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Banach Algebra of Bounded Complex Linear Operators

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Summary. This article is an extension of [16].

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The terminology and notation used here are introduced in the following articles: [18], [8], [20], [5], [7], [6], [3], [1], [17], [13], [19], [14], [2], [4], [15], [10], [11], [9], and [12].

One can prove the following propositions:

(1) Let X, Y, Z be complex linear spaces, f be a linear operator from X into Y, and g be a linear operator from Y into Z. Then $g \cdot f$ is a linear operator from X into Z.

(2) Let X, Y, Z be complex normed spaces, f be a bounded linear operator from X into Y, and g be a bounded linear operator from Y into Z. Then

(i) $g \cdot f$ is a bounded linear operator from X into Z, and

(ii) for every vector x of X holds $\|(g \cdot f)(x)\| \leqslant (\mathrm{BdLinOpsNorm}(Y, Z))(g) \cdot (\mathrm{BdLinOpsNorm}(X, Y))(f) \cdot \|x\|$ and $(\mathrm{BdLinOpsNorm}(X, Z))(g \cdot f) \leqslant (\mathrm{BdLinOpsNorm}(Y, Z))(g) \cdot (\mathrm{BdLinOpsNorm}(X, Y))(f)$.

Let X be a complex normed space and let f, g be bounded linear operators from X into X. Then $g \cdot f$ is a bounded linear operator from X into X.

Let X be a complex normed space and let f, g be elements of $\mathrm{BdLinOps}(X, X)$. The functor $f + g$ yields an element of $\mathrm{BdLinOps}(X, X)$ and is defined by:

(Def. 1) $f + g = (\mathrm{Add\_}(\mathrm{BdLinOps}(X, X), \mathrm{CVSpLinOps}(X, X)))(f, g)$.

Let X be a complex normed space and let f, g be elements of $\mathrm{BdLinOps}(X, X)$. The functor $g \cdot f$ yields an element of $\mathrm{BdLinOps}(X, X)$ and is defined as follows:

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(Def. 2) $g \cdot f = \mathrm{modetrans}(g, X, X) \cdot \mathrm{modetrans}(f, X, X)$.

Let X be a complex normed space, let f be an element of $\mathrm{BdLinOps}(X, X)$, and let z be a complex number. The functor $z \cdot f$ yields an element of $\mathrm{BdLinOps}(X, X)$ and is defined by:

(Def. 3) $z \cdot f = (\mathrm{Mult}_{\_}(\mathrm{BdLinOps}(X, X), \mathrm{CVSpLinOps}(X, X)))(z, f)$.

Let X be a complex normed space. The functor $\mathrm{FuncMult}(X)$ yields a binary operation on $\mathrm{BdLinOps}(X, X)$ and is defined as follows:

(Def. 4) For all elements f, g of $\mathrm{BdLinOps}(X, X)$ holds $(\mathrm{FuncMult}(X))(f, g) = f \cdot g$.

The following proposition is true

(3) For every complex normed space X holds $\mathrm{id}_{\text{the carrier of } X}$ is a bounded linear operator from X into X.

Let X be a complex normed space. The functor $\mathrm{FuncUnit}(X)$ yielding an element of $\mathrm{BdLinOps}(X, X)$ is defined by:

(Def. 5) $\mathrm{FuncUnit}(X) = \mathrm{id}_{\text{the carrier of } X}$.

The following propositions are true:

(4) Let X be a complex normed space and f, g, h be bounded linear operators from X into X. Then $h = f \cdot g$ if and only if for every vector x of X holds $h(x) = f(g(x))$.

(5) For every complex normed space X and for all bounded linear operators f, g, h from X into X holds $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

(6) Let X be a complex normed space and f be a bounded linear operator from X into X. Then $f \cdot \mathrm{id}_{\text{the carrier of } X} = f$ and $\mathrm{id}_{\text{the carrier of } X} \cdot f = f$.

(7) For every complex normed space X and for all elements f, g, h of $\mathrm{BdLinOps}(X, X)$ holds $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

(8) For every complex normed space X and for every element f of $\mathrm{BdLinOps}(X, X)$ holds $f \cdot \mathrm{FuncUnit}(X) = f$ and $\mathrm{FuncUnit}(X) \cdot f = f$.

(9) For every complex normed space X and for all elements f, g, h of $\mathrm{BdLinOps}(X, X)$ holds $f \cdot (g + h) = f \cdot g + f \cdot h$.

(10) For every complex normed space X and for all elements f, g, h of $\mathrm{BdLinOps}(X, X)$ holds $(g + h) \cdot f = g \cdot f + h \cdot f$.

(11) Let X be a complex normed space, f, g be elements of $\mathrm{BdLinOps}(X, X)$, and a, b be complex numbers. Then $(a \cdot b) \cdot (f \cdot g) = a \cdot f \cdot (b \cdot g)$.

(12) Let X be a complex normed space, f, g be elements of $\mathrm{BdLinOps}(X, X)$, and a be a complex number. Then $a \cdot (f \cdot g) = (a \cdot f) \cdot g$.

Let X be a complex normed space.

The functor $\mathrm{RingOfBoundedLinearOperators}(X)$ yields a double loop structure and is defined by:

(Def. 6) $\mathrm{RingOfBoundedLinearOperators}(X) = \langle \mathrm{BdLinOps}(X, X),$

Add_(BdLinOps(X, X), CVSpLinOps(X, X)), FuncMult(X), FuncUnit(X), Zero_(BdLinOps(X, X), CVSpLinOps(X, X))⟩.

Let X be a complex normed space.

Note that RingOfBoundedLinearOperators(X) is non empty and strict.

Next we state two propositions:

(13) Let X be a complex normed space and x, y, z be elements of RingOfBoundedLinearOperators(X). Then $x + y = y + x$ and $(x + y) + z = x + (y + z)$ and $x + 0_{\mathrm{RingOfBoundedLinearOperators}(X)} = x$ and there exists an element t of RingOfBoundedLinearOperators(X) such that $x + t = 0_{\mathrm{RingOfBoundedLinearOperators}(X)}$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{RingOfBoundedLinearOperators}(X)} = x$ and $\mathbf{1}_{\mathrm{RingOfBoundedLinearOperators}(X)} \cdot x = x$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$.

(14) For every complex normed space X holds RingOfBoundedLinearOperators(X) is a ring.

Let X be a complex normed space.

Observe that RingOfBoundedLinearOperators(X) is Abelian, add-associative, right zeroed, right complementable, associative, left unital, right unital, and distributive.

Let X be a complex normed space. The functor CAlgBdLinOps(X) yields a complex algebra structure and is defined by:

(Def. 7) CAlgBdLinOps(X) = ⟨BdLinOps(X, X), FuncMult(X), Add_(BdLinOps(X, X), CVSpLinOps(X, X)), Mult_(BdLinOps(X, X), CVSpLinOps(X, X)), FuncUnit(X), Zero_(BdLinOps(X, X), CVSpLinOps(X, X))⟩.

Let X be a complex normed space. Note that CAlgBdLinOps(X) is non empty and strict.

The following proposition is true

(15) Let X be a complex normed space, x, y, z be elements of CAlgBdLinOps(X), and a, b be complex numbers. Then $x+y = y+x$ and $(x + y) + z = x + (y + z)$ and $x + 0_{\mathrm{CAlgBdLinOps}(X)} = x$ and there exists an element t of CAlgBdLinOps(X) such that $x+t = 0_{\mathrm{CAlgBdLinOps}(X)}$ and $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ and $x\cdot\mathbf{1}_{\mathrm{CAlgBdLinOps}(X)} = x$ and $\mathbf{1}_{\mathrm{CAlgBdLinOps}(X)}\cdot x = x$ and $x\cdot(y+z) = x\cdot y+x\cdot z$ and $(y+z)\cdot x = y\cdot x+z\cdot x$ and $a\cdot(x\cdot y) = (a\cdot x)\cdot y$ and $a\cdot(x+y) = a\cdot x+a\cdot y$ and $(a+b)\cdot x = a\cdot x+b\cdot x$ and $(a\cdot b)\cdot x = a\cdot(b\cdot x)$ and $(a \cdot b) \cdot (x \cdot y) = a \cdot x \cdot (b \cdot y)$.

A complex BL algebra is an Abelian add-associative right zeroed right complementable associative complex algebra-like non empty complex algebra structure.

We now state the proposition

(16) For every complex normed space X holds CAlgBdLinOps(X) is a complex BL algebra.

Let us note that Complex-l1-Space is complete.

Let us mention that Complex-l1-Space is non trivial.

Let us note that there exists a complex Banach space which is non trivial.

The following two propositions are true:

(17) For every non trivial complex normed space X there exists a vector w of X such that $\|w\| = 1$.

(18) For every non trivial complex normed space X holds $(\mathrm{BdLinOpsNorm}(X, X))(\mathrm{id}_{\text{the carrier of } X}) = 1$.

We introduce normed complex algebra structures which are extensions of complex algebra structure and complex normed space structure and are systems

$\langle$ a carrier, a multiplication, an addition, an external multiplication, a unity, a zero, a norm $\rangle$,

where the carrier is a set, the multiplication and the addition are binary operations on the carrier, the external multiplication is a function from [: $\mathbb{C}$, the carrier :] into the carrier, the unity and the zero are elements of the carrier, and the norm is a function from the carrier into $\mathbb{R}$.

One can check that there exists a normed complex algebra structure which is non empty.

Let X be a complex normed space. The functor CNAlgBdLinOps(X) yields a normed complex algebra structure and is defined by:

(Def. 8) $\mathrm{CNAlgBdLinOps}(X) = \langle \mathrm{BdLinOps}(X, X), \mathrm{FuncMult}(X), \mathrm{Add}\_(\mathrm{BdLinOps}(X, X), \mathrm{CVSpLinOps}(X, X)), \mathrm{Mult}\_(\mathrm{BdLinOps}(X, X), \mathrm{CVSpLinOps}(X, X)), \mathrm{FuncUnit}(X), \mathrm{Zero}\_(\mathrm{BdLinOps}(X, X), \mathrm{CVSpLinOps}(X, X)), \mathrm{BdLinOpsNorm}(X, X)\rangle$.

Let X be a complex normed space. Note that CNAlgBdLinOps(X) is non empty and strict.

The following propositions are true:

(19) Let X be a complex normed space, x, y, z be elements of CNAlgBdLinOps(X), and a, b be complex numbers. Then $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x+0_{\mathrm{CNAlgBdLinOps}(X)} = x$ and there exists an element t of CNAlgBdLinOps(X) such that $x+t = 0_{\mathrm{CNAlgBdLinOps}(X)}$ and $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ and $x\cdot \mathbf{1}_{\mathrm{CNAlgBdLinOps}(X)} = x$ and $\mathbf{1}_{\mathrm{CNAlgBdLinOps}(X)}\cdot x = x$ and $x\cdot(y+z) = x\cdot y+x\cdot z$ and $(y+z)\cdot x = y\cdot x+z\cdot x$ and $a\cdot(x\cdot y) = (a\cdot x)\cdot y$ and $(a\cdot b)\cdot(x\cdot y) = a\cdot x\cdot(b\cdot y)$ and $a\cdot(x+y) = a\cdot x+a\cdot y$ and $(a+b)\cdot x = a\cdot x+b\cdot x$ and $(a\cdot b)\cdot x = a\cdot(b\cdot x)$ and $1_{\mathbb{C}}\cdot x = x$.

(20) Let X be a complex normed space. Then CNAlgBdLinOps(X) is complex normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, complex algebra-like, and complex linear space-like.

Let us observe that there exists a non empty normed complex algebra structure which is complex normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, complex algebra-like, complex linear space-like, and strict.

A normed complex algebra is a complex normed space-like Abelian add-associative right zeroed right complementable associative complex algebra-like complex linear space-like non empty normed complex algebra structure.

Let X be a complex normed space. One can check that CNAlgBdLinOps(X) is complex normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, complex algebra-like, and complex linear space-like.

Let X be a non empty normed complex algebra structure. We say that X is Banach Algebra-like1 if and only if:

(Def. 9) For all elements x, y of X holds $\|x \cdot y\| \leqslant \|x\| \cdot \|y\|$.

We say that X is Banach Algebra-like2 if and only if:

(Def. 10) $\|\mathbf{1}_X\| = 1$.

We say that X is Banach Algebra-like3 if and only if:

(Def. 11) For every complex number a and for all elements x, y of X holds $a \cdot (x \cdot y) = x \cdot (a \cdot y)$.

Let X be a normed complex algebra. We say that X is Banach Algebra-like if and only if the condition (Def. 12) is satisfied.

(Def. 12) X is Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left unital, left distributive, and complete.

One can verify that every normed complex algebra which is Banach Algebra-like is also Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left distributive, left unital, and complete and every normed complex algebra which is Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left distributive, left unital, and complete is also Banach Algebra-like.

Let X be a non trivial complex Banach space. One can verify that CNAlgBdLinOps(X) is Banach Algebra-like.

One can check that there exists a normed complex algebra which is Banach Algebra-like.

A complex Banach algebra is a Banach Algebra-like normed complex algebra.

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