Continuous Functions on Real and Complex Normed Linear Spaces

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Summary. This article is an extension of [18].

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The notation and terminology used here are introduced in the following papers: [25], [28], [29], [4], [30], [6], [14], [5], [2], [24], [10], [26], [27], [19], [15], [12], [13], [11], [31], [20], [3], [1], [16], [21], [17], [23], [7], [8], [22], [18], and [9].

For simplicity, we use the following convention: $n$ denotes a natural number, $r, s$ denote real numbers, $z$ denotes a complex number, $C_1, C_2, C_3$ denote complex normed spaces, and $R_1$ denotes a real normed space.

Let $C_4$ be a complex linear space and let $s_1$ be a sequence of $C_4$. The functor $-s_1$ yields a sequence of $C_4$ and is defined by:

(Def. 1) For every $n$ holds $(-s_1)(n) = -s_1(n)$.

The following propositions are true:

1. For all sequences $s_2, s_3$ of $C_1$ holds $s_2 - s_3 = s_2 + (-s_3)$.
2. For every sequence $s_1$ of $C_1$ holds $-s_1 = (-1_C) \cdot s_1$.

Let us consider $C_2, C_3$ and let $f$ be a partial function from $C_2$ to $C_3$. The functor $\|f\|$ yielding a partial function from the carrier of $C_2$ to $R$ is defined by:

(Def. 2) $\text{dom} \|f\| = \text{dom} f$ and for every point $c$ of $C_2$ such that $c \in \text{dom} \|f\|$ holds $\|f\|(c) = \|f_c\|$. 

Let us consider $C_1, R_1$ and let $f$ be a partial function from $C_1$ to $R_1$. The functor $\|f\|$ yielding a partial function from the carrier of $C_1$ to $R$ is defined as follows:

(Def. 3) $\text{dom} \|f\| = \text{dom} f$ and for every point $c$ of $C_1$ such that $c \in \text{dom} \|f\|$ holds $\|f\|(c) = \|f_c\|$. 

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Let us consider $R_1$, $C_1$ and let $f$ be a partial function from $R_1$ to $C_1$. The functor $\|f\|$ yielding a partial function from the carrier of $R_1$ to $\mathbb{R}$ is defined by:

(Def. 4) \[ \text{dom}\|f\| = \text{dom} f \text{ and for every point } c \text{ of } R_1 \text{ such that } c \in \text{dom}\|f\| \text{ holds } \|f\|(c) = \|f_c\|. \]

Let us consider $C_1$ and let $x_0$ be a point of $C_1$. A subset of $C_1$ is called a neighbourhood of $x_0$ if:

(Def. 5) There exists a real number $g$ such that $0 < g$ and \{ $y; y$ ranges over points of $C_1$: $\|y - x_0\| < g$ \} $\subseteq$ it.

Next we state two propositions:

(3) Let $x_0$ be a point of $C_1$ and $g$ be a real number. If $0 < g$, then \{ $y; y$ ranges over points of $C_1$: $\|y - x_0\| < g$ \} is a neighbourhood of $x_0$.

(4) For every point $x_0$ of $C_1$ and for every neighbourhood $N$ of $x_0$ holds $x_0 \in N$.

Let us consider $C_1$ and let $X$ be a subset of $C_1$. We say that $X$ is compact if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let $s_4$ be a sequence of $C_1$. Suppose \text{rng } s_4 \subseteq X$. Then there exists a sequence $s_5$ of $C_1$ such that $s_5$ is a subsequence of $s_4$ and convergent and $\lim s_5 \in X$.

Let us consider $C_1$ and let $X$ be a subset of $C_1$. We say that $X$ is closed if and only if:

(Def. 7) For every sequence $s_4$ of $C_1$ such that $\text{rng } s_4 \subseteq X$ and $s_4$ is convergent holds $\lim s_4 \in X$.

Let us consider $C_1$ and let $X$ be a subset of $C_1$. We say that $X$ is open if and only if:

(Def. 8) $X^c$ is closed.

Let us consider $C_2$, $C_3$, let $f$ be a partial function from $C_2$ to $C_3$, and let $s_1$ be a sequence of $C_2$. Let us assume that $\text{rng } s_1 \subseteq \text{dom } f$. The functor $f \cdot s_1$ yields a sequence of $C_3$ and is defined by:

(Def. 9) $f \cdot s_1 = (f \text{ qua function}) \cdot (s_1)$.

Let us consider $C_1$, $R_1$, let $f$ be a partial function from $C_1$ to $R_1$, and let $s_1$ be a sequence of $C_1$. Let us assume that $\text{rng } s_1 \subseteq \text{dom } f$. The functor $f \cdot s_1$ yielding a sequence of $R_1$ is defined by:

(Def. 10) $f \cdot s_1 = (f \text{ qua function}) \cdot (s_1)$.

Let us consider $C_1$, $R_1$, let $f$ be a partial function from $R_1$ to $C_1$, and let $s_1$ be a sequence of $R_1$. Let us assume that $\text{rng } s_1 \subseteq \text{dom } f$. The functor $f \cdot s_1$ yields a sequence of $C_1$ and is defined by:

(Def. 11) $f \cdot s_1 = (f \text{ qua function}) \cdot (s_1)$.
\[ f \cdot s_1 \text{ yields a complex sequence and is defined as follows:} \]

(Def. 12) \[ f \cdot s_1 = (f \text{ qua function}) \cdot (s_1). \]

Let us consider \( R_1 \), let \( f \) be a partial function from the carrier of \( R_1 \) to \( \mathbb{C} \), and let \( s_1 \) be a sequence of \( R_1 \). Let us assume that \( \text{rng} \, s_1 \subseteq \text{dom} \, f \). The functor \( f \cdot s_1 \) yielding a complex sequence is defined by:

(Def. 13) \[ f \cdot s_1 = (f \text{ qua function}) \cdot (s_1). \]

Let us consider \( C_1 \), let \( f \) be a partial function from the carrier of \( C_1 \) to \( \mathbb{R} \), and let \( s_1 \) be a sequence of \( C_1 \). Let us assume that \( \text{rng} \, s_1 \subseteq \text{dom} \, f \). The functor \( f \cdot s_1 \) yielding a sequence of real numbers is defined as follows:

(Def. 14) \[ f \cdot s_1 = (f \text{ qua function}) \cdot (s_1). \]

Let us consider \( C_2, C_3 \), let \( f \) be a partial function from \( C_2 \) to \( C_3 \), and let \( x_0 \) be a point of \( C_2 \). We say that \( f \) is continuous in \( x_0 \) if and only if the conditions (Def. 15) are satisfied.

(Def. 15)(i) \[ x_0 \in \text{dom} \, f, \] and
(ii) for every sequence \( s_1 \) of \( C_2 \) such that \( \text{rng} \, s_1 \subseteq \text{dom} \, f \) and \( s_1 \) is convergent and \( \lim s_1 = x_0 \) holds \( f \cdot s_1 \) is convergent and \( f_{x_0} = \lim(f \cdot s_1) \).

Let us consider \( C_1, R_1 \), let \( f \) be a partial function from \( C_1 \) to \( R_1 \), and let \( x_0 \) be a point of \( C_1 \). We say that \( f \) is continuous in \( x_0 \) if and only if the conditions (Def. 16) are satisfied.

(Def. 16)(i) \[ x_0 \in \text{dom} \, f, \] and
(ii) for every sequence \( s_1 \) of \( C_1 \) such that \( \text{rng} \, s_1 \subseteq \text{dom} \, f \) and \( s_1 \) is convergent and \( \lim s_1 = x_0 \) holds \( f \cdot s_1 \) is convergent and \( f_{x_0} = \lim(f \cdot s_1) \).

Let us consider \( R_1 \), let us consider \( C_1 \), let \( f \) be a partial function from \( R_1 \) to \( C_1 \), and let \( x_0 \) be a point of \( R_1 \). We say that \( f \) is continuous in \( x_0 \) if and only if the conditions (Def. 17) are satisfied.

(Def. 17)(i) \[ x_0 \in \text{dom} \, f, \] and
(ii) for every sequence \( s_1 \) of \( R_1 \) such that \( \text{rng} \, s_1 \subseteq \text{dom} \, f \) and \( s_1 \) is convergent and \( \lim s_1 = x_0 \) holds \( f \cdot s_1 \) is convergent and \( f_{x_0} = \lim(f \cdot s_1) \).

Let us consider \( C_1 \), let \( f \) be a partial function from the carrier of \( C_1 \) to \( \mathbb{C} \), and let \( x_0 \) be a point of \( C_1 \). We say that \( f \) is continuous in \( x_0 \) if and only if the conditions (Def. 18) are satisfied.

(Def. 18)(i) \[ x_0 \in \text{dom} \, f, \] and
(ii) for every sequence \( s_1 \) of \( C_1 \) such that \( \text{rng} \, s_1 \subseteq \text{dom} \, f \) and \( s_1 \) is convergent and \( \lim s_1 = x_0 \) holds \( f \cdot s_1 \) is convergent and \( f_{x_0} = \lim(f \cdot s_1) \).

Let us consider \( C_1 \), let \( f \) be a partial function from the carrier of \( C_1 \) to \( \mathbb{R} \), and let \( x_0 \) be a point of \( C_1 \). We say that \( f \) is continuous in \( x_0 \) if and only if the conditions (Def. 19) are satisfied.

(Def. 19)(i) \[ x_0 \in \text{dom} \, f, \] and
(ii) for every sequence \( s_1 \) of \( C_1 \) such that \( \text{rng} \, s_1 \subseteq \text{dom} \, f \) and \( s_1 \) is convergent and \( \lim s_1 = x_0 \) holds \( f \cdot s_1 \) is convergent and \( f_{x_0} = \lim(f \cdot s_1) \).
Let us consider $R_1$, let $f$ be a partial function from the carrier of $R_1$ to $\mathbb{C}$, and let $x_0$ be a point of $R_1$. We say that $f$ is continuous in $x_0$ if and only if the conditions (Def. 20) are satisfied.

(Def. 20)(i) $x_0 \in \text{dom } f$, and

(ii) for every sequence $s_1$ of $R_1$ such that rng $s_1 \subseteq \text{dom } f$ and $s_1$ is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f x_0 = \lim(f \cdot s_1)$.

The following propositions are true:

(5) For every sequence $s_1$ of $C_2$ and for every partial function $h$ from $C_2$ to $C_3$ such that rng $s_1 \subseteq \text{dom } h$ holds $s_1(n) \in \text{dom } h$.

(6) For every sequence $s_1$ of $C_1$ and for every partial function $h$ from $C_1$ to $R_1$ such that rng $s_1 \subseteq \text{dom } h$ holds $s_1(n) \in \text{dom } h$.

(7) For every sequence $s_1$ of $R_1$ and for every partial function $h$ from $R_1$ to $C_1$ such that rng $s_1 \subseteq \text{dom } h$ holds $s_1(n) \in \text{dom } h$.

(8) For every sequence $s_1$ of $C_1$ and for every set $x$ holds $x \in \text{rng } s_1$ iff there exists $n$ such that $x = s_1(n)$.

(9) For all sequences $s_1$, $s_2$ of $C_1$ such that $s_2$ is a subsequence of $s_1$ holds rng $s_2 \subseteq \text{rng } s_1$.

(10) Let $f$ be a partial function from $C_2$ to $C_3$ and $C_5$ be a sequence of $C_2$.

If rng $C_5 \subseteq \text{dom } f$, then for every $n$ holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(11) Let $f$ be a partial function from $C_1$ to $R_1$ and $C_5$ be a sequence of $C_1$.

If rng $C_5 \subseteq \text{dom } f$, then for every $n$ holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(12) Let $f$ be a partial function from $R_1$ to $C_1$ and $R_2$ be a sequence of $R_1$.

If rng $R_2 \subseteq \text{dom } f$, then for every $n$ holds $(f \cdot R_2)(n) = f_{R_2(n)}$.

(13) Let $f$ be a partial function from the carrier of $C_1$ to $\mathbb{C}$ and $C_5$ be a sequence of $C_1$. If rng $C_5 \subseteq \text{dom } f$, then for every $n$ holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(14) Let $f$ be a partial function from the carrier of $C_1$ to $\mathbb{R}$ and $C_5$ be a sequence of $C_1$. If rng $C_5 \subseteq \text{dom } f$, then for every $n$ holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(15) Let $f$ be a partial function from the carrier of $R_1$ to $\mathbb{C}$ and $R_2$ be a sequence of $R_1$. If rng $R_2 \subseteq \text{dom } f$, then for every $n$ holds $(f \cdot R_2)(n) = f_{R_2(n)}$.

(16) Let $h$ be a partial function from $C_2$ to $C_3$, $C_5$ be a sequence of $C_2$, and $N_1$ be an increasing sequence of naturals. If rng $C_5 \subseteq \text{dom } h$, then $(h \cdot C_5) \cdot N_1 = h \cdot (C_5 \cdot N_1)$.

(17) Let $h$ be a partial function from $C_1$ to $R_1$, $C_6$ be a sequence of $C_1$, and $N_1$ be an increasing sequence of naturals. If rng $C_6 \subseteq \text{dom } h$, then $(h \cdot C_6) \cdot N_1 = h \cdot (C_6 \cdot N_1)$.

(18) Let $h$ be a partial function from $R_1$ to $C_1$, $R_3$ be a sequence of $R_1$,
and $N_1$ be an increasing sequence of naturals. If $\text{rng} \, R_3 \subseteq \text{dom} \, h$, then 
$(h \cdot R_3) \cdot N_1 = h \cdot (R_3 \cdot N_1)$.

(19) Let $h$ be a partial function from the carrier of $C_1$ to $\mathbb{C}$, $C_6$ be a sequence of $C_1$, and $N_1$ be an increasing sequence of naturals. If $\text{rng} \, C_6 \subseteq \text{dom} \, h$, then 
$(h \cdot C_6) \cdot N_1 = h \cdot (C_6 \cdot N_1)$.

(20) Let $h$ be a partial function from the carrier of $C_1$ to $\mathbb{R}$, $C_6$ be a sequence of $C_1$, and $N_1$ be an increasing sequence of naturals. If $\text{rng} \, C_6 \subseteq \text{dom} \, h$, then 
$(h \cdot C_6) \cdot N_1 = h \cdot (C_6 \cdot N_1)$.

(21) Let $h$ be a partial function from the carrier of $R_1$ to $\mathbb{C}$, $R_3$ be a sequence of $R_1$, and $N_1$ be an increasing sequence of naturals. If $\text{rng} \, R_3 \subseteq \text{dom} \, h$, then 
$(h \cdot R_3) \cdot N_1 = h \cdot (R_3 \cdot N_1)$.

(22) Let $h$ be a partial function from $C_2$ to $C_3$ and $C_7$, $C_8$ be sequences of $C_2$. If $\text{rng} \, C_7 \subseteq \text{dom} \, h$ and $C_8$ is a subsequence of $C_7$, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(23) Let $h$ be a partial function from $C_1$ to $R_1$ and $C_7$, $C_8$ be sequences of $C_1$. If $\text{rng} \, C_7 \subseteq \text{dom} \, h$ and $C_8$ is a subsequence of $C_7$, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(24) Let $h$ be a partial function from $R_1$ to $C_1$ and $R_4$, $R_5$ be sequences of $R_1$. If $\text{rng} \, R_4 \subseteq \text{dom} \, h$ and $R_5$ is a subsequence of $R_4$, then $h \cdot R_5$ is a subsequence of $h \cdot R_4$.

(25) Let $s_1$ be a complex sequence, $n$ be a natural number, and $N_2$ be an increasing sequence of naturals. Then $(s_1 \cdot N_2)(n) = s_1(N_2(n))$.

(26) Let $h$ be a partial function from the carrier of $C_1$ to $\mathbb{C}$ and $C_7$, $C_8$ be sequences of $C_1$. If $\text{rng} \, C_7 \subseteq \text{dom} \, h$ and $C_8$ is a subsequence of $C_7$, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(27) Let $h$ be a partial function from the carrier of $C_1$ to $\mathbb{R}$ and $C_7$, $C_8$ be sequences of $C_1$. If $\text{rng} \, C_7 \subseteq \text{dom} \, h$ and $C_8$ is a subsequence of $C_7$, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(28) Let $h$ be a partial function from the carrier of $R_1$ to $\mathbb{C}$ and $R_4$, $R_5$ be sequences of $R_1$. If $\text{rng} \, R_4 \subseteq \text{dom} \, h$ and $R_5$ is a subsequence of $R_4$, then $h \cdot R_5$ is a subsequence of $h \cdot R_4$.

(29) Let $f$ be a partial function from $C_2$ to $C_3$ and $x_0$ be a point of $C_2$. Then $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:

(i) $x_0 \in \text{dom} \, f$, and

(ii) for every $r$ such that $0 < r$ there exists $s$ such that $0 < s$ and for every point $x_1$ of $C_2$ such that $x_1 \in \text{dom} \, f$ and $\|x_1 - x_0\| < s$ holds $\|f(x_1) - f(x_0)\| < r$.

(30) Let $f$ be a partial function from $C_1$ to $R_1$ and $x_0$ be a point of $C_1$. Then $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:

(i) $x_0 \in \text{dom} \, f$, and
(ii) for every \( r \) such that \( 0 < r \) there exists \( s \) such that \( 0 < s \) and for every point \( x_1 \) of \( C_1 \) such that \( x_1 \in \text{dom } f \) and \( \|x_1 - x_0\| < s \) holds \( \|f_{x_1} - f_{x_0}\| < r \).

(31) Let \( f \) be a partial function from \( R_1 \) to \( C_1 \) and \( x_0 \) be a point of \( R_1 \). Then \( f \) is continuous in \( x_0 \) if and only if the following conditions are satisfied:

(i) \( x_0 \in \text{dom } f \), and

(ii) for every \( r \) such that \( 0 < r \) there exists \( s \) such that \( 0 < s \) and for every point \( x_1 \) of \( R_1 \) such that \( x_1 \in \text{dom } f \) and \( \|x_1 - x_0\| < s \) holds \( \|f_{x_1} - f_{x_0}\| < r \).

(32) Let \( f \) be a partial function from the carrier of \( C_1 \) to \( \mathbb{R} \) and \( x_0 \) be a point of \( C_1 \). Then \( f \) is continuous in \( x_0 \) if and only if the following conditions are satisfied:

(i) \( x_0 \in \text{dom } f \), and

(ii) for every \( r \) such that \( 0 < r \) there exists \( s \) such that \( 0 < s \) and for every point \( x_1 \) of \( C_1 \) such that \( x_1 \in \text{dom } f \) and \( \|x_1 - x_0\| < s \) holds \( |f_{x_1} - f_{x_0}| < r \).

(33) Let \( f \) be a partial function from the carrier of \( C_1 \) to \( \mathbb{C} \) and \( x_0 \) be a point of \( C_1 \). Then \( f \) is continuous in \( x_0 \) if and only if the following conditions are satisfied:

(i) \( x_0 \in \text{dom } f \), and

(ii) for every \( r \) such that \( 0 < r \) there exists \( s \) such that \( 0 < s \) and for every point \( x_1 \) of \( C_1 \) such that \( x_1 \in \text{dom } f \) and \( \|x_1 - x_0\| < s \) holds \( |f_{x_1} - f_{x_0}| < r \).

(34) Let \( f \) be a partial function from the carrier of \( R_1 \) to \( \mathbb{C} \) and \( x_0 \) be a point of \( R_1 \). Then \( f \) is continuous in \( x_0 \) if and only if the following conditions are satisfied:

(i) \( x_0 \in \text{dom } f \), and

(ii) for every \( r \) such that \( 0 < r \) there exists \( s \) such that \( 0 < s \) and for every point \( x_1 \) of \( R_1 \) such that \( x_1 \in \text{dom } f \) and \( \|x_1 - x_0\| < s \) holds \( |f_{x_1} - f_{x_0}| < r \).

(35) Let \( f \) be a partial function from \( C_2 \) to \( C_3 \) and \( x_0 \) be a point of \( C_2 \). Then \( f \) is continuous in \( x_0 \) if and only if the following conditions are satisfied:

(i) \( x_0 \in \text{dom } f \), and

(ii) for every neighbourhood \( N_3 \) of \( f_{x_0} \) there exists a neighbourhood \( N \) of \( x_0 \) such that for every point \( x_1 \) of \( C_2 \) such that \( x_1 \in \text{dom } f \) and \( x_1 \in N \) holds \( f_{x_1} \in N_3 \).

(36) Let \( f \) be a partial function from \( C_1 \) to \( R_1 \) and \( x_0 \) be a point of \( C_1 \). Then \( f \) is continuous in \( x_0 \) if and only if the following conditions are satisfied:

(i) \( x_0 \in \text{dom } f \), and

(ii) for every neighbourhood \( N_3 \) of \( f_{x_0} \) there exists a neighbourhood \( N \) of \( x_0 \) such that for every point \( x_1 \) of \( C_1 \) such that \( x_1 \in \text{dom } f \) and \( x_1 \in N \) holds \( f_{x_1} \in N_3 \).

(37) Let \( f \) be a partial function from \( R_1 \) to \( C_1 \) and \( x_0 \) be a point of \( R_1 \). Then \( f \) is continuous in \( x_0 \) if and only if the following conditions are satisfied:
Let $f$ be a partial function from $C_2$ to $C_3$ and $x_0$ be a point of $C_2$. Then $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:

(i) $x_0 \in \text{dom } f$, and

(ii) for every neighbourhood $N_3$ of $f(x_0)$ there exists a neighbourhood $N$ of $x_0$ such that for every point $x_1$ of $R_1$ such that $x_1 \in \text{dom } f$ and $x_1 \in N$ holds $f(x_1) \in N_3$.

(38) Let $f$ be a partial function from $C_2$ to $C_3$ and $x_0$ be a point of $C_2$. Then $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:

(i) $x_0 \in \text{dom } f$, and

(ii) for every neighbourhood $N_3$ of $f(x_0)$ there exists a neighbourhood $N$ of $x_0$ such that $f^0N \subseteq N_3$.

(39) Let $f$ be a partial function from $C_1$ to $R_1$ and $x_0$ be a point of $C_1$. Then $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:

(i) $x_0 \in \text{dom } f$, and

(ii) for every neighbourhood $N_3$ of $f(x_0)$ there exists a neighbourhood $N$ of $x_0$ such that $f^0N \subseteq N_3$.

(40) Let $f$ be a partial function from $R_1$ to $C_1$ and $x_0$ be a point of $R_1$. Then $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:

(i) $x_0 \in \text{dom } f$, and

(ii) for every neighbourhood $N_3$ of $f(x_0)$ there exists a neighbourhood $N$ of $x_0$ such that $f^0N \subseteq N_3$.

(41) Let $f$ be a partial function from $C_2$ to $C_3$ and $x_0$ be a point of $C_2$. Suppose $x_0 \in \text{dom } f$ and there exists a neighbourhood $N$ of $x_0$ such that $\text{dom } f \cap N = \{x_0\}$. Then $f$ is continuous in $x_0$.

(42) Let $f$ be a partial function from $C_1$ to $R_1$ and $x_0$ be a point of $C_1$. Suppose $x_0 \in \text{dom } f$ and there exists a neighbourhood $N$ of $x_0$ such that $\text{dom } f \cap N = \{x_0\}$. Then $f$ is continuous in $x_0$.

(43) Let $f$ be a partial function from $R_1$ to $C_1$ and $x_0$ be a point of $R_1$. Suppose $x_0 \in \text{dom } f$ and there exists a neighbourhood $N$ of $x_0$ such that $\text{dom } f \cap N = \{x_0\}$. Then $f$ is continuous in $x_0$.

(44) Let $h_1, h_2$ be partial functions from $C_2$ to $C_3$ and $s_1$ be a sequence of $C_2$. If $\text{rng } s_1 \subseteq \text{dom } h_1 \cap \text{dom } h_2$, then $(h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1$ and $(h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1$.

(45) Let $h_1, h_2$ be partial functions from $C_1$ to $R_1$ and $s_1$ be a sequence of $C_1$. If $\text{rng } s_1 \subseteq \text{dom } h_1 \cap \text{dom } h_2$, then $(h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1$ and $(h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1$.

(46) Let $h_1, h_2$ be partial functions from $R_1$ to $C_1$ and $s_1$ be a sequence of $R_1$. If $\text{rng } s_1 \subseteq \text{dom } h_1 \cap \text{dom } h_2$, then $(h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1$ and $(h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1$.

(47) Let $h$ be a partial function from $C_2$ to $C_3$, $s_1$ be a sequence of $C_2$, and $z$ be a complex number. If $\text{rng } s_1 \subseteq \text{dom } h$, then $(z \cdot h) \cdot s_1 = z \cdot (h \cdot s_1)$.
(48) Let $h$ be a partial function from $C_1$ to $R_1$, $s_1$ be a sequence of $C_1$, and $r$ be a real number. If $\text{rng} \ s_1 \subseteq \text{dom} \ h$, then $(r \cdot h) \cdot s_1 = r \cdot (h \cdot s_1)$.

(49) Let $h$ be a partial function from $R_1$ to $C_1$, $s_1$ be a sequence of $R_1$, and $z$ be a complex number. If $\text{rng} \ s_1 \subseteq \text{dom} \ h$, then $(z \cdot h) \cdot s_1 = z \cdot (h \cdot s_1)$.

(50) Let $h$ be a partial function from $C_2$ to $C_3$ and $s_1$ be a sequence of $C_2$. If $\text{rng} \ s_1 \subseteq \text{dom} \ h$, then $\|h \cdot s_1\| = \|h\| \cdot s_1$ and $-h \cdot s_1 = (-h) \cdot s_1$.

(51) Let $h$ be a partial function from $C_1$ to $R_1$ and $s_1$ be a sequence of $C_1$. If $\text{rng} \ s_1 \subseteq \text{dom} \ h$, then $\|h \cdot s_1\| = \|h\| \cdot s_1$ and $-h \cdot s_1 = (-h) \cdot s_1$.

(52) Let $h$ be a partial function from $R_1$ to $C_1$ and $s_1$ be a sequence of $R_1$. If $\text{rng} \ s_1 \subseteq \text{dom} \ h$, then $\|h \cdot s_1\| = \|h\| \cdot s_1$ and $-h \cdot s_1 = (-h) \cdot s_1$.

(53) Let $f_1$, $f_2$ be partial functions from $C_2$ to $C_3$ and $x_0$ be a point of $C_2$. Suppose $f_1$ is continuous in $x_0$ and $f_2$ is continuous in $x_0$. Then $f_1 + f_2$ is continuous in $x_0$ and $f_1 - f_2$ is continuous in $x_0$.

(54) Let $f_1$, $f_2$ be partial functions from $C_1$ to $R_1$ and $x_0$ be a point of $C_1$. Suppose $f_1$ is continuous in $x_0$ and $f_2$ is continuous in $x_0$. Then $f_1 + f_2$ is continuous in $x_0$ and $f_1 - f_2$ is continuous in $x_0$.

(55) Let $f_1$, $f_2$ be partial functions from $R_1$ to $C_1$ and $x_0$ be a point of $R_1$. Suppose $f_1$ is continuous in $x_0$ and $f_2$ is continuous in $x_0$. Then $f_1 + f_2$ is continuous in $x_0$ and $f_1 - f_2$ is continuous in $x_0$.

(56) Let $f$ be a partial function from $C_2$ to $C_3$, $x_0$ be a point of $C_2$, and $z$ be a complex number. If $f$ is continuous in $x_0$, then $z \cdot f$ is continuous in $x_0$.

(57) Let $f$ be a partial function from $C_1$ to $R_1$, $x_0$ be a point of $C_1$, and $r$ be a real number. If $f$ is continuous in $x_0$, then $r \cdot f$ is continuous in $x_0$.

(58) Let $f$ be a partial function from $R_1$ to $C_1$, $x_0$ be a point of $R_1$, and $z$ be a complex number. If $f$ is continuous in $x_0$, then $z \cdot f$ is continuous in $x_0$.

(59) Let $f$ be a partial function from $C_2$ to $C_3$ and $x_0$ be a point of $C_2$. If $f$ is continuous in $x_0$, then $\|f\|$ is continuous in $x_0$ and $-f$ is continuous in $x_0$.

(60) Let $f$ be a partial function from $C_1$ to $R_1$ and $x_0$ be a point of $C_1$. If $f$ is continuous in $x_0$, then $\|f\|$ is continuous in $x_0$ and $-f$ is continuous in $x_0$.

(61) Let $f$ be a partial function from $R_1$ to $C_1$ and $x_0$ be a point of $R_1$. If $f$ is continuous in $x_0$, then $\|f\|$ is continuous in $x_0$ and $-f$ is continuous in $x_0$.

Let $C_2$, $C_3$ be complex normed spaces, let $f$ be a partial function from $C_2$ to $C_3$, and let $X$ be a set. We say that $f$ is continuous on $X$ if and only if:

(Def. 21) $X \subseteq \text{dom} \ f$ and for every point $x_0$ of $C_2$ such that $x_0 \in X$ holds $f|X$ is continuous in $x_0$.

Let $C_1$ be a complex normed space, let $R_1$ be a real normed space, let $f$ be a
partial function from $C_1$ to $R_1$, and let $X$ be a set. We say that $f$ is continuous on $X$ if and only if:

(Def. 22) $X \subseteq \text{dom } f$ and for every point $x_0$ of $C_1$ such that $x_0 \in X$ holds $f|X$ is continuous in $x_0$.

Let $R_1$ be a real normed space, let $C_1$ be a complex normed space, let $g$ be a partial function from $R_1$ to $C_1$, and let $X$ be a set. We say that $g$ is continuous on $X$ if and only if:

(Def. 23) $X \subseteq \text{dom } g$ and for every point $x_0$ of $R_1$ such that $x_0 \in X$ holds $g|X$ is continuous in $x_0$.

Let $C_1$ be a complex normed space, let $f$ be a partial function from the carrier of $C_1$ to $C$, and let $X$ be a set. We say that $f$ is continuous on $X$ if and only if:

(Def. 24) $X \subseteq \text{dom } f$ and for every point $x_0$ of $C_1$ such that $x_0 \in X$ holds $f|X$ is continuous in $x_0$.

Let $R_1$ be a real normed space, let $f$ be a partial function from the carrier of $R_1$ to $C$, and let $X$ be a set. We say that $f$ is continuous on $X$ if and only if:

(Def. 25) $X \subseteq \text{dom } f$ and for every point $x_0$ of $R_1$ such that $x_0 \in X$ holds $f|X$ is continuous in $x_0$.

In the sequel $X$, $X_1$ denote sets.

The following propositions are true:

(62) Let $f$ be a partial function from $C_2$ to $C_3$. Then $f$ is continuous on $X$ if and only if the following conditions are satisfied:

(i) $X \subseteq \text{dom } f$, and
(ii) for every sequence $s_4$ of $C_2$ such that $\text{rng } s_4 \subseteq X$ and $s_4$ is convergent and $\lim s_4 \in X$ holds $f \cdot s_4$ is convergent and $f_{\lim s_4} = \lim(f \cdot s_4)$.

(63) Let $f$ be a partial function from $C_1$ to $R_1$. Then $f$ is continuous on $X$ if and only if the following conditions are satisfied:

(i) $X \subseteq \text{dom } f$, and
(ii) for every sequence $s_4$ of $C_1$ such that $\text{rng } s_4 \subseteq X$ and $s_4$ is convergent and $\lim s_4 \in X$ holds $f \cdot s_4$ is convergent and $f_{\lim s_4} = \lim(f \cdot s_4)$.

(64) Let $f$ be a partial function from $R_1$ to $C_1$. Then $f$ is continuous on $X$ if and only if the following conditions are satisfied:

(i) $X \subseteq \text{dom } f$, and
(ii) for every sequence $s_4$ of $R_1$ such that $\text{rng } s_4 \subseteq X$ and $s_4$ is convergent and $\lim s_4 \in X$ holds $f \cdot s_4$ is convergent and $f_{\lim s_4} = \lim(f \cdot s_4)$. 
(65) Let $f$ be a partial function from $C_2$ to $C_3$. Then $f$ is continuous on $X$ if and only if the following conditions are satisfied:

(i) $X \subseteq \text{dom } f$, and

(ii) for every point $x_0$ of $C_2$ and for every $r$ such that $x_0 \in X$ and $0 < r$
there exists $s$ such that $0 < s$ and for every point $x_1$ of $C_2$ such that
$x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(66) Let $f$ be a partial function from $C_1$ to $R_1$. Then $f$ is continuous on $X$ if and only if the following conditions are satisfied:

(i) $X \subseteq \text{dom } f$, and

(ii) for every point $x_0$ of $C_1$ and for every $r$ such that $x_0 \in X$ and $0 < r$
there exists $s$ such that $0 < s$ and for every point $x_1$ of $C_1$ such that
$x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(67) Let $f$ be a partial function from $R_1$ to $C_1$. Then $f$ is continuous on $X$ if and only if the following conditions are satisfied:

(i) $X \subseteq \text{dom } f$, and

(ii) for every point $x_0$ of $R_1$ and for every $r$ such that $x_0 \in X$ and $0 < r$
there exists $s$ such that $0 < s$ and for every point $x_1$ of $R_1$ such that
$x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(68) Let $f$ be a partial function from the carrier of $C_1$ to $C$. Then $f$ is continuous on $X$ if and only if the following conditions are satisfied:

(i) $X \subseteq \text{dom } f$, and

(ii) for every point $x_0$ of $C_1$ and for every $r$ such that $x_0 \in X$ and $0 < r$
there exists $s$ such that $0 < s$ and for every point $x_1$ of $C_1$ such that
$x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(69) Let $f$ be a partial function from the carrier of $C_1$ to $R$. Then $f$ is continuous on $X$ if and only if the following conditions are satisfied:

(i) $X \subseteq \text{dom } f$, and

(ii) for every point $x_0$ of $C_1$ and for every $r$ such that $x_0 \in X$ and $0 < r$
there exists $s$ such that $0 < s$ and for every point $x_1$ of $C_1$ such that
$x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(70) Let $f$ be a partial function from the carrier of $R_1$ to $C$. Then $f$ is continuous on $X$ if and only if the following conditions are satisfied:

(i) $X \subseteq \text{dom } f$, and

(ii) for every point $x_0$ of $R_1$ and for every $r$ such that $x_0 \in X$ and $0 < r$
there exists $s$ such that $0 < s$ and for every point $x_1$ of $R_1$ such that
$x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(71) For every partial function $f$ from $C_2$ to $C_3$ holds $f$ is continuous on $X$
iff $f | X$ is continuous on $X$.

(72) For every partial function $f$ from $C_1$ to $R_1$ holds $f$ is continuous on $X$
iff $f | X$ is continuous on $X$. 
For every partial function $f$ from $R_1$ to $C_1$ holds $f$ is continuous on $X$ iff $f|X$ is continuous on $X$.

Let $f$ be a partial function from the carrier of $C_1$ to $\mathbb{C}$. Then $f$ is continuous on $X$ if and only if $f|X$ is continuous on $X$.

Let $f$ be a partial function from the carrier of $C_1$ to $\mathbb{R}$. Then $f$ is continuous on $X$ if and only if $f|X$ is continuous on $X$.

Let $f$ be a partial function from the carrier of $R_1$ to $\mathbb{C}$. Then $f$ is continuous on $X$ if and only if $f|X$ is continuous on $X$.

For every partial function $f$ from $C_2$ to $C_3$ such that $f$ is continuous on $X$ and $X_1 \subseteq X$ holds $f$ is continuous on $X_1$.

For every partial function $f$ from $C_1$ to $R_1$ such that $f$ is continuous on $X$ and $X_1 \subseteq X$ holds $f$ is continuous on $X_1$.

For every partial function $f$ from $R_1$ to $C_1$ such that $f$ is continuous on $X$ and $X_1 \subseteq X$ holds $f$ is continuous on $X_1$.

For every partial function $f$ from $C_2$ to $C_3$ and for every point $x_0$ of $C_2$ such that $x_0 \in \text{dom } f$ holds $f$ is continuous on $\{x_0\}$.

For every partial function $f$ from $R_1$ to $C_1$ and for every point $x_0$ of $R_1$ such that $x_0 \in \text{dom } f$ holds $f$ is continuous on $\{x_0\}$.

Let $f_1$, $f_2$ be partial functions from $R_1$ to $C_3$. Suppose $f_1$ is continuous on $X$ and $f_2$ is continuous on $X$. Then $f_1 + f_2$ is continuous on $X$ and $f_1 - f_2$ is continuous on $X$.

Let $f_1$, $f_2$ be partial functions from $C_1$ to $R_1$. Suppose $f_1$ is continuous on $X$ and $f_2$ is continuous on $X$. Then $f_1 + f_2$ is continuous on $X$ and $f_1 - f_2$ is continuous on $X$.

Let $f_1$, $f_2$ be partial functions from $R_1$ to $C_1$. Suppose $f_1$ is continuous on $X$ and $f_2$ is continuous on $X$. Then $f_1 + f_2$ is continuous on $X$ and $f_1 - f_2$ is continuous on $X$.

Let $f_1$, $f_2$ be partial functions from $C_2$ to $C_3$. Suppose $f_1$ is continuous on $X$ and $f_2$ is continuous on $X_1$. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

Let $f_1$, $f_2$ be partial functions from $C_1$ to $R_1$. Suppose $f_1$ is continuous on $X$ and $f_2$ is continuous on $X_1$. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

Let $f_1$, $f_2$ be partial functions from $R_1$ to $C_1$. Suppose $f_1$ is continuous on $X$ and $f_2$ is continuous on $X_1$. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

For every partial function $f$ from $C_2$ to $C_3$ such that $f$ is continuous on...
\(X\) holds \(zf\) is continuous on \(X\).

(90) For every partial function \(f\) from \(C_1\) to \(R_1\) such that \(f\) is continuous on \(X\) holds \(rf\) is continuous on \(X\).

(91) For every partial function \(f\) from \(R_1\) to \(C_1\) such that \(f\) is continuous on \(X\) holds \(zf\) is continuous on \(X\).

(92) Let \(f\) be a partial function from \(C_2\) to \(C_3\). If \(f\) is continuous on \(X\), then \(||f||\) is continuous on \(X\) and \(-f\) is continuous on \(X\).

(93) Let \(f\) be a partial function from \(C_1\) to \(R_1\). If \(f\) is continuous on \(X\), then \(||f||\) is continuous on \(X\) and \(-f\) is continuous on \(X\).

(94) Let \(f\) be a partial function from \(R_1\) to \(C_1\). If \(f\) is continuous on \(X\), then \(||f||\) is continuous on \(X\) and \(-f\) is continuous on \(X\).

(95) Let \(f\) be a partial function from \(C_2\) to \(C_3\). Suppose \(f\) is total and for all points \(x_1, x_2\) of \(C_2\) holds \(f_{x_1}+x_2 = f_{x_1}+f_{x_2}\) and there exists a point \(x_0\) of \(C_2\) such that \(f\) is continuous in \(x_0\). Then \(f\) is continuous on the carrier of \(C_2\).

(96) Let \(f\) be a partial function from \(C_1\) to \(R_1\). Suppose \(f\) is total and for all points \(x_1, x_2\) of \(C_1\) holds \(f_{x_1}+x_2 = f_{x_1}+f_{x_2}\) and there exists a point \(x_0\) of \(C_1\) such that \(f\) is continuous in \(x_0\). Then \(f\) is continuous on the carrier of \(C_1\).

(97) Let \(f\) be a partial function from \(R_1\) to \(C_1\). Suppose \(f\) is total and for all points \(x_1, x_2\) of \(R_1\) holds \(f_{x_1}+x_2 = f_{x_1}+f_{x_2}\) and there exists a point \(x_0\) of \(R_1\) such that \(f\) is continuous in \(x_0\). Then \(f\) is continuous on the carrier of \(R_1\).

(98) For every partial function \(f\) from \(C_2\) to \(C_3\) such that \(\text{dom} f\) is compact and \(f\) is continuous on \(\text{dom} f\) holds \(\text{rng} f\) is compact.

(99) For every partial function \(f\) from \(C_1\) to \(R_1\) such that \(\text{dom} f\) is compact and \(f\) is continuous on \(\text{dom} f\) holds \(\text{rng} f\) is compact.

(100) For every partial function \(f\) from \(R_1\) to \(C_1\) such that \(\text{dom} f\) is compact and \(f\) is continuous on \(\text{dom} f\) holds \(\text{rng} f\) is compact.

(101) Let \(f\) be a partial function from the carrier of \(C_1\) to \(\mathbb{C}\). If \(\text{dom} f\) is compact and \(f\) is continuous on \(\text{dom} f\), then \(\text{rng} f\) is compact.

(102) Let \(f\) be a partial function from the carrier of \(C_1\) to \(\mathbb{R}\). If \(\text{dom} f\) is compact and \(f\) is continuous on \(\text{dom} f\), then \(\text{rng} f\) is compact.

(103) Let \(f\) be a partial function from the carrier of \(R_1\) to \(\mathbb{C}\). If \(\text{dom} f\) is compact and \(f\) is continuous on \(\text{dom} f\), then \(\text{rng} f\) is compact.

(104) Let \(Y\) be a subset of \(C_2\) and \(f\) be a partial function from \(C_2\) to \(C_3\). If \(Y \subseteq \text{dom} f\) and \(Y\) is compact and \(f\) is continuous on \(Y\), then \(f^0Y\) is compact.

(105) Let \(Y\) be a subset of \(C_1\) and \(f\) be a partial function from \(C_1\) to \(R_1\).
If $Y \subseteq \text{dom } f$ and $Y$ is compact and $f$ is continuous on $Y$, then $f^o Y$ is compact.

(106) Let $Y$ be a subset of $R_1$ and $f$ be a partial function from $R_1$ to $C_1$. If $Y \subseteq \text{dom } f$ and $Y$ is compact and $f$ is continuous on $Y$, then $f^o Y$ is compact.

(107) Let $f$ be a partial function from the carrier of $C_1$ to $\mathbb{R}$. Suppose $\text{dom } f \neq \emptyset$ and $f$ is compact and $f$ is continuous on $\text{dom } f$. Then there exist points $x_1, x_2$ of $C_1$ such that $x_1 \in \text{dom } f$ and $x_2 \in \text{dom } f$ and $\| f \|_{x_1} = \sup \text{rng } f$ and $f_{x_2} = \inf \text{rng } f$.

(108) Let $f$ be a partial function from $C_2$ to $C_3$. Suppose $\text{dom } f \neq \emptyset$ and $\text{dom } f$ is compact and $f$ is continuous on $\text{dom } f$. Then there exist points $x_1, x_2$ of $C_2$ such that $x_1 \in \text{dom } f$ and $x_2 \in \text{dom } f$ and $\| f \|_{x_1} = \sup \text{rng } f$ and $\| f \|_{x_2} = \inf \text{rng } f$.

(109) Let $f$ be a partial function from $C_1$ to $R_1$. Suppose $\text{dom } f \neq \emptyset$ and $\text{dom } f$ is compact and $f$ is continuous on $\text{dom } f$. Then there exist points $x_1, x_2$ of $C_1$ such that $x_1 \in \text{dom } f$ and $x_2 \in \text{dom } f$ and $\| f \|_{x_1} = \sup \text{rng } f$ and $\| f \|_{x_2} = \inf \text{rng } f$.

(110) Let $f$ be a partial function from $R_1$ to $C_1$. Suppose $\text{dom } f \neq \emptyset$ and $\text{dom } f$ is compact and $f$ is continuous on $\text{dom } f$. Then there exist points $x_1, x_2$ of $R_1$ such that $x_1 \in \text{dom } f$ and $x_2 \in \text{dom } f$ and $\| f \|_{x_1} = \sup \text{rng } f$ and $\| f \|_{x_2} = \inf \text{rng } f$.

(111) For every partial function $f$ from $C_2$ to $C_3$ holds $\| f \|_{X} = \| f \|_{X}$.

(112) For every partial function $f$ from $C_1$ to $R_1$ holds $\| f \|_{X} = \| f \|_{X}$.

(113) For every partial function $f$ from $R_1$ to $C_1$ holds $\| f \|_{X} = \| f \|_{X}$.

(114) Let $f$ be a partial function from $C_2$ to $C_3$ and $Y$ be a subset of $C_2$. Suppose $Y \neq \emptyset$ and $Y \subseteq \text{dom } f$ and $Y$ is compact and $f$ is continuous on $Y$. Then there exist points $x_1, x_2$ of $C_2$ such that $x_1 \in Y$ and $x_2 \in Y$ and $\| f \|_{x_1} = \sup (\| f \|_{x}Y)$ and $\| f \|_{x_2} = \inf (\| f \|_{x}Y)$.

(115) Let $f$ be a partial function from $C_1$ to $R_1$ and $Y$ be a subset of $C_1$. Suppose $Y \neq \emptyset$ and $Y \subseteq \text{dom } f$ and $Y$ is compact and $f$ is continuous on $Y$. Then there exist points $x_1, x_2$ of $C_1$ such that $x_1 \in Y$ and $x_2 \in Y$ and $\| f \|_{x_1} = \sup (\| f \|_{x}Y)$ and $\| f \|_{x_2} = \inf (\| f \|_{x}Y)$.

(116) Let $f$ be a partial function from $R_1$ to $C_1$ and $Y$ be a subset of $R_1$. Suppose $Y \neq \emptyset$ and $Y \subseteq \text{dom } f$ and $Y$ is compact and $f$ is continuous on $Y$. Then there exist points $x_1, x_2$ of $R_1$ such that $x_1 \in Y$ and $x_2 \in Y$ and $\| f \|_{x_1} = \sup (\| f \|_{x}Y)$ and $\| f \|_{x_2} = \inf (\| f \|_{x}Y)$.

(117) Let $f$ be a partial function from the carrier of $C_1$ to $\mathbb{R}$ and $Y$ be a subset of $C_1$. Suppose $Y \neq \emptyset$ and $Y \subseteq \text{dom } f$ and $Y$ is compact and $f$ is continuous on $Y$. Then there exist points $x_1, x_2$ of $C_1$ such that $x_1 \in Y$ and $x_2 \in Y$ and $f_{x_1} = \sup (f_{x}Y)$ and $f_{x_2} = \inf (f_{x}Y)$. 
Let $C_2$, $C_3$ be complex normed spaces, let $X$ be a set, and let $f$ be a partial function from $C_2$ to $C_3$. We say that $f$ is Lipschitzian on $X$ if and only if:

(Def. 27) \( X \subseteq \text{dom } f \) and there exists $r$ such that $0 < r$ and for all points $x_1$, $x_2$ of $C_2$ such that $x_1 \in X$ and $x_2 \in X$ holds \( \|f_{x_1} - f_{x_2}\| \leq r \cdot \|x_1 - x_2\| \).

Let $C_1$ be a complex normed space, let $R_1$ be a real normed space, let $X$ be a set, and let $f$ be a partial function from $C_1$ to $R_1$. We say that $f$ is Lipschitzian on $X$ if and only if:

(Def. 28) \( X \subseteq \text{dom } f \) and there exists $r$ such that $0 < r$ and for all points $x_1$, $x_2$ of $C_2$ such that $x_1 \in X$ and $x_2 \in X$ holds \( \|f_{x_1} - f_{x_2}\| \leq r \cdot \|x_1 - x_2\| \).

Let $R_1$ be a real normed space, let $C_1$ be a complex normed space, let $X$ be a set, and let $f$ be a partial function from the carrier of $C_1$ to $C$. We say that $f$ is Lipschitzian on $X$ if and only if:

(Def. 29) \( X \subseteq \text{dom } f \) and there exists $r$ such that $0 < r$ and for all points $x_1$, $x_2$ of $R_1$ such that $x_1 \in X$ and $x_2 \in X$ holds \( \|f_{x_1} - f_{x_2}\| \leq r \cdot \|x_1 - x_2\| \).

Let $C_1$ be a complex normed space, let $X$ be a set, and let $f$ be a partial function from the carrier of $C_1$ to $R$. We say that $f$ is Lipschitzian on $X$ if and only if:

(Def. 30) \( X \subseteq \text{dom } f \) and there exists $r$ such that $0 < r$ and for all points $x_1$, $x_2$ of $C_1$ such that $x_1 \in X$ and $x_2 \in X$ holds \( |f_{x_1} - f_{x_2}| \leq r \cdot \|x_1 - x_2\| \).

Let $C_1$ be a complex normed space, let $X$ be a set, and let $f$ be a partial function from the carrier of $C_1$ to $\mathbb{R}$. We say that $f$ is Lipschitzian on $X$ if and only if:

(Def. 31) \( X \subseteq \text{dom } f \) and there exists $r$ such that $0 < r$ and for all points $x_1$, $x_2$ of $C_1$ such that $x_1 \in X$ and $x_2 \in X$ holds \( |f_{x_1} - f_{x_2}| \leq r \cdot \|x_1 - x_2\| \).

Let $R_1$ be a real normed space, let $X$ be a set, and let $f$ be a partial function from the carrier of $R_1$ to $\mathbb{C}$. We say that $f$ is Lipschitzian on $X$ if and only if:

(Def. 32) \( X \subseteq \text{dom } f \) and there exists $r$ such that $0 < r$ and for all points $x_1$, $x_2$ of $R_1$ such that $x_1 \in X$ and $x_2 \in X$ holds \( |f_{x_1} - f_{x_2}| \leq r \cdot \|x_1 - x_2\| \).

Next we state a number of propositions:

(118) For every partial function $f$ from $C_2$ to $C_3$ such that $f$ is Lipschitzian on $X$ and $X_1 \subseteq X$ holds $f$ is Lipschitzian on $X_1$.

(119) For every partial function $f$ from $C_1$ to $R_1$ such that $f$ is Lipschitzian on $X$ and $X_1 \subseteq X$ holds $f$ is Lipschitzian on $X_1$.

(120) For every partial function $f$ from $R_1$ to $C_1$ such that $f$ is Lipschitzian on $X$ and $X_1 \subseteq X$ holds $f$ is Lipschitzian on $X_1$.

(121) Let $f_1$, $f_2$ be partial functions from $C_2$ to $C_3$. Suppose $f_1$ is Lipschitzian on $X$ and $f_2$ is Lipschitzian on $X_1$. Then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(122) Let $f_1$, $f_2$ be partial functions from $C_1$ to $R_1$. Suppose $f_1$ is Lipschitzian on $X$ and $f_2$ is Lipschitzian on $X_1$. Then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$. 
Let $f_1, f_2$ be partial functions from $R_1$ to $C_1$. Suppose $f_1$ is Lipschitzian on $X$ and $f_2$ is Lipschitzian on $X_1$. Then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

Let $f_1, f_2$ be partial functions from $C_2$ to $C_3$. Suppose $f_1$ is Lipschitzian on $X$ and $f_2$ is Lipschitzian on $X_1$. Then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

Let $f_1, f_2$ be partial functions from $C_1$ to $R_1$. Suppose $f_1$ is Lipschitzian on $X$ and $f_2$ is Lipschitzian on $X_1$. Then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

Let $f_1, f_2$ be partial functions from $R_1$ to $C_1$. Suppose $f_1$ is Lipschitzian on $X$ and $f_2$ is Lipschitzian on $X_1$. Then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

For every partial function $f$ from $C_2$ to $C_3$ such that $f$ is Lipschitzian on $X$ holds $z f$ is Lipschitzian on $X$.

For every partial function $f$ from $C_1$ to $R_1$ such that $f$ is Lipschitzian on $X$ holds $z f$ is Lipschitzian on $X$.

For every partial function $f$ from $C_1$ to $C_3$ such that $f$ is Lipschitzian on $X$. Then $-f$ is Lipschitzian on $X$ and $\|f\|$ is Lipschitzian on $X$.

Let $f$ be a partial function from $C_1$ to $R_1$. Suppose $f$ is Lipschitzian on $X$. Then $-f$ is Lipschitzian on $X$ and $\|f\|$ is Lipschitzian on $X$.

Let $f$ be a partial function from $C_1$ to $R_1$. Suppose $f$ is Lipschitzian on $X$. Then $-f$ is Lipschitzian on $X$ and $\|f\|$ is Lipschitzian on $X$.

Let $f$ be a partial function from $R_1$ to $C_1$. Suppose $f$ is Lipschitzian on $X$. Then $-f$ is Lipschitzian on $X$ and $\|f\|$ is Lipschitzian on $X$.

Let $X$ be a set and $f$ be a partial function from $C_2$ to $C_3$. If $X \subseteq \text{dom } f$ and $f$ is a constant on $X$, then $f$ is Lipschitzian on $X$.

Let $X$ be a set and $f$ be a partial function from $C_1$ to $R_1$. If $X \subseteq \text{dom } f$ and $f$ is a constant on $X$, then $f$ is Lipschitzian on $X$.

Let $X$ be a set and $f$ be a partial function from $R_1$ to $C_1$. If $X \subseteq \text{dom } f$ and $f$ is a constant on $X$, then $f$ is Lipschitzian on $X$.

For every subset $Y$ of $C_1$ holds $\text{id}_Y$ is Lipschitzian on $Y$.

For every partial function $f$ from $C_2$ to $C_3$ such that $f$ is Lipschitzian on $X$ holds $f$ is continuous on $X$.

For every partial function $f$ from $C_1$ to $R_1$ such that $f$ is Lipschitzian on $X$ holds $f$ is continuous on $X$.

For every partial function $f$ from $R_1$ to $C_1$ such that $f$ is Lipschitzian on $X$ holds $f$ is continuous on $X$.

Let $f$ be a partial function from the carrier of $C_1$ to $\mathbb{C}$. If $f$ is Lipschitzian on $X$, then $f$ is continuous on $X$.

Let $f$ be a partial function from the carrier of $C_1$ to $\mathbb{R}$. If $f$ is Lipschitzian on $X$, then $f$ is continuous on $X$.

Let $f$ be a partial function from the carrier of $R_1$ to $\mathbb{C}$. If $f$ is Lipschitzian on $X$, then $f$ is continuous on $X$. 

For every partial function $f$ from $C_2$ to $C_3$ such that there exists a point $r$ of $C_3$ such that $\text{rng } f = \{ r \}$ holds $f$ is continuous on $\text{dom } f$.

For every partial function $f$ from $C_1$ to $R_1$ such that there exists a point $r$ of $R_1$ such that $\text{rng } f = \{ r \}$ holds $f$ is continuous on $\text{dom } f$.

For every partial function $f$ from $R_1$ to $C_1$ such that there exists a point $r$ of $C_1$ such that $\text{rng } f = \{ r \}$ holds $f$ is continuous on $\text{dom } f$.

For every partial function $f$ from $C_2$ to $C_3$ such that $X \subseteq \text{dom } f$ and $f$ is a constant on $X$ holds $f$ is continuous on $X$.

For every partial function $f$ from $C_1$ to $R_1$ such that $X \subseteq \text{dom } f$ and $f$ is a constant on $X$ holds $f$ is continuous on $X$.

Let $f$ be a partial function from $C_1$ to $C_1$. Suppose that for every point $x_0$ of $C_1$ such that $x_0 \in \text{dom } f$ holds $f_{x_0} = x_0$. Then $f$ is continuous on $\text{dom } f$.

For every partial function $f$ from $C_1$ to $C_1$ such that $f = \text{id}_{\text{dom } f}$ holds $f$ is continuous on $\text{dom } f$.

Let $f$ be a partial function from $C_1$ to $C_1$ and $Y$ be a subset of $C_1$. If $Y \subseteq \text{dom } f$ and $f|Y = \text{id}_Y$, then $f$ is continuous on $Y$.

Let $f$ be a partial function from $C_1$ to $C_1$, $z$ be a complex number, and $p$ be a point of $C_1$. Suppose $X \subseteq \text{dom } f$ and for every point $x_0$ of $C_1$ such that $x_0 \in X$ holds $f_{x_0} = z \cdot x_0 + p$. Then $f$ is continuous on $X$.

Let $f$ be a partial function from the carrier of $C_1$ to $\mathbb{R}$. Suppose that for every point $x_0$ of $C_1$ such that $x_0 \in \text{dom } f$ holds $f_{x_0} = \| x_0 \|$. Then $f$ is continuous on $\text{dom } f$.

Let $f$ be a partial function from the carrier of $C_1$ to $\mathbb{R}$. Suppose $X \subseteq \text{dom } f$ and for every point $x_0$ of $C_1$ such that $x_0 \in X$ holds $f_{x_0} = \| x_0 \|$. Then $f$ is continuous on $X$.

REFERENCES


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