The Uniform Continuity of Functions on Normed Linear Spaces

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Summary. In this article, the basic properties of uniform continuity of functions on normed linear spaces are described.

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The notation and terminology used in this paper are introduced in the following articles: [15], [18], [19], [1], [20], [3], [2], [7], [14], [16], [9], [13], [4], [17], [6], [5], [11], [21], [10], [12], and [8].

1. The Uniform Continuity of Functions on Normed Linear Spaces

For simplicity, we follow the rules: $X$, $X_1$ are sets, $s$, $r$, $p$ are real numbers, $S$, $T$ are real normed spaces, $f$, $f_1$, $f_2$ are partial functions from $S$ to $T$, $x_1$, $x_2$ are points of $S$, and $Y$ is a subset of $S$.

Let us consider $X$, $S$, $T$ and let us consider $f$. We say that $f$ is uniformly continuous on $X$ if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $X \subseteq \text{dom } f$, and

(ii) for every $r$ such that $0 < r$ there exists $s$ such that $0 < s$ and for all $x_1$, $x_2$ such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

Let us consider $X$, $S$ and let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$. We say that $f$ is uniformly continuous on $X$ if and only if the conditions (Def. 2) are satisfied.

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(Def. 2)(i) \( X \subseteq \text{dom} \ f \), and
(ii) for every \( r \) such that \( 0 < r \) there exists \( s \) such that \( 0 < s \) and for all \( x_1, x_2 \) such that \( x_1 \in X \) and \( x_2 \in X \) and \( \|x_1 - x_2\| < s \) holds \( |f_{x_1} - f_{x_2}| < r \).

The following propositions are true:

1. If \( f \) is uniformly continuous on \( X \) and \( X_1 \subseteq X \), then \( f \) is uniformly continuous on \( X_1 \).
2. If \( f_1 \) is uniformly continuous on \( X \) and \( f_2 \) is uniformly continuous on \( X_1 \), then \( f_1 + f_2 \) is uniformly continuous on \( X \cap X_1 \).
3. If \( f_1 \) is uniformly continuous on \( X \) and \( f_2 \) is uniformly continuous on \( X_1 \), then \( f_1 - f_2 \) is uniformly continuous on \( X \cap X_1 \).
4. If \( f \) is uniformly continuous on \( X \), then \( p f \) is uniformly continuous on \( X \).
5. If \( f \) is uniformly continuous on \( X \), then \(-f\) is uniformly continuous on \( X \).
6. If \( f \) is uniformly continuous on \( X \), then \( \|f\| \) is uniformly continuous on \( X \).
7. If \( f \) is uniformly continuous on \( X \), then \( f \) is continuous on \( X \).
8. Let \( f \) be a partial function from the carrier of \( S \) to \( \mathbb{R} \). If \( f \) is uniformly continuous on \( X \), then \( f \) is continuous on \( X \).
9. If \( f \) is Lipschitzian on \( X \), then \( f \) is uniformly continuous on \( X \).
10. For all \( f, Y \) such that \( Y \) is compact and \( f \) is continuous on \( Y \) holds \( f \) is uniformly continuous on \( Y \).
11. If \( Y \subseteq \text{dom} \ f \) and \( Y \) is compact and \( f \) is uniformly continuous on \( Y \), then \( f^o Y \) is compact.
12. Let \( f \) be a partial function from the carrier of \( S \) to \( \mathbb{R} \) and given \( Y \). Suppose \( Y \neq \emptyset \) and \( Y \subseteq \text{dom} \ f \) and \( Y \) is compact and \( f \) is uniformly continuous on \( Y \). Then there exist \( x_1, x_2 \) such that \( x_1 \in Y \) and \( x_2 \in Y \) and \( f_{x_1} = \sup(f^o Y) \) and \( f_{x_2} = \inf(f^o Y) \).
13. If \( X \subseteq \text{dom} \ f \) and \( f \) is a constant on \( X \), then \( f \) is uniformly continuous on \( X \).

2. The Contraction Mapping Principle on Normed Linear Spaces

Let \( M \) be a real Banach space. A function from the carrier of \( M \) into the carrier of \( M \) is said to be a contraction of \( M \) if:

(Def. 3) There exists a real number \( L \) such that \( 0 < L \) and \( L < 1 \) and for all points \( x, y \) of \( M \) holds \( \|it(x) - it(y)\| \leq L \cdot \|x - y\| \).

The following two propositions are true:
(14) Let $X$ be a real Banach space and $f$ be a function from $X$ into $X$. Suppose $f$ is a contraction of $X$. Then there exists a point $x_3$ of $X$ such that $f(x_3) = x_3$ and for every point $x$ of $X$ such that $f(x) = x$ holds $x_3 = x$.

(15) Let $X$ be a real Banach space and $f$ be a function from $X$ into $X$. Given a natural number $n_0$ such that $f^{n_0}$ is a contraction of $X$. Then there exists a point $x_3$ of $X$ such that $f(x_3) = x_3$ and for every point $x$ of $X$ such that $f(x) = x$ holds $x_3 = x$.

References


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