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Differentiable Functions on Normed Linear Spaces. Part II

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Summary. A continuation of [7], the basic properties of the differentiable functions on normed linear spaces are described.

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The terminology and notation used in this paper have been introduced in the following articles: [16], [3], [19], [5], [4], [1], [15], [6], [17], [18], [9], [8], [2], [20], [12], [14], [10], [13], [7], and [11].

For simplicity, we adopt the following rules: S, T denote non trivial real normed spaces, x_0 denotes a point of S, f denotes a partial function from S to T, h denotes a convergent to 0 sequence of S, and c denotes a constant sequence of S.

Let X, Y, Z be real normed spaces, let f be an element of BdLinOps(X, Y), and let g be an element of BdLinOps(Y, Z). The functor $g \cdot f$ yielding an element of BdLinOps(X, Z) is defined by:

(Def. 1) $g \cdot f = \mathrm{modetrans}(g, Y, Z) \cdot \mathrm{modetrans}(f, X, Y)$.

Let X, Y, Z be real normed spaces, let f be a point of RNormSpaceOfBoundedLinearOperators(X, Y), and let g be a point of RNormSpaceOfBoundedLinearOperators(Y, Z). The functor $g \cdot f$ yields a point of RNormSpaceOfBoundedLinearOperators(X, Z) and is defined by:

(Def. 2) $g \cdot f = \mathrm{modetrans}(g, Y, Z) \cdot \mathrm{modetrans}(f, X, Y)$.

Next we state three propositions:

(1) Let x_0 be a point of S. Suppose f is differentiable in x_0. Then there exists a neighbourhood N of x_0 such that

(i) $N \subseteq \mathrm{dom}\, f$, and

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(ii) for every point z of S and for every convergent to 0 sequence h of real numbers and for every c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h \cdot z + c) \subseteq N$ holds $h^{-1} (f \cdot (h \cdot z + c) - f \cdot c)$ is convergent and $f'(x_0)(z) = \lim(h^{-1} (f \cdot (h \cdot z + c) - f \cdot c))$.

(2) Let x_0 be a point of S. Suppose f is differentiable in x_0. Let z be a point of S, h be a convergent to 0 sequence of real numbers, and given c. Suppose $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h \cdot z + c) \subseteq \operatorname{dom} f$. Then $h^{-1} (f \cdot (h \cdot z + c) - f \cdot c)$ is convergent and $f'(x_0)(z) = \lim(h^{-1} (f \cdot (h \cdot z + c) - f \cdot c))$.

(3) Let x_0 be a point of S and N be a neighbourhood of x_0. Suppose $N \subseteq \operatorname{dom} f$. Let z be a point of S and d_1 be a point of T. Then the following statements are equivalent

(i) for every convergent to 0 sequence h of real numbers and for every c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h \cdot z + c) \subseteq N$ holds $h^{-1} (f \cdot (h \cdot z + c) - f \cdot c)$ is convergent and $d_1 = \lim(h^{-1} (f \cdot (h \cdot z + c) - f \cdot c))$,

(ii) for every real number e such that $e > 0$ there exists a real number d such that $d > 0$ and for every real number h such that $|h| < d$ and $h \neq 0$ and $h \cdot z + x_0 \in N$ holds $\|h^{-1} \cdot (f_{h \cdot z + x_0} - f_{x_0}) - d_1\| < e$.

Let us consider S, T, let us consider f, let x_0 be a point of S, and let z be a point of S. We say that f is Gateaux differentiable in x_0, z if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exists a neighbourhood N of x_0 such that

(i) $N \subseteq \operatorname{dom} f$, and

(ii) there exists a point d_1 of T such that for every real number e such that $e > 0$ there exists a real number d such that $d > 0$ and for every real number h such that $|h| < d$ and $h \neq 0$ and $h \cdot z + x_0 \in N$ holds $\|h^{-1} \cdot (f_{h \cdot z + x_0} - f_{x_0}) - d_1\| < e$.

One can prove the following proposition

(4) For every real normed space X and for all points x, y of X holds $\|x - y\| > 0$ iff $x \neq y$ and for every real normed space X and for all points x, y of X holds $\|x - y\| = \|y - x\|$ and for every real normed space X and for all points x, y of X holds $\|x - y\| = 0$ iff $x = y$ and for every real normed space X and for all points x, y of X holds $\|x - y\| \neq 0$ iff $x \neq y$ and for every real normed space X and for all points x, y, z of X and for every real number e such that $e > 0$ holds if $\|x - z\| < \frac{e}{2}$ and $\|z - y\| < \frac{e}{2}$, then $\|x - y\| < e$ and for every real normed space X and for all points x, y, z of X and for every real number e such that $e > 0$ holds if $\|x - z\| < \frac{e}{2}$ and $\|y - z\| < \frac{e}{2}$, then $\|x - y\| < e$ and for every real normed space X and for every point x of X such that for every real number e such that $e > 0$ holds $\|x\| < e$ holds $x = 0_X$ and for every real normed space X and for all points x, y of X such that for every real number e such that $e > 0$ holds $\|x - y\| < e$ holds $x = y$.

Let us consider S, T, let us consider f, let x_0 be a point of S, and let z be a point of S. Let us assume that f is Gateaux differentiable in x_0, z. The functor $\mathrm{GateauxDiff}_z(f, x_0)$ yields a point of T and is defined by the condition (Def. 4).

(Def. 4) There exists a neighbourhood N of x_0 such that

(i) $N \subseteq \mathrm{dom}\, f$, and

(ii) for every real number e such that $e > 0$ there exists a real number d such that $d > 0$ and for every real number h such that $|h| < d$ and $h \neq 0$ and $h \cdot z + x_0 \in N$ holds $\|h^{-1} \cdot (f_{h \cdot z + x_0} - f_{x_0}) - \mathrm{GateauxDiff}_z(f, x_0)\| < e$.

We now state two propositions:

(5) Let x_0 be a point of S and z be a point of S. Then f is Gateaux differentiable in x_0, z if and only if there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and there exists a point d_1 of T such that for every convergent to 0 sequence h of real numbers and for every c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h \cdot z + c) \subseteq N$ holds $h^{-1}\,(f \cdot (h \cdot z + c) - f \cdot c)$ is convergent and $d_1 = \lim(h^{-1}\,(f \cdot (h \cdot z + c) - f \cdot c))$.

(6) Let x_0 be a point of S. Suppose f is differentiable in x_0. Let z be a point of S. Then

(i) f is Gateaux differentiable in x_0, z,

(ii) $\mathrm{GateauxDiff}_z(f, x_0) = f'(x_0)(z)$, and

(iii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and for every convergent to 0 sequence h of real numbers and for every c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h \cdot z + c) \subseteq N$ holds $h^{-1}\,(f \cdot (h \cdot z + c) - f \cdot c)$ is convergent and $\mathrm{GateauxDiff}_z(f, x_0) = \lim(h^{-1}\,(f \cdot (h \cdot z + c) - f \cdot c))$.

In the sequel U is a non trivial real normed space.

Next we state several propositions:

(7) Let R be a rest of S, T. Suppose $R_{0_S} = 0_T$. Let e be a real number. Suppose $e > 0$. Then there exists a real number d such that $d > 0$ and for every point h of S such that $\|h\| < d$ holds $\|R_h\| \leqslant e \cdot \|h\|$.

(8) Let R be a rest of T, U. Suppose $R_{0_T} = 0_U$. Let L be a bounded linear operator from S into T. Then $R \cdot L$ is a rest of S, U.

(9) For every rest R of S, T and for every bounded linear operator L from T into U holds $L \cdot R$ is a rest of S, U.

(10) Let R_1 be a rest of S, T. Suppose $(R_1)_{0_S} = 0_T$. Let R_2 be a rest of T, U. If $(R_2)_{0_T} = 0_U$, then $R_2 \cdot R_1$ is a rest of S, U.

(11) Let R_1 be a rest of S, T. Suppose $(R_1)_{0_S} = 0_T$. Let R_2 be a rest of T, U. Suppose $(R_2)_{0_T} = 0_U$. Let L be a bounded linear operator from S into T. Then $R_2 \cdot (L + R_1)$ is a rest of S, U.

(12) Let R_1 be a rest of S, T. Suppose $(R_1)_{0_S} = 0_T$. Let R_2 be a rest of T, U. Suppose $(R_2)_{0_T} = 0_U$. Let L_1 be a bounded linear operator from S into T and L_2 be a bounded linear operator from T into U. Then

$L_2 \cdot R_1 + R_2 \cdot (L_1 + R_1)$ is a rest of S, U.

(13) Let f_1 be a partial function from S to T. Suppose f_1 is differentiable in x_0. Let f_2 be a partial function from T to U. Suppose f_2 is differentiable in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is differentiable in x_0 and $(f_2 \cdot f_1)'(x_0) = f_2{}'((f_1)_{x_0}) \cdot f_1{}'(x_0)$.

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