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The Nagata-Smirnov Theorem. Part I[1]

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Summary. In this paper we define a discrete subset family of a topological space and basis sigma locally finite and sigma discrete. First, we prove an auxiliary fact for discrete family and sigma locally finite and sigma discrete basis. We also show the necessary condition for the Nagata Smirnov theorem: every metrizable space is T_3 and has a sigma locally finite basis. Also, we define a sufficient condition for a T_3 topological space to be T_4. We introduce the concept of pseudo metric.

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The terminology and notation used in this paper have been introduced in the following articles: [9], [27], [28], [32], [20], [5], [12], [8], [21], [15], [2], [17], [14], [18], [19], [6], [10], [11], [24], [23], [4], [33], [1], [3], [25], [16], [26], [7], [13], [29], [31], [34], [30], and [22].

In this paper T, T_1 denote non empty topological spaces and P_1 denotes a non empty metric structure.

Let T be a topological space and let F be a family of subsets of T. We say that F is discrete if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let p be a point of T. Then there exists an open subset O of T such that $p \in O$ and for all subsets A, B of T such that $A \in F$ and $B \in F$ holds if O meets A and O meets B, then $A = B$.

Let T be a non empty topological space. Note that there exists a family of subsets of T which is discrete.

Let us consider T. One can check that there exists a family of subsets of T which is empty and discrete.

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For simplicity, we adopt the following convention: F, G, H denote families of subsets of T, A, B denote subsets of T, O, U denote open subsets of T, p denotes a point of T, and x, X denote sets.

The following propositions are true:

(1) For every F such that there exists A such that $F = \{A\}$ holds F is discrete.

(2) For all F, G such that $F \subseteq G$ and G is discrete holds F is discrete.

(3) For all F, G such that F is discrete holds $F \cap G$ is discrete.

(4) For all F, G such that F is discrete holds $F \setminus G$ is discrete.

(5) For all F, G, H such that F is discrete and G is discrete and $F \Cap G = H$ holds H is discrete.

(6) For all F, A, B such that F is discrete and $A \in F$ and $B \in F$ holds $A = B$ or A misses B.

(7) If F is discrete, then for every p there exists O such that $p \in O$ and $\{O\} \Cap F \setminus \{\emptyset\}$ is trivial.

(8) F is discrete if and only if the following conditions are satisfied:

(i) for every p there exists O such that $p \in O$ and $\{O\} \Cap F \setminus \{\emptyset\}$ is trivial, and

(ii) for all A, B such that $A \in F$ and $B \in F$ holds $A = B$ or A misses B.

Let us consider T and let F be a discrete family of subsets of T. Observe that $\operatorname{clf} F$ is discrete.

Next we state three propositions:

(9) For every F such that F is discrete and for all A, B such that $A \in F$ and $B \in F$ holds $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

(10) For every F such that F is discrete holds $\overline{\bigcup F} = \bigcup \operatorname{clf} F$.

(11) For every F such that F is discrete holds F is locally finite.

Let T be a topological space. A family sequence of T is a function from $\mathbb{N}$ into $2^{2^{\text{the carrier of } T}}$.

In the sequel U_1 denotes a family sequence of T, r denotes a real number, n denotes a natural number, and f denotes a function.

Let us consider T, U_1, n. Then $U_1(n)$ is a family of subsets of T.

Let us consider T, U_1. Then $\bigcup U_1$ is a family of subsets of T.

Let T be a non empty topological space and let U_1 be a family sequence of T. We say that U_1 is sigma-discrete if and only if:

(Def. 2) For every natural number n holds $U_1(n)$ is discrete.

Let T be a non empty topological space. Note that there exists a family sequence of T which is sigma-discrete.

Let T be a non empty topological space and let U_1 be a family sequence of T. We say that U_1 is sigma-locally-finite if and only if:

(Def. 3) For every natural number n holds $U_1(n)$ is locally finite.

Let us consider T and let F be a family of subsets of T. We say that F is sigma-discrete if and only if:

(Def. 4) There exists a sigma-discrete family sequence f of T such that $F = \bigcup f$.

Let X be a set. We introduce X is uncountable as an antonym of X is countable.

One can verify that every set which is uncountable is also non empty.

Let T be a non empty topological space. One can check that there exists a family sequence of T which is sigma-locally-finite.

Next we state two propositions:

(12) For every U_1 such that U_1 is sigma-discrete holds U_1 is sigma-locally-finite.

(13) Let A be an uncountable set. Then there exists a family F of subsets of $\{[\!: A, A :\!]\}_{\rm top}$ such that F is locally finite and F is not sigma-discrete.

Let T be a non empty topological space and let U_1 be a family sequence of T. We say that U_1 is Basis-sigma-discrete if and only if:

(Def. 5) U_1 is sigma-discrete and $\bigcup U_1$ is a basis of T.

Let T be a non empty topological space and let U_1 be a family sequence of T. We say that U_1 is Basis-sigma-locally finite if and only if:

(Def. 6) U_1 is sigma-locally-finite and $\bigcup U_1$ is a basis of T.

The following propositions are true:

(14) Let r be a real number. Suppose P_1 is a non empty metric space. Let x be an element of P_1. Then $\Omega_{(P_1)} \setminus \overline{\rm Ball}(x, r) \in$ the open set family of P_1.

(15) For every T such that T is metrizable holds T is a T_3 space and a T_1 space.

(16) For every T such that T is metrizable holds there exists a family sequence of T which is Basis-sigma-locally finite.

(17) For every function U from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ such that for every n holds $U(n)$ is open holds $\bigcup U$ is open.

(18) Suppose that for all A, U such that A is closed and U is open and $A \subseteq U$ there exists a function W from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ such that $A \subseteq \bigcup W$ and $\bigcup W \subseteq U$ and for every n holds $\overline{W(n)} \subseteq U$ and $W(n)$ is open. Then T is a T_4 space.

(19) Let given T. Suppose T is a T_3 space. Let B_1 be a family sequence of T. Suppose $\bigcup B_1$ is a basis of T. Let U be a subset of T and p be a point of T. Suppose U is open and $p \in U$. Then there exists a subset O of T such that $p \in O$ and $\overline{O} \subseteq U$ and $O \in \bigcup B_1$.

(20) For every T such that T is a T_3 space and a T_1 space and there exists a family sequence of T which is Basis-sigma-locally finite holds T is a T_4

space.

Let us consider T and let F, G be real maps of T. The functor $F+G$ yielding a real map of T is defined as follows:

(Def. 7) For every element t of T holds $(F + G)(t) = F(t) + G(t)$.

Next we state four propositions:

(21) Let f be a real map of T. Suppose f is continuous. Let F be a real map of $[: T, T :]$. Suppose that for all elements x, y of the carrier of T holds $F(\langle x, y\rangle) = |f(x) - f(y)|$. Then F is continuous.

(22) For all real maps F, G of T such that F is continuous and G is continuous holds $F + G$ is continuous.

(23) Let A_1 be a binary operation on $\mathbb{R}^{\text{the carrier of } T}$. Suppose that for all real maps f_1, f_2 of T holds $A_1(f_1, f_2) = f_1 + f_2$. Then A_1 is commutative and associative and has a unity.

(24) Let A_1 be a binary operation on $\mathbb{R}^{\text{the carrier of } T}$. Suppose that for all real maps f_1, f_2 of T holds $A_1(f_1, f_2) = f_1 + f_2$. Let m'_1 be an element of $\mathbb{R}^{\text{the carrier of } T}$. If m'_1 is a unity w.r.t. A_1, then m'_1 is continuous.

Let T, T_1 be non empty topological spaces, let S_1 be a function from the carrier of T into $2^{\text{the carrier of } T}$, and let F_1 be a function from the carrier of T into (the carrier of T_1)$^{\text{the carrier of } T}$. The functor $F_1 \approx S_1$ yields a map from T into T_1 and is defined by:

(Def. 8) For every point p of T holds $(F_1 \approx S_1)(p) = F_1(p)(p)$.

The following propositions are true:

(25) Let A_1 be a binary operation on $\mathbb{R}^{\text{the carrier of } T}$. Suppose that for all real maps f_1, f_2 of T holds $A_1(f_1, f_2) = f_1 + f_2$. Let F be a finite sequence of elements of $\mathbb{R}^{\text{the carrier of } T}$. Suppose that for every n such that $0 \neq n$ and $n \leqslant \operatorname{len} F$ holds $F(n)$ is a continuous real map of T. Then $A_1 \odot F$ is a continuous real map of T.

(26) Let F be a function from the carrier of T into (the carrier of T_1)$^{\text{the carrier of } T}$. Suppose that for every point p of T holds $F(p)$ is a continuous map from T into T_1. Let S be a function from the carrier of T into $2^{\text{the carrier of } T}$. Suppose that for every point p of T holds $p \in S(p)$ and $S(p)$ is open and for all points p, q of T such that $p \in S(q)$ holds $F(p)(p) = F(q)(p)$. Then $F \approx S$ is continuous.

In the sequel m denotes a function from $[:$ the carrier of T, the carrier of $T :]$ into $\mathbb{R}$.

Let us consider X, r and let f be a function from X into $\mathbb{R}$. The functor $\min(r, f)$ yielding a function from X into $\mathbb{R}$ is defined as follows:

(Def. 9) For every x such that $x \in X$ holds $(\min(r, f))(x) = \min(r, f(x))$.

One can prove the following proposition

(27) For every real number r and for every real map f of T such that f is continuous holds $\min(r, f)$ is continuous.

Let X be a set and let f be a function from $[: X, X :]$ into $\mathbb{R}$. We say that f is a pseudometric of if and only if:

(Def. 10) f is Reflexive, symmetric, and triangle.

One can prove the following propositions:

(28) Let f be a function from $[: X, X :]$ into $\mathbb{R}$. Then f is a pseudometric of if and only if for all elements a, b, c of X holds $f(a, a) = 0$ and $f(a, c) \leqslant f(a, b) + f(c, b)$.

(29) For every function f from $[: X, X :]$ into $\mathbb{R}$ such that f is a pseudometric of and for all elements x, y of X holds $f(x, y) \geqslant 0$.

(30) For all r, m such that $r > 0$ and m is a pseudometric of holds $\min(r, m)$ is a pseudometric of.

(31) For all r, m such that $r > 0$ and m is a metric of the carrier of T holds $\min(r, m)$ is a metric of the carrier of T.

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