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Algebraic Properties of Homotopies

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The notation and terminology used here are introduced in the following papers: [21], [9], [25], [1], [20], [14], [24], [22], [2], [5], [27], [6], [7], [18], [11], [19], [10], [17], [26], [8], [15], [23], [12], [4], [3], [16], and [13].

1. Preliminaries

The scheme *ExFunc3CondD* deals with a non empty set $\mathcal{A}$, three unary functors $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ yielding sets, and three unary predicates $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, and states that:

> There exists a function f such that $\mathrm{dom}\, f = \mathcal{A}$ and for every element c of $\mathcal{A}$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ and if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$ and if $\mathcal{R}[c]$, then $f(c) = \mathcal{H}(c)$

provided the parameters meet the following conditions:

- For every element c of $\mathcal{A}$ holds if $\mathcal{P}[c]$, then not $\mathcal{Q}[c]$ and if $\mathcal{P}[c]$, then not $\mathcal{R}[c]$ and if $\mathcal{Q}[c]$, then not $\mathcal{R}[c]$, and
- For every element c of $\mathcal{A}$ holds $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ or $\mathcal{R}[c]$.

Let n be a natural number. Observe that every element of $\mathcal{E}^n_{\mathrm{T}}$ is function-like and relation-like.

Let n be a natural number. Observe that every element of $\mathcal{E}^n_{\mathrm{T}}$ is finite sequence-like.

We now state a number of propositions:

(1) The carrier of $[\!:\mathbb{I}, \mathbb{I}:\!] = [\!:[0,1], [0,1]:\!]$.

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(2) For every real number x such that $x \leqslant \frac{1}{2}$ holds $2 \cdot x - 1 \leqslant 1 - 2 \cdot x$.

(3) For every real number x such that $x \geqslant \frac{1}{2}$ holds $2 \cdot x - 1 \geqslant 1 - 2 \cdot x$.

(4) For all real numbers x, a, b, c, d such that $a \neq b$ holds $\frac{d-c}{b-a} \cdot (x-a) + c = (1 - \frac{x-a}{b-a}) \cdot c + \frac{x-a}{b-a} \cdot d$.

(5) For all real numbers a, b, x such that $a \leqslant x$ and $x \leqslant b$ holds $\frac{x-a}{b-a} \in$ the carrier of $[0, 1]_{\mathrm{T}}$.

(6) For every point x of $\mathbb{I}$ such that $x \leqslant \frac{1}{2}$ holds $2 \cdot x$ is a point of $\mathbb{I}$.

(7) For every point x of $\mathbb{I}$ such that $x \geqslant \frac{1}{2}$ holds $2 \cdot x - 1$ is a point of $\mathbb{I}$.

(8) For all points p, q of $\mathbb{I}$ holds $p \cdot q$ is a point of $\mathbb{I}$.

(9) For every point x of $\mathbb{I}$ holds $\frac{1}{2} \cdot x$ is a point of $\mathbb{I}$.

(10) For every point x of $\mathbb{I}$ such that $x \geqslant \frac{1}{2}$ holds $x - \frac{1}{4}$ is a point of $\mathbb{I}$.

(12)[3] $\mathrm{id}_{\mathbb{I}}$ is a path from $0_{\mathbb{I}}$ to $1_{\mathbb{I}}$.

(13) For all points a, b, c, d of $\mathbb{I}$ such that $a \leqslant b$ and $c \leqslant d$ holds $[\![[a, b], [c, d]]\!]$ is a compact non empty subset of $[\![\mathbb{I}, \mathbb{I}]\!]$.

2. Affine Maps

One can prove the following four propositions:

(14) Let S, T be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $S = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant 2 \cdot p_{\mathbf{1}} - 1\}$ and $T = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant p_{\mathbf{1}}\}$. Then $(\mathrm{AffineMap}(1, 0, \frac{1}{2}, \frac{1}{2}))^{\circ} S = T$.

(15) Let S, T be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $S = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$ and $T = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant p_{\mathbf{1}}\}$. Then $(\mathrm{AffineMap}(1, 0, \frac{1}{2}, \frac{1}{2}))^{\circ} S = T$.

(16) Let S, T be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $S = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ and $T = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant -p_{\mathbf{1}}\}$. Then $(\mathrm{AffineMap}(1, 0, \frac{1}{2}, -\frac{1}{2}))^{\circ} S = T$.

(17) Let S, T be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $S = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ and $T = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}\}$. Then $(\mathrm{AffineMap}(1, 0, \frac{1}{2}, -\frac{1}{2}))^{\circ} S = T$.

3. Real-Membered Structures

Let T be a 1-sorted structure. We say that T is real-membered if and only if:

(Def. 1) The carrier of T is real-membered.

We now state the proposition

[3]The proposition (11) has been removed.

(18) For every non empty 1-sorted structure T holds T is real-membered iff every element of T is real.

Let us mention that $\mathbb{I}$ is real-membered.

One can verify that there exists a 1-sorted structure which is non empty and real-membered and there exists a topological space which is non empty and real-membered.

Let T be a real-membered 1-sorted structure. Note that every element of T is real.

Let T be a real-membered topological structure. Note that every subspace of T is real-membered.

Let S, T be real-membered non empty topological spaces and let p be an element of $[: S, T :]$. One can check that $p_{\mathbf{1}}$ is real and $p_{\mathbf{2}}$ is real.

Let T be a non empty subspace of $[: \mathbb{I}, \mathbb{I} :]$ and let x be a point of T. One can check that $x_{\mathbf{1}}$ is real and $x_{\mathbf{2}}$ is real.

One can check that $\mathbb{R}^{\mathbf{1}}$ is real-membered.

4. Closed Subsets of Euclidean Topological Spaces

The following propositions are true:

(19) $\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(20) $\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(21) $\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(22) $\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(23) $\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}} \ \wedge \ p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(24) There exists a map f from $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$ into $\mathcal{E}^2_{\mathrm{T}}$ such that for all real numbers x, y holds $f(\langle x, y \rangle) = \langle x, y \rangle$.

(25) $\{p; p$ ranges over points of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$: $p_{\mathbf{2}} \leqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ is a closed subset of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$.

(26) $\{p; p$ ranges over points of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$: $p_{\mathbf{2}} \leqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed subset of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$.

(27) $\{p; p$ ranges over points of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}} \ \wedge \ p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed subset of $[: \mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}} :]$.

(28) $\{p; p$ ranges over points of $[: \mathbb{I}, \mathbb{I} :]$: $p_{\mathbf{2}} \leqslant 1 - 2 \cdot p_{\mathbf{1}}\}$ is a closed non empty subset of $[: \mathbb{I}, \mathbb{I} :]$.

(29) $\{p; p$ ranges over points of $[: \mathbb{I}, \mathbb{I} :]$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}} \ \wedge \ p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed non empty subset of $[: \mathbb{I}, \mathbb{I} :]$.

(30) $\{p; p$ ranges over points of $[: \mathbb{I}, \mathbb{I} :]$: $p_{\mathbf{2}} \leqslant 2 \cdot p_{\mathbf{1}} - 1\}$ is a closed non empty subset of $[: \mathbb{I}, \mathbb{I} :]$.

(31) Let S, T be non empty topological spaces and p be a point of $[\!: S, T :\!]$. Then $p_{\mathbf{1}}$ is a point of S and $p_{\mathbf{2}}$ is a point of T.

(32) For all subsets A, B of $[\!: \mathbb{I}, \mathbb{I} :\!]$ such that $A = [\!: [0, \frac{1}{2}], [0,1] :\!]$ and $B = [\!: [\frac{1}{2}, 1], [0,1] :\!]$ holds $\Omega_{[: \mathbb{I}, \mathbb{I} :] \restriction A} \cup \Omega_{[: \mathbb{I}, \mathbb{I} :] \restriction B} = \Omega_{[: \mathbb{I}, \mathbb{I} :]}$.

(33) For all subsets A, B of $[\!: \mathbb{I}, \mathbb{I} :\!]$ such that $A = [\!: [0, \frac{1}{2}], [0,1] :\!]$ and $B = [\!: [\frac{1}{2}, 1], [0,1] :\!]$ holds $\Omega_{[: \mathbb{I}, \mathbb{I} :] \restriction A} \cap \Omega_{[: \mathbb{I}, \mathbb{I} :] \restriction B} = [\!: \{\frac{1}{2}\}, [0,1] :\!]$.

5. Compact Spaces

Let T be a topological structure. Note that $\emptyset_T$ is compact.

Let T be a topological structure. Observe that there exists a subset of T which is empty and compact.

Next we state three propositions:

(34) For every topological structure T holds $\emptyset$ is an empty compact subset of T.

(35) Let T be a topological structure and a, b be real numbers. If $a > b$, then $[a, b]$ is an empty compact subset of T.

(36) For all points a, b, c, d of $\mathbb{I}$ holds $[\!: [a,b], [c,d] :\!]$ is a compact subset of $[\!: \mathbb{I}, \mathbb{I} :\!]$.

6. Continuous Maps

Let a, b, c, d be real numbers. The functor $\mathrm{L}_{01}(a,b,c,d)$ yielding a map from $[a, b]_{\mathrm{T}}$ into $[c, d]_{\mathrm{T}}$ is defined by:

(Def. 2) $\mathrm{L}_{01}(a,b,c,d) = \mathrm{L}_{01}(c_{[c,d]_{\mathrm{T}}}, d_{[c,d]_{\mathrm{T}}}) \cdot \mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}})$.

The following propositions are true:

(37) For all real numbers a, b, c, d such that $a < b$ and $c < d$ holds $(\mathrm{L}_{01}(a,b,c,d))(a) = c$ and $(\mathrm{L}_{01}(a,b,c,d))(b) = d$.

(38) For all real numbers a, b, c, d such that $a < b$ and $c \leqslant d$ holds $\mathrm{L}_{01}(a,b,c,d)$ is a continuous map from $[a, b]_{\mathrm{T}}$ into $[c, d]_{\mathrm{T}}$.

(39) Let a, b, c, d be real numbers. Suppose $a < b$ and $c \leqslant d$. Let x be a real number. If $a \leqslant x$ and $x \leqslant b$, then $(\mathrm{L}_{01}(a,b,c,d))(x) = \frac{d-c}{b-a} \cdot (x - a) + c$.

(40) Let f_1, f_2 be maps from $[\!: \mathbb{I}, \mathbb{I} :\!]$ into $\mathbb{I}$. Suppose f_1 is continuous and f_2 is continuous and for every point p of $[\!: \mathbb{I}, \mathbb{I} :\!]$ holds $f_1(p) \cdot f_2(p)$ is a point of $\mathbb{I}$. Then there exists a map g from $[\!: \mathbb{I}, \mathbb{I} :\!]$ into $\mathbb{I}$ such that

(i) for every point p of $[\!: \mathbb{I}, \mathbb{I} :\!]$ and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 \cdot r_2$, and

(ii) g is continuous.

(41) Let f_1, f_2 be maps from $[:\mathbb{I}, \mathbb{I}:]$ into $\mathbb{I}$. Suppose f_1 is continuous and f_2 is continuous and for every point p of $[:\mathbb{I}, \mathbb{I}:]$ holds $f_1(p) + f_2(p)$ is a point of $\mathbb{I}$. Then there exists a map g from $[:\mathbb{I}, \mathbb{I}:]$ into $\mathbb{I}$ such that

(i) for every point p of $[:\mathbb{I}, \mathbb{I}:]$ and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 + r_2$, and

(ii) g is continuous.

(42) Let f_1, f_2 be maps from $[:\mathbb{I}, \mathbb{I}:]$ into $\mathbb{I}$. Suppose f_1 is continuous and f_2 is continuous and for every point p of $[:\mathbb{I}, \mathbb{I}:]$ holds $f_1(p) - f_2(p)$ is a point of $\mathbb{I}$. Then there exists a map g from $[:\mathbb{I}, \mathbb{I}:]$ into $\mathbb{I}$ such that

(i) for every point p of $[:\mathbb{I}, \mathbb{I}:]$ and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 - r_2$, and

(ii) g is continuous.

7. Paths

We follow the rules: T denotes a non empty topological space and a, b, c, d denote points of T.

The following three propositions are true:

(43) For every path P from a to b such that P is continuous holds $P \cdot \mathrm{L}_{01}(1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}})$ is a continuous map from $\mathbb{I}$ into T.

(44) Let X be a non empty topological structure, a, b be points of X, and P be a path from a to b. If $P(0) = a$ and $P(1) = b$, then $(P \cdot \mathrm{L}_{01}(1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}}))(0) = b$ and $(P \cdot \mathrm{L}_{01}(1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}}))(1) = a$.

(45) Let P be a path from a to b. Suppose P is continuous and $P(0) = a$ and $P(1) = b$. Then $-P$ is continuous and $(-P)(0) = b$ and $(-P)(1) = a$.

Let T be a topological structure and let a, b be points of T. We say that a, b are connected if and only if:

(Def. 3) There exists a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$.

Let T be a non empty topological space and let a, b be points of T. Let us notice that the predicate a, b are connected is reflexive and symmetric.

We now state several propositions:

(46) If a, b are connected and b, c are connected, then a, c are connected.

(47) For every arcwise connected topological structure T and for all points a, b of T holds a, b are connected.

(48) For every path A from a to a holds A, A are homotopic.

(49) If a, b are connected, then for every path A from a to b holds A, A are homotopic.

(50) If a, b are connected, then for every path A from a to b holds $A = --A$.

(51) Let T be a non empty arcwise connected topological space, a, b be points of T, and A be a path from a to b. Then $A = --A$.

(52) If a, b are connected, then every path from a to b is continuous.

8. Reexamination of a Path Concept

Let T be a non empty arcwise connected topological space, let a, b, c be points of T, let P be a path from a to b, and let Q be a path from b to c. Then $P + Q$ can be characterized by the condition:

(Def. 4) For every point t of $\mathbb{I}$ holds if $t \leqslant \frac{1}{2}$, then $(P + Q)(t) = P(2 \cdot t)$ and if $\frac{1}{2} \leqslant t$, then $(P + Q)(t) = Q(2 \cdot t - 1)$.

Let T be a non empty arcwise connected topological space, let a, b be points of T, and let P be a path from a to b. Then $-P$ can be characterized by the condition:

(Def. 5) For every point t of $\mathbb{I}$ holds $(-P)(t) = P(1 - t)$.

9. Reparametrizations

Let T be a non empty topological space, let a, b be points of T, let P be a path from a to b, and let f be a continuous map from $\mathbb{I}$ into $\mathbb{I}$. Let us assume that $f(0) = 0$ and $f(1) = 1$ and a, b are connected. The functor $\mathrm{RePar}(P, f)$ yields a path from a to b and is defined by:

(Def. 6) $\mathrm{RePar}(P, f) = P \cdot f$.

Next we state two propositions:

(53) Let P be a path from a to b and f be a continuous map from $\mathbb{I}$ into $\mathbb{I}$. Suppose $f(0) = 0$ and $f(1) = 1$ and a, b are connected. Then $\mathrm{RePar}(P, f)$, P are homotopic.

(54) Let T be a non empty arcwise connected topological space, a, b be points of T, P be a path from a to b, and f be a continuous map from $\mathbb{I}$ into $\mathbb{I}$. If $f(0) = 0$ and $f(1) = 1$, then $\mathrm{RePar}(P, f)$, P are homotopic.

The map $1^{\mathrm{st}}\mathrm{RP}$ from $\mathbb{I}$ into $\mathbb{I}$ is defined as follows:

(Def. 7) For every point t of $\mathbb{I}$ holds if $t \leqslant \frac{1}{2}$, then $(1^{\mathrm{st}}\mathrm{RP})(t) = 2 \cdot t$ and if $t > \frac{1}{2}$, then $(1^{\mathrm{st}}\mathrm{RP})(t) = 1$.

Let us note that $1^{\mathrm{st}}\mathrm{RP}$ is continuous.

One can prove the following proposition

(55) $(1^{\mathrm{st}}\mathrm{RP})(0) = 0$ and $(1^{\mathrm{st}}\mathrm{RP})(1) = 1$.

The map $2^{\mathrm{nd}}\mathrm{RP}$ from $\mathbb{I}$ into $\mathbb{I}$ is defined by:

(Def. 8) For every point t of $\mathbb{I}$ holds if $t \leqslant \frac{1}{2}$, then $(2^{\mathrm{nd}}\mathrm{RP})(t) = 0$ and if $t > \frac{1}{2}$, then $(2^{\mathrm{nd}}\mathrm{RP})(t) = 2 \cdot t - 1$.

One can verify that 2^{nd}RP is continuous.

One can prove the following proposition

(56) $(2^{\mathrm{nd}}\mathrm{RP})(0) = 0$ and $(2^{\mathrm{nd}}\mathrm{RP})(1) = 1$.

The map 3^{rd}RP from $\mathbb{I}$ into $\mathbb{I}$ is defined by the condition (Def. 9).

(Def. 9) Let x be a point of $\mathbb{I}$. Then

(i) if $x \leqslant \frac{1}{2}$, then $(3^{\mathrm{rd}}\mathrm{RP})(x) = \frac{1}{2} \cdot x$,

(ii) if $x > \frac{1}{2}$ and $x \leqslant \frac{3}{4}$, then $(3^{\mathrm{rd}}\mathrm{RP})(x) = x - \frac{1}{4}$, and

(iii) if $x > \frac{3}{4}$, then $(3^{\mathrm{rd}}\mathrm{RP})(x) = 2 \cdot x - 1$.

Let us note that 3^{rd}RP is continuous.

We now state four propositions:

(57) $(3^{\mathrm{rd}}\mathrm{RP})(0) = 0$ and $(3^{\mathrm{rd}}\mathrm{RP})(1) = 1$.

(58) Let P be a path from a to b and Q be a constant path from b to b. If a, b are connected, then $\mathrm{RePar}(P, 1^{\mathrm{st}}\mathrm{RP}) = P + Q$.

(59) Let P be a path from a to b and Q be a constant path from a to a. If a, b are connected, then $\mathrm{RePar}(P, 2^{\mathrm{nd}}\mathrm{RP}) = Q + P$.

(60) Let P be a path from a to b, Q be a path from b to c, and R be a path from c to d. Suppose a, b are connected and b, c are connected and c, d are connected. Then $\mathrm{RePar}(P + Q + R, 3^{\mathrm{rd}}\mathrm{RP}) = P + (Q + R)$.

10. Decomposition of the Unit Square

The subset LowerLeftUnitTriangle of $[\!: \mathbb{I}, \mathbb{I} :\!]$ is defined as follows:

(Def. 10) For every set x holds $x \in$ LowerLeftUnitTriangle iff there exist points a, b of $\mathbb{I}$ such that $x = \langle a, b \rangle$ and $b \leqslant 1 - 2 \cdot a$.

We introduce IAA as a synonym of LowerLeftUnitTriangle.

The subset UpperUnitTriangle of $[\!: \mathbb{I}, \mathbb{I} :\!]$ is defined by:

(Def. 11) For every set x holds $x \in$ UpperUnitTriangle iff there exist points a, b of $\mathbb{I}$ such that $x = \langle a, b \rangle$ and $b \geqslant 1 - 2 \cdot a$ and $b \geqslant 2 \cdot a - 1$.

We introduce IBB as a synonym of UpperUnitTriangle.

The subset LowerRightUnitTriangle of $[\!: \mathbb{I}, \mathbb{I} :\!]$ is defined as follows:

(Def. 12) For every set x holds $x \in$ LowerRightUnitTriangle iff there exist points a, b of $\mathbb{I}$ such that $x = \langle a, b \rangle$ and $b \leqslant 2 \cdot a - 1$.

We introduce ICC as a synonym of LowerRightUnitTriangle.

The following propositions are true:

(61) $\mathrm{IAA} = \{p; p$ ranges over points of $[\!: \mathbb{I}, \mathbb{I} :\!]$: $p_{\mathbf{2}} \leqslant 1 - 2 \cdot p_{\mathbf{1}}\}$.

(62) $\mathrm{IBB} = \{p; p$ ranges over points of $[\!: \mathbb{I}, \mathbb{I} :\!]$: $p_{\mathbf{2}} \geqslant 1 - 2 \cdot p_{\mathbf{1}} \wedge p_{\mathbf{2}} \geqslant 2 \cdot p_{\mathbf{1}} - 1\}$.

(63) $\mathrm{ICC} = \{p; p$ ranges over points of $[\!: \mathbb{I}, \mathbb{I} :\!]$: $p_{\mathbf{2}} \leqslant 2 \cdot p_{\mathbf{1}} - 1\}$.

One can check the following observations:

∗ IAA is closed and non empty,

* IBB is closed and non empty, and
* ICC is closed and non empty.

Next we state a number of propositions:

(64) $\mathrm{IAA} \cup \mathrm{IBB} \cup \mathrm{ICC} = [: [0,1], [0,1] :]$.

(65) $\mathrm{IAA} \cap \mathrm{IBB} = \{p; p$ ranges over points of $[: \mathbb{I}, \mathbb{I} :]$: $p_{\mathbf{2}} = 1 - 2 \cdot p_{\mathbf{1}}\}$.

(66) $\mathrm{ICC} \cap \mathrm{IBB} = \{p; p$ ranges over points of $[: \mathbb{I}, \mathbb{I} :]$: $p_{\mathbf{2}} = 2 \cdot p_{\mathbf{1}} - 1\}$.

(67) For every point x of $[: \mathbb{I}, \mathbb{I} :]$ such that $x \in \mathrm{IAA}$ holds $x_{\mathbf{1}} \leqslant \frac{1}{2}$.

(68) For every point x of $[: \mathbb{I}, \mathbb{I} :]$ such that $x \in \mathrm{ICC}$ holds $x_{\mathbf{1}} \geqslant \frac{1}{2}$.

(69) For every point x of $\mathbb{I}$ holds $\langle 0, x \rangle \in \mathrm{IAA}$.

(70) For every set s such that $\langle 0, s \rangle \in \mathrm{IBB}$ holds $s = 1$.

(71) For every set s such that $\langle s, 1 \rangle \in \mathrm{ICC}$ holds $s = 1$.

(72) $\langle 0, 1 \rangle \in \mathrm{IBB}$.

(73) For every point x of $\mathbb{I}$ holds $\langle x, 1 \rangle \in \mathrm{IBB}$.

(74) $\langle \frac{1}{2}, 0 \rangle \in \mathrm{ICC}$ and $\langle 1, 1 \rangle \in \mathrm{ICC}$.

(75) $\langle \frac{1}{2}, 0 \rangle \in \mathrm{IBB}$.

(76) For every point x of $\mathbb{I}$ holds $\langle 1, x \rangle \in \mathrm{ICC}$.

(77) For every point x of $\mathbb{I}$ such that $x \geqslant \frac{1}{2}$ holds $\langle x, 0 \rangle \in \mathrm{ICC}$.

(78) For every point x of $\mathbb{I}$ such that $x \leqslant \frac{1}{2}$ holds $\langle x, 0 \rangle \in \mathrm{IAA}$.

(79) For every point x of $\mathbb{I}$ such that $x < \frac{1}{2}$ holds $\langle x, 0 \rangle \notin \mathrm{IBB}$ and $\langle x, 0 \rangle \notin \mathrm{ICC}$.

(80) $\mathrm{IAA} \cap \mathrm{ICC} = \{\langle \frac{1}{2}, 0 \rangle\}$.

11. Properties of a Homotopy

We use the following convention: X denotes a non empty arcwise connected topological space and a_1, b_1, c_1, d_1 denote points of X.

One can prove the following propositions:

(81) Let P be a path from a to b, Q be a path from b to c, and R be a path from c to d. Suppose a, b are connected and b, c are connected and c, d are connected. Then $(P + Q) + R$, $P + (Q + R)$ are homotopic.

(82) Let P be a path from a_1 to b_1, Q be a path from b_1 to c_1, and R be a path from c_1 to d_1. Then $(P + Q) + R$, $P + (Q + R)$ are homotopic.

(83) Let P_1, P_2 be paths from a to b and Q_1, Q_2 be paths from b to c. Suppose a, b are connected and b, c are connected and P_1, P_2 are homotopic and Q_1, Q_2 are homotopic. Then $P_1 + Q_1$, $P_2 + Q_2$ are homotopic.

(84) Let P_1, P_2 be paths from a_1 to b_1 and Q_1, Q_2 be paths from b_1 to c_1. Suppose P_1, P_2 are homotopic and Q_1, Q_2 are homotopic. Then $P_1 + Q_1$, $P_2 + Q_2$ are homotopic.

(85) Let P, Q be paths from a to b. Suppose a, b are connected and P, Q are homotopic. Then $-P$, $-Q$ are homotopic.

(86) For all paths P, Q from a_1 to b_1 such that P, Q are homotopic holds $-P$, $-Q$ are homotopic.

(87) Let P, Q, R be paths from a to b. Suppose P, Q are homotopic and Q, R are homotopic. Then P, R are homotopic.

(88) Let P be a path from a to b and Q be a constant path from b to b. If a, b are connected, then $P+Q$, P are homotopic.

(89) For every path P from a_1 to b_1 and for every constant path Q from b_1 to b_1 holds $P+Q$, P are homotopic.

(90) Let P be a path from a to b and Q be a constant path from a to a. If a, b are connected, then $Q+P$, P are homotopic.

(91) For every path P from a_1 to b_1 and for every constant path Q from a_1 to a_1 holds $Q+P$, P are homotopic.

(92) Let P be a path from a to b and Q be a constant path from a to a. If a, b are connected, then $P+-P$, Q are homotopic.

(93) For every path P from a_1 to b_1 and for every constant path Q from a_1 to a_1 holds $P+-P$, Q are homotopic.

(94) Let P be a path from b to a and Q be a constant path from a to a. If b, a are connected, then $-P+P$, Q are homotopic.

(95) For every path P from b_1 to a_1 and for every constant path Q from a_1 to a_1 holds $-P+P$, Q are homotopic.

(96) For all constant paths P, Q from a to a holds P, Q are homotopic.

Let T be a non empty topological space, let a, b be points of T, and let P, Q be paths from a to b. Let us assume that P, Q are homotopic. A map from $[:\mathbb{I}, \mathbb{I}:]$ into T is said to be a homotopy between P and Q if it satisfies the conditions (Def. 13).

(Def. 13)(i) It is continuous, and

(ii) for every point s of $\mathbb{I}$ holds $\mathrm{it}(s, 0) = P(s)$ and $\mathrm{it}(s, 1) = Q(s)$ and for every point t of $\mathbb{I}$ holds $\mathrm{it}(0, t) = a$ and $\mathrm{it}(1, t) = b$.

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