition

(5) For every sequence v_1 of linfty-Space such that v_1 is Cauchy sequence by norm holds v_1 is convergent.

2. The Banach Space of Bounded Functions

Let X be a non empty set, let Y be a real normed space, and let I_1 be a function from X into the carrier of Y. We say that I_1 is bounded if and only if:

(Def. 4) There exists a real number K such that $0 \leqslant K$ and for every element x of X holds $\|I_1(x)\| \leqslant K$.

The following proposition is true

(6) Let X be a non empty set, Y be a real normed space, and f be a function from X into the carrier of Y. If for every element x of X holds $f(x) = 0_Y$, then f is bounded.

Let X be a non empty set and let Y be a real normed space. Note that there exists a function from X into the carrier of Y which is bounded.

Let X be a non empty set and let Y be a real normed space. The functor $\mathrm{BdFuncs}(X, Y)$ yields a subset of $\mathrm{RealVectSpace}(X, Y)$ and is defined by:

(Def. 5) For every set x holds $x \in \mathrm{BdFuncs}(X, Y)$ iff x is a bounded function from X into the carrier of Y.

Let X be a non empty set and let Y be a real normed space. Observe that $\mathrm{BdFuncs}(X, Y)$ is non empty.

The following propositions are true:

(7) For every non empty set X and for every real normed space Y holds $\mathrm{BdFuncs}(X, Y)$ is linearly closed.

(8) For every non empty set X and for every real normed space Y holds $\langle \mathrm{BdFuncs}(X, Y), \mathrm{Zero\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Add\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Mult\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y))\rangle$ is a subspace of $\mathrm{RealVectSpace}(X, Y)$.

Let X be a non empty set and let Y be a real normed space. One can verify that $\langle \mathrm{BdFuncs}(X, Y), \mathrm{Zero\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Add\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Mult\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y))\rangle$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

One can prove the following proposition

(9) For every non empty set X and for every real normed space Y holds $\langle \mathrm{BdFuncs}(X, Y), \mathrm{Zero\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Add\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Mult\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y))\rangle$ is a real linear space.

Let X be a non empty set and let Y be a real normed space. The set of bounded real sequences from X into Y yields a real linear space and is defined as follows:

(Def. 6) The set of bounded real sequences from X into $Y = \langle \mathrm{BdFuncs}(X, Y), \mathrm{Zero\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Add\_}(\mathrm{BdFuncs}(X, Y),$

RealVectSpace(X, Y)), Mult_(BdFuncs(X, Y), RealVectSpace(X, Y)))⟩.

Let X be a non empty set and let Y be a real normed space. Observe that the set of bounded real sequences from X into Y is strict.

One can prove the following three propositions:

(10) Let X be a non empty set, Y be a real normed space, f, g, h be vectors of the set of bounded real sequences from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element x of X holds $h'(x) = f'(x) + g'(x)$.

(11) Let X be a non empty set, Y be a real normed space, f, h be vectors of the set of bounded real sequences from X into Y, and f', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $h' = h$. Let a be a real number. Then $h = a \cdot f$ if and only if for every element x of X holds $h'(x) = a \cdot f'(x)$.

(12) Let X be a non empty set and Y be a real normed space. Then $0_{\text{the set of bounded real sequences from } X \text{ into } Y} = X \longmapsto 0_Y$.

Let X be a non empty set, let Y be a real normed space, and let f be a set. Let us assume that $f \in \text{BdFuncs}(X, Y)$. The functor modetrans$(f, X, Y)$ yields a bounded function from X into the carrier of Y and is defined as follows:

(Def. 7) modetrans$(f, X, Y) = f$.

Let X be a non empty set, let Y be a real normed space, and let u be a function from X into the carrier of Y. The functor PreNorms(u) yielding a non empty subset of $\mathbb{R}$ is defined as follows:

(Def. 8) PreNorms$(u) = \{\|u(t)\| : t$ ranges over elements of $X\}$.

Next we state three propositions:

(13) Let X be a non empty set, Y be a real normed space, and g be a bounded function from X into the carrier of Y. Then PreNorms(g) is non empty and upper bounded.

(14) Let X be a non empty set, Y be a real normed space, and g be a function from X into the carrier of Y. Then g is bounded if and only if PreNorms(g) is upper bounded.

(15) Let X be a non empty set and Y be a real normed space. Then there exists a function N_1 from BdFuncs(X, Y) into $\mathbb{R}$ such that for every set f if $f \in \text{BdFuncs}(X, Y)$, then $N_1(f) = \sup \text{PreNorms}(\text{modetrans}(f, X, Y))$.

Let X be a non empty set and let Y be a real normed space. The functor BdFuncsNorm(X, Y) yielding a function from BdFuncs(X, Y) into $\mathbb{R}$ is defined by:

(Def. 9) For every set x such that $x \in \text{BdFuncs}(X, Y)$ holds BdFuncsNorm$(X, Y)(x) = \sup \text{PreNorms}(\text{modetrans}(x, X, Y))$.

One can prove the following two propositions:

(16) Let X be a non empty set, Y be a real normed space, and f be a bounded function from X into the carrier of Y. Then $\mathrm{modetrans}(f, X, Y) = f$.

(17) Let X be a non empty set, Y be a real normed space, and f be a bounded function from X into the carrier of Y. Then $\mathrm{BdFuncsNorm}(X, Y)(f) = \sup \mathrm{PreNorms}(f)$.

Let X be a non empty set and let Y be a real normed space. The real normed space of bounded functions from X into Y yielding a non empty normed structure is defined as follows:

(Def. 10) The real normed space of bounded functions from X into Y = $\langle \mathrm{BdFuncs}(X, Y), \mathrm{Zero\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Add\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Mult\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{BdFuncsNorm}(X, Y)\rangle$.

We now state several propositions:

(18) Let X be a non empty set and Y be a real normed space. Then $X \longmapsto 0_Y = 0_{\text{the real normed space of bounded functions from } X \text{ into } Y}$.

(19) Let X be a non empty set, Y be a real normed space, f be a point of the real normed space of bounded functions from X into Y, and g be a bounded function from X into the carrier of Y. If $g = f$, then for every element t of X holds $\|g(t)\| \leqslant \|f\|$.

(20) Let X be a non empty set, Y be a real normed space, and f be a point of the real normed space of bounded functions from X into Y. Then $0 \leqslant \|f\|$.

(21) Let X be a non empty set, Y be a real normed space, and f be a point of the real normed space of bounded functions from X into Y. Suppose $f = 0_{\text{the real normed space of bounded functions from } X \text{ into } Y}$. Then $0 = \|f\|$.

(22) Let X be a non empty set, Y be a real normed space, f, g, h be points of the real normed space of bounded functions from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element x of X holds $h'(x) = f'(x) + g'(x)$.

(23) Let X be a non empty set, Y be a real normed space, f, h be points of the real normed space of bounded functions from X into Y, and f', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $h' = h$. Let a be a real number. Then $h = a \cdot f$ if and only if for every element x of X holds $h'(x) = a \cdot f'(x)$.

(24) Let X be a non empty set, Y be a real normed space, f, g be points of the real normed space of bounded functions from X into Y, and a be a real number. Then

(i) $\|f\| = 0$ iff $f = 0_{\text{the real normed space of bounded functions from } X \text{ into } Y}$,

(ii) $\|a \cdot f\| = |a| \cdot \|f\|$, and

(iii) $\|f + g\| \leqslant \|f\| + \|g\|$.

(25) Let X be a non empty set and Y be a real normed space. Then the real normed space of bounded functions from X into Y is real normed space-like.

(26) Let X be a non empty set and Y be a real normed space. Then the real normed space of bounded functions from X into Y is a real normed space.

Let X be a non empty set and let Y be a real normed space. Observe that the real normed space of bounded functions from X into Y is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

We now state three propositions:

(27) Let X be a non empty set, Y be a real normed space, f, g, h be points of the real normed space of bounded functions from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f - g$ if and only if for every element x of X holds $h'(x) = f'(x) - g'(x)$.

(28) Let X be a non empty set and Y be a real normed space. Suppose Y is complete. Let s_1 be a sequence of the real normed space of bounded functions from X into Y. If s_1 is Cauchy sequence by norm, then s_1 is convergent.

(29) Let X be a non empty set and Y be a real Banach space. Then the real normed space of bounded functions from X into Y is a real Banach space.

Let X be a non empty set and let Y be a real Banach space. One can verify that the real normed space of bounded functions from X into Y is complete.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.
[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[7] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.
[8] Noboru Endou, Yasumasa Suzuki, and Yasunari Shidama. Real linear space of real sequences. *Formalized Mathematics*, 11(**3**):249–253, 2003.
[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[10] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[11] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[12] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.

[13] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.
[14] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[16] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[17] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[18] Yasunari Shidama. Banach space of bounded linear operators. *Formalized Mathematics*, 12(**1**):39–48, 2003.
[19] Yasumasa Suzuki, Noboru Endou, and Yasunari Shidama. Banach space of absolute summable real sequences. *Formalized Mathematics*, 11(**4**):377–380, 2003.
[20] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[21] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[22] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[23] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[24] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(**4**):341–347, 2003.
[25] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.
[26] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[27] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[28] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[29] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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