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The Nagata-Smirnov Theorem. Part II[1]

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Summary. In this paper we show some auxiliary facts for sequence function to be pseudo-metric. Next we prove the Nagata-Smirnov theorem that every topological space is metrizable if and only if it has σ-locally finite basis. We attach also the proof of the Bing's theorem that every topological space is metrizable if and only if its basis is σ-discrete.

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The terminology and notation used in this paper have been introduced in the following articles: [9], [27], [28], [32], [20], [5], [12], [8], [21], [15], [2], [17], [14], [18], [19], [6], [10], [11], [24], [23], [4], [33], [1], [3], [25], [16], [26], [7], [13], [29], [31], [34], [30], and [22].

For simplicity, we adopt the following convention: i, k, m, n denote natural numbers, r, s denote real numbers, X denotes a set, T, T_1, T_2 denote non empty topological spaces, p denotes a point of T, A denotes a subset of T, A' denotes a non empty subset of T, p_1 denotes an element of [: the carrier of T, the carrier of T :], p_2 denotes a function from [: the carrier of T, the carrier of T :] into $\mathbb{R}$, p'_1 denotes a real map of [: T, T :], f denotes a real map of T, F_2 denotes a sequence of partial functions from [: the carrier of T, the carrier of T :] into $\mathbb{R}$, and s_1 denotes a sequence of real numbers.

The following proposition is true

(1) For every i such that $i > 0$ there exist n, m such that $i = 2^n \cdot (2 \cdot m + 1)$.

The function PairFunc from [: $\mathbb{N}$, $\mathbb{N}$:] into $\mathbb{N}$ is defined by:

(Def. 1) For all n, m holds PairFunc($\langle n, m \rangle$) $= 2^n \cdot (2 \cdot m + 1) - 1$.

We now state the proposition

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(2) PairFunc is bijective.

Let X be a set, let f be a function from $[: X, X :]$ into $\mathbb{R}$, and let x be an element of X. The functor $\rho(f, x)$ yielding a function from X into $\mathbb{R}$ is defined as follows:

(Def. 2) For every element y of X holds $(\rho(f, x))(y) = f(x, y)$.

The following two propositions are true:

(3) Let D be a subset of $[: T_1, T_2 :]$. Suppose D is open. Let x_1 be a point of T_1, x_2 be a point of T_2, X_1 be a subset of T_1, and X_2 be a subset of T_2. Then

(i) if $X_1 = \pi_1(($the carrier of $T_1) \times$ the carrier of $T_2)^\circ(D \cap [:$ the carrier of T_1, $\{x_2\} :])$, then X_1 is open, and

(ii) if $X_2 = \pi_2(($the carrier of $T_1) \times$ the carrier of $T_2)^\circ(D \cap [: \{x_1\}$, the carrier of $T_2 :])$, then X_2 is open.

(4) For every p_2 such that for every p_1' such that $p_2 = p_1'$ holds p_1' is continuous and for every point x of T holds $\rho(p_2, x)$ is continuous.

Let X be a non empty set, let f be a function from $[: X, X :]$ into $\mathbb{R}$, and let A be a subset of X. The functor $\inf(f, A)$ yielding a function from X into $\mathbb{R}$ is defined by:

(Def. 3) For every element x of X holds $(\inf(f, A))(x) = \inf((\rho(f, x))^\circ A)$.

One can prove the following propositions:

(5) Let X be a non empty set and f be a function from $[: X, X :]$ into $\mathbb{R}$. Suppose f is a pseudometric of. Let A be a non empty subset of X and x be an element of X. Then $(\inf(f, A))(x) \geqslant 0$.

(6) Let X be a non empty set and f be a function from $[: X, X :]$ into $\mathbb{R}$. Suppose f is a pseudometric of. Let A be a subset of X and x be an element of X. If $x \in A$, then $(\inf(f, A))(x) = 0$.

(7) Let given p_2. Suppose p_2 is a pseudometric of. Let x, y be points of T and A be a non empty subset of T. Then $|(\inf(p_2, A))(x) - (\inf(p_2, A))(y)| \leqslant p_2(x, y)$.

(8) Let given p_2. Suppose p_2 is a pseudometric of and for every p holds $\rho(p_2, p)$ is continuous. Let A be a non empty subset of T. Then $\inf(p_2, A)$ is continuous.

(9) For every function f from $[: X, X :]$ into $\mathbb{R}$ such that f is a metric of X holds f is a pseudometric of.

(10) Let given p_2. Suppose p_2 is a metric of the carrier of T and for every non empty subset A of T holds $\overline{A} = \{p; p$ ranges over points of T: $(\inf(p_2, A))(p) = 0\}$. Then T is metrizable.

(11) Let given F_2. Suppose for every n there exists p_2 such that $F_2(n) = p_2$ and p_2 is a pseudometric of and for every p_1 holds $F_2 \# p_1$ is summable.

Let given p_2. If for every p_1 holds $p_2(p_1) = \sum(F_2\#p_1)$, then p_2 is a pseudometric of.

(12) For all n, s_1 such that for every m such that $m \leqslant n$ holds $s_1(m) \leqslant r$ and for every m such that $m \leqslant n$ holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) \leqslant r \cdot (m+1)$.

(13) For every k holds $|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(k)| \leqslant (\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(14) Let F_1 be a sequence of partial functions from the carrier of T into $\mathbb{R}$. Suppose that

(i) for every n there exists f such that $F_1(n) = f$ and f is continuous and for every p holds $f(p) \geqslant 0$, and

(ii) there exists s_1 such that s_1 is summable and for all n, p holds $(F_1\#p)(n) \leqslant s_1(n)$.

Let given f. If for every p holds $f(p) = \sum(F_1\#p)$, then f is continuous.

(15) Let given s, F_2. Suppose that for every n there exists p_2 such that $F_2(n) = p_2$ and p_2 is a pseudometric of and for every p_1 holds $p_2(p_1) \leqslant s$ and for every p'_1 such that $p_2 = p'_1$ holds p'_1 is continuous. Let given p_2. Suppose that for every p_1 holds $p_2(p_1) = \sum(((\frac{1}{2})^{\kappa})_{\kappa\in\mathbb{N}}\,(F_2\#p_1))$. Then p_2 is a pseudometric of and for every p'_1 such that $p_2 = p'_1$ holds p'_1 is continuous.

(16) Let given p_2. Suppose p_2 is a pseudometric of and for every p'_1 such that $p_2 = p'_1$ holds p'_1 is continuous. Let A be a non empty subset of T and given p. If $p \in \overline{A}$, then $(\inf(p_2, A))(p) = 0$.

(17) Let given T. Suppose T is a T_1 space. Let given s, F_2. Suppose that

(i) for every n there exists p_2 such that $F_2(n) = p_2$ and p_2 is a pseudometric of and for every p_1 holds $p_2(p_1) \leqslant s$ and for every p'_1 such that $p_2 = p'_1$ holds p'_1 is continuous, and

(ii) for all p, A' such that $p \notin A'$ and A' is closed there exists n such that for every p_2 such that $F_2(n) = p_2$ holds $(\inf(p_2, A'))(p) > 0$.

Then there exists p_2 such that p_2 is a metric of the carrier of T and for every p_1 holds $p_2(p_1) = \sum(((\frac{1}{2})^{\kappa})_{\kappa\in\mathbb{N}}\,(F_2\#p_1))$ and T is metrizable.

(18) Let D be a non empty set, p, q be finite sequences of elements of D, and B be a binary operation on D. Suppose that

(i) p is one-to-one,

(ii) q is one-to-one,

(iii) $\operatorname{rng} q \subseteq \operatorname{rng} p$,

(iv) B is commutative and associative, and

(v) B has a unity or $\operatorname{len} q \geqslant 1$ and $\operatorname{len} p > \operatorname{len} q$.

Then there exists a finite sequence r of elements of D such that r is one-to-one and $\operatorname{rng} r = \operatorname{rng} p \setminus \operatorname{rng} q$ and $B \odot p = B(B \odot q,\, B \odot r)$.

(19) Let given T. Then T is a T_3 space and a T_1 space and there exists a family sequence of T which is Basis-sigma-locally finite if and only if T is metrizable.

(20) Suppose T is metrizable. Let F_3 be a family of subsets of T. Suppose F_3 is a cover of T and open. Then there exists a family sequence U_1 of T such that $\bigcup U_1$ is open and $\bigcup U_1$ is a cover of T and $\bigcup U_1$ is finer than F_3 and U_1 is sigma-discrete.

(21) For every T such that T is metrizable holds there exists a family sequence of T which is Basis-sigma-discrete.

(22) For every T holds T is a T_3 space and a T_1 space and there exists a family sequence of T which is Basis-sigma-discrete iff T is metrizable.

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