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The Operation of Addition of Relational Structures

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Summary. The article contains the formalization of the addition operator on relational structures as defined by A. Wroński [8] (as a generalization of Troelstra's sum or Jaśkowski's star addition). The ordering relation of $A \otimes B$ is given by

$$\leqslant_{A \otimes B} = \leqslant_A \cup \leqslant_B \cup (\leqslant_A \circ \leqslant_B),$$

where the carrier is defined as the set-theoretical union of carriers of A and B. Main part – Section 3 – is devoted to the Mizar translation of Theorem 1 (iv–xiii), p. 66 of [8].

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The terminology and notation used in this paper are introduced in the following articles: [4], [6], [7], [5], [2], [3], and [1].

1. Preliminaries

One can prove the following proposition

(1) Let x, y, A, B be sets. Suppose $x \in A \cup B$ and $y \in A \cup B$. Then $x \in A \setminus B$ and $y \in A \setminus B$ or $x \in B$ and $y \in B$ or $x \in A \setminus B$ and $y \in B$ or $x \in B$ and $y \in A \setminus B$.

Let R, S be relational structures. The predicate $R \approx S$ is defined by the condition (Def. 1).

(Def. 1) Let x, y be sets. Suppose $x \in$ (the carrier of R) $\cap$ (the carrier of S) and $y \in$ (the carrier of R) $\cap$ (the carrier of S). Then $\langle x, y\rangle \in$ the internal relation of R if and only if $\langle x, y\rangle \in$ the internal relation of S.

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2. The Wroński's Operation

Let R, S be relational structures. The functor $R \otimes S$ yields a strict relational structure and is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of $R \otimes S =$ (the carrier of R) $\cup$ (the carrier of S), and

(ii) the internal relation of $R \otimes S =$ (the internal relation of R) $\cup$ (the internal relation of S) $\cup$ (the internal relation of R) $\cdot$ (the internal relation of S).

Let R be a relational structure and let S be a non empty relational structure. Observe that $R \otimes S$ is non empty.

Let R be a non empty relational structure and let S be a relational structure. Observe that $R \otimes S$ is non empty.

One can prove the following two propositions:

(2) Let R, S be relational structures. Then

(i) the internal relation of $R \subseteq$ the internal relation of $R \otimes S$, and

(ii) the internal relation of $S \subseteq$ the internal relation of $R \otimes S$.

(3) For all relational structures R, S such that R is reflexive and S is reflexive holds $R \otimes S$ is reflexive.

3. Properties of the Addition

Next we state a number of propositions:

(4) Let R, S be relational structures and a, b be sets. Suppose that

(i) $\langle a, b \rangle \in$ the internal relation of $R \otimes S$,

(ii) $a \in$ the carrier of R,

(iii) $b \in$ the carrier of R,

(iv) $R \approx S$, and

(v) R is transitive.

Then $\langle a, b \rangle \in$ the internal relation of R.

(5) Let R, S be relational structures and a, b be sets. Suppose that

(i) $\langle a, b \rangle \in$ the internal relation of $R \otimes S$,

(ii) $a \in$ the carrier of S,

(iii) $b \in$ the carrier of S,

(iv) $R \approx S$, and

(v) S is transitive.

Then $\langle a, b \rangle \in$ the internal relation of S.

(6) Let R, S be relational structures and a, b be sets. Then

(i) if $\langle a, b \rangle \in$ the internal relation of R, then $\langle a, b \rangle \in$ the internal relation of $R \otimes S$, and

(ii) if $\langle a, b \rangle \in$ the internal relation of S, then $\langle a, b \rangle \in$ the internal relation of $R \otimes S$.

(7) Let R, S be non empty relational structures and x be an element of $R \otimes S$. Then $x \in$ the carrier of R or $x \in$ (the carrier of S) $\setminus$ (the carrier of R).

(8) Let R, S be non empty relational structures, x, y be elements of R, and a, b be elements of $R \otimes S$. Suppose $x = a$ and $y = b$ and $R \approx S$ and R is transitive. Then $x \leqslant y$ if and only if $a \leqslant b$.

(9) Let R, S be non empty relational structures, a, b be elements of $R \otimes S$, and c, d be elements of S. Suppose $a = c$ and $b = d$ and $R \approx S$ and S is transitive. Then $a \leqslant b$ if and only if $c \leqslant d$.

(10) Let R, S be antisymmetric reflexive transitive non empty relational structures with l.u.b.'s and x be a set. If $x \in$ the carrier of R, then x is an element of $R \otimes S$.

(11) Let R, S be antisymmetric reflexive transitive non empty relational structures with l.u.b.'s and x be a set. If $x \in$ the carrier of S, then x is an element of $R \otimes S$.

(12) Let R, S be non empty relational structures and x be a set. Suppose $x \in$ (the carrier of R) $\cap$ (the carrier of S). Then x is an element of R.

(13) Let R, S be non empty relational structures and x be a set. Suppose $x \in$ (the carrier of R) $\cap$ (the carrier of S). Then x is an element of S.

(14) Let R, S be antisymmetric reflexive transitive non empty relational structures with l.u.b.'s and x, y be elements of $R \otimes S$. Suppose $x \in$ the carrier of R and $y \in$ the carrier of S and $R \approx S$. Then $x \leqslant y$ if and only if there exists an element a of $R \otimes S$ such that $a \in$ (the carrier of R) $\cap$ (the carrier of S) and $x \leqslant a$ and $a \leqslant y$.

(15) Let R, S be non empty relational structures, a, b be elements of R, and c, d be elements of S. Suppose $a = c$ and $b = d$ and $R \approx S$ and R is transitive and S is transitive. Then $a \leqslant b$ if and only if $c \leqslant d$.

(16) Let R be an antisymmetric reflexive transitive non empty relational structure with l.u.b.'s, D be a lower directed subset of R, and x, y be elements of R. If $x \in D$ and $y \in D$, then $x \sqcup y \in D$.

(17) Let R, S be relational structures and a, b be sets. Suppose that
- (i) (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R,
- (ii) $\langle a, b \rangle \in$ the internal relation of $R \otimes S$, and
- (iii) $a \in$ the carrier of S.

Then $b \in$ the carrier of S.

(18) Let R, S be relational structures and a, b be elements of $R \otimes S$. Suppose (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R and $a \leqslant b$ and $a \in$ the carrier of S. Then $b \in$ the carrier of S.

(19) Let R, S be antisymmetric reflexive transitive non empty relational structures with l.u.b.'s, x, y be elements of R, and a, b be elements of

S. Suppose that
(i) (the carrier of R) $\cap$ (the carrier of S) is a lower directed subset of S,
(ii) (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R,
(iii) $R \approx S$,
(iv) $x = a$, and
(v) $y = b$.
Then $x \sqcup y = a \sqcup b$.

(20) Let R, S be lower-bounded antisymmetric reflexive transitive non empty relational structures with l.u.b.'s. Suppose (the carrier of R) $\cap$ (the carrier of S) is a non empty lower directed subset of S. Then $\bot_S \in$ the carrier of R.

(21) Let R, S be relational structures and a, b be sets. Suppose that
(i) (the carrier of R) $\cap$ (the carrier of S) is a lower subset of S,
(ii) $\langle a, b \rangle \in$ the internal relation of $R \otimes S$, and
(iii) $b \in$ the carrier of R.
Then $a \in$ the carrier of R.

(22) Let x, y be sets and R, S be relational structures. Suppose $\langle x, y \rangle \in$ the internal relation of $R \otimes S$ and (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R. Then
(i) $x \in$ the carrier of R and $y \in$ the carrier of R, or
(ii) $x \in$ the carrier of S and $y \in$ the carrier of S, or
(iii) $x \in$ (the carrier of R) $\setminus$ (the carrier of S) and $y \in$ (the carrier of S) $\setminus$ (the carrier of R).

(23) Let R, S be relational structures and a, b be elements of $R \otimes S$. Suppose (the carrier of R) $\cap$ (the carrier of S) is a lower subset of S and $a \leqslant b$ and $b \in$ the carrier of R. Then $a \in$ the carrier of R.

(24) Let R, S be relational structures. Suppose that
(i) $R \approx S$,
(ii) (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R,
(iii) (the carrier of R) $\cap$ (the carrier of S) is a lower subset of S,
(iv) R is transitive and antisymmetric, and
(v) S is transitive and antisymmetric.
Then $R \otimes S$ is antisymmetric.

(25) Let R, S be relational structures. Suppose that
(i) (the carrier of R) $\cap$ (the carrier of S) is an upper subset of R,
(ii) (the carrier of R) $\cap$ (the carrier of S) is a lower subset of S,
(iii) $R \approx S$,
(iv) R is transitive, and
(v) S is transitive.
Then $R \otimes S$ is transitive.

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