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The Hall Marriage Theorem

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Summary. The Marriage Theorem, as credited to Philip Hall [7], gives the necessary and sufficient condition allowing us to select a distinct element from each of a finite collection $\{A_i\}$ of n finite subsets. This selection, called a set of different representatives (SDR), exists if and only if the marriage condition (or Hall condition) is satisfied:

$$\forall_{J\subseteq\{1,\ldots,n\}} |\bigcup_{i\in J} A_i| \geqslant |J|.$$

The proof which is given in this article (according to Richard Rado, 1967) is based on the lemma that for finite sequences with non-trivial elements which satisfy Hall property there exists a reduction (see Def. 5) such that Hall property again holds (see Th. 29 for details).

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The notation and terminology used here are introduced in the following papers: [9], [5], [10], [11], [4], [8], [2], [6], [1], and [3].

1. Preliminaries

One can prove the following proposition

(1) For all finite sets X, Y holds $\operatorname{card}(X\cup Y)+\operatorname{card}(X\cap Y) = \operatorname{card} X+\operatorname{card} Y$.

In this article we present several logical schemes. The scheme *Regr11* deals with a natural number $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every natural number k such that $1 \leqslant k$ and $k \leqslant \mathcal{A}$ holds $\mathcal{P}[k]$

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provided the parameters meet the following conditions:

- $\mathcal{P}[\mathcal{A}]$ and $\mathcal{A} \geqslant 2$, and
- For every natural number k such that $1 \leqslant k$ and $k < \mathcal{A}$ and $\mathcal{P}[k+1]$ holds $\mathcal{P}[k]$.

The scheme *Regr2* concerns a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[1]$

provided the parameters meet the following requirements:

- There exists a natural number n such that $n > 1$ and $\mathcal{P}[n]$, and
- For every natural number k such that $k \geqslant 1$ and $\mathcal{P}[k+1]$ holds $\mathcal{P}[k]$.

Let F be a non empty set. One can check that there exists a finite sequence of elements of 2^F which is non empty and non-empty.

We now state the proposition

(2) Let F be a non empty set, f be a non-empty finite sequence of elements of 2^F, and i be a natural number. If $i \in \operatorname{dom} f$, then $f(i) \neq \emptyset$.

Let F be a finite set, let A be a finite sequence of elements of 2^F, and let i be a natural number. Note that $A(i)$ is finite.

2. Union of Finite Sequences

Let F be a set, let A be a finite sequence of elements of 2^F, and let J be a set. The functor $\bigcup_J A$ yields a set and is defined as follows:

(Def. 1) For every set x holds $x \in \bigcup_J A$ iff there exists a set j such that $j \in J$ and $j \in \operatorname{dom} A$ and $x \in A(j)$.

Next we state two propositions:

(3) For every set F and for every finite sequence A of elements of 2^F and for every set J holds $\bigcup_J A \subseteq F$.

(4) Let F be a finite set, A be a finite sequence of elements of 2^F, and J, K be sets. If $J \subseteq K$, then $\bigcup_J A \subseteq \bigcup_K A$.

Let F be a finite set, let A be a finite sequence of elements of 2^F, and let J be a set. One can verify that $\bigcup_J A$ is finite.

The following propositions are true:

(5) Let F be a finite set, A be a finite sequence of elements of 2^F, and i be a natural number. If $i \in \operatorname{dom} A$, then $\bigcup_{\{i\}} A = A(i)$.

(6) Let F be a finite set, A be a finite sequence of elements of 2^F, and i, j be natural numbers. If $i \in \operatorname{dom} A$ and $j \in \operatorname{dom} A$, then $\bigcup_{\{i,j\}} A = A(i) \cup A(j)$.

(7) Let J be a set, F be a finite set, A be a finite sequence of elements of 2^F, and i be a natural number. If $i \in J$ and $i \in \operatorname{dom} A$, then $A(i) \subseteq \bigcup_J A$.

(8) Let J be a set, F be a finite set, i be a natural number, and A be a finite sequence of elements of 2^F. If $i \in J$ and $i \in \operatorname{dom} A$, then $\bigcup_J A = \bigcup_{J \setminus \{i\}} A \cup A(i)$.

(9) Let J_1, J_2 be sets, F be a finite set, i be a natural number, and A be a finite sequence of elements of 2^F. If $i \in \operatorname{dom} A$, then $\bigcup_{\{i\} \cup J_1 \cup J_2} A = A(i) \cup \bigcup_{J_1 \cup J_2} A$.

(10) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x, y be sets. If $x \neq y$ and $x \in A(i)$ and $y \in A(i)$, then $(A(i) \setminus \{x\}) \cup (A(i) \setminus \{y\}) = A(i)$.

3. Cut Operation for Finite Sequences

Let F be a finite set, let A be a finite sequence of elements of 2^F, let i be a natural number, and let x be a set. The functor $\operatorname{Cut}(A, i, x)$ yielding a finite sequence of elements of 2^F is defined by the conditions (Def. 2).

(Def. 2)(i) $\operatorname{dom} \operatorname{Cut}(A, i, x) = \operatorname{dom} A$, and

(ii) for every natural number k such that $k \in \operatorname{dom} \operatorname{Cut}(A, i, x)$ holds if $i = k$, then $(\operatorname{Cut}(A, i, x))(k) = A(k) \setminus \{x\}$ and if $i \neq k$, then $(\operatorname{Cut}(A, i, x))(k) = A(k)$.

The following propositions are true:

(11) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x be a set. If $i \in \operatorname{dom} A$ and $x \in A(i)$, then $\operatorname{card}(\operatorname{Cut}(A, i, x))(i) = \operatorname{card} A(i) - 1$.

(12) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x, J be sets. Then $\bigcup_{J \setminus \{i\}} \operatorname{Cut}(A, i, x) = \bigcup_{J \setminus \{i\}} A$.

(13) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x, J be sets. If $i \notin J$, then $\bigcup_J A = \bigcup_J \operatorname{Cut}(A, i, x)$.

(14) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x, J be sets. If $i \in \operatorname{dom} \operatorname{Cut}(A, i, x)$ and $J \subseteq \operatorname{dom} \operatorname{Cut}(A, i, x)$ and $i \in J$, then $\bigcup_J \operatorname{Cut}(A, i, x) = \bigcup_{J \setminus \{i\}} A \cup (A(i) \setminus \{x\})$.

4. System of Different Representatives and Hall Property

Let F be a finite set, let X be a finite sequence of elements of 2^F, and let A be a set. We say that A is a system of different representatives of X if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exists a finite sequence f of elements of F such that $f = A$ and $\operatorname{dom} X = \operatorname{dom} f$ and for every natural number i such that $i \in \operatorname{dom} f$ holds $f(i) \in X(i)$ and f is one-to-one.

Let F be a finite set and let A be a finite sequence of elements of 2^F. We say that A satisfies Hall condition if and only if:

(Def. 4) For every finite set J such that $J \subseteq \operatorname{dom} A$ holds $\operatorname{card} J \leqslant \operatorname{card} \bigcup_J A$.

Next we state four propositions:

(15) Let F be a finite set and A be a non empty finite sequence of elements of 2^F. If A satisfies Hall condition, then A is non-empty.

(16) Let F be a finite set, A be a finite sequence of elements of 2^F, and i be a natural number. If $i \in \operatorname{dom} A$ and A satisfies Hall condition, then $\operatorname{card} A(i) \geqslant 1$.

(17) Let F be a non empty finite set and A be a non empty finite sequence of elements of 2^F. Suppose for every natural number i such that $i \in \operatorname{dom} A$ holds $\operatorname{card} A(i) = 1$ and A satisfies Hall condition. Then there exists a set which is a system of different representatives of A.

(18) Let F be a finite set and A be a finite sequence of elements of 2^F such that there exists a set which is a system of different representatives of A. Then A satisfies Hall condition.

5. Reductions and Singlifications of Finite Sequences

Let F be a set, let A be a finite sequence of elements of 2^F, and let i be a natural number. A finite sequence of elements of 2^F is said to be a reduction of A at i-th position if:

(Def. 5) $\operatorname{dom} \mathrm{it} = \operatorname{dom} A$ and for every natural number j such that $j \in \operatorname{dom} A$ and $j \neq i$ holds $A(j) = \mathrm{it}(j)$ and $\mathrm{it}(i) \subseteq A(i)$.

Let F be a set and let A be a finite sequence of elements of 2^F. A finite sequence of elements of 2^F is said to be a reduction of A if:

(Def. 6) $\operatorname{dom} \mathrm{it} = \operatorname{dom} A$ and for every natural number i such that $i \in \operatorname{dom} A$ holds $\mathrm{it}(i) \subseteq A(i)$.

Let F be a set, let A be a finite sequence of elements of 2^F, and let i be a natural number. Let us assume that $i \in \operatorname{dom} A$ and $A(i) \neq \emptyset$. A reduction of A is called a singlification of A at i-th position if:

(Def. 7) $\overline{\overline{\mathrm{it}(i)}} = 1$.

One can prove the following propositions:

(19) Let F be a finite set, A be a finite sequence of elements of 2^F, and i be a natural number. Then every reduction of A at i-th position is a reduction of A.

(20) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x be a set. If $i \in \operatorname{dom} A$ and $x \in A(i)$, then $\mathrm{Cut}(A, i, x)$ is a reduction of A at i-th position.

(21) Let F be a finite set, A be a finite sequence of elements of 2^F, i be a natural number, and x be a set. If $i \in \operatorname{dom} A$ and $x \in A(i)$, then $\mathrm{Cut}(A, i, x)$ is a reduction of A.

(22) Let F be a finite set, A be a finite sequence of elements of 2^F, and B be a reduction of A. Then every reduction of B is a reduction of A.

(23) Let F be a non empty finite set, A be a non-empty finite sequence of elements of 2^F, i be a natural number, and B be a singlification of A at i-th position. If $i \in \operatorname{dom} A$, then $B(i) \neq \emptyset$.

(24) Let F be a non empty finite set, A be a non-empty finite sequence of elements of 2^F, i, j be natural numbers, B be a singlification of A at i-th position, and C be a singlification of B at j-th position. Suppose $i \in \operatorname{dom} A$ and $j \in \operatorname{dom} A$ and $C(i) \neq \emptyset$ and $B(j) \neq \emptyset$. Then C is a singlification of A at j-th position and a singlification of A at i-th position.

(25) Let F be a set, A be a finite sequence of elements of 2^F, and i be a natural number. Then A is a reduction of A at i-th position.

(26) For every set F holds every finite sequence A of elements of 2^F is a reduction of A.

Let F be a non empty set and let A be a finite sequence of elements of 2^F. Let us assume that A is non-empty. A reduction of A is called a singlification of A if:

(Def. 8) For every natural number i such that $i \in \operatorname{dom} A$ holds $\overline{\overline{\mathrm{it}(i)}} = 1$.

We now state the proposition

(27) Let F be a non empty finite set, A be a non empty non-empty finite sequence of elements of 2^F, and f be a function. Then f is a singlification of A if and only if the following conditions are satisfied:

(i) $\operatorname{dom} f = \operatorname{dom} A$, and

(ii) for every natural number i such that $i \in \operatorname{dom} A$ holds f is a singlification of A at i-th position.

Let F be a non empty finite set, let A be a non empty finite sequence of elements of 2^F, and let k be a natural number. Note that every singlification of A at k-th position is non empty.

Let F be a non empty finite set and let A be a non empty finite sequence of elements of 2^F. One can check that every singlification of A is non empty.

6. Rado's Proof of the Hall Marriage Theorem

One can prove the following propositions:

(28) Let F be a non empty finite set, A be a non empty finite sequence of elements of 2^F, X be a set, and B be a reduction of A. Suppose X is a

system of different representatives of B. Then X is a system of different representatives of A.

(29) Let F be a finite set and A be a finite sequence of elements of 2^F. Suppose A satisfies Hall condition. Let i be a natural number. If $\operatorname{card} A(i) \geqslant 2$, then there exists a set x such that $x \in A(i)$ and $\operatorname{Cut}(A, i, x)$ satisfies Hall condition.

(30) Let F be a finite set, A be a finite sequence of elements of 2^F, and i be a natural number. If $i \in \operatorname{dom} A$ and A satisfies Hall condition, then there exists a singlification of A at i-th position which satisfies Hall condition.

(31) Let F be a non empty finite set and A be a non empty finite sequence of elements of 2^F. If A satisfies Hall condition, then there exists a singlification of A which satisfies Hall condition.

(32) Let F be a non empty finite set and A be a non empty finite sequence of elements of 2^F. Then there exists a set which is a system of different representatives of A if and only if A satisfies Hall condition.

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