FORMALIZED MATHEMATICS
Volume 12, Number 3, 2004
University of Białystok

The Continuous Functions on Normed Linear Spaces

Takaya Nishiyama
Shinshu University
Nagano

Keiji Ohkubo
Shinshu University
Nagano

Yasunari Shidama
Shinshu University
Nagano

Summary. In this article, the basic properties of the continuous function on normed linear spaces are described.

MML Identifier: NFCONT_1.

The articles [16], [19], [20], [2], [21], [4], [9], [3], [1], [11], [15], [5], [17], [18], [10], [7], [8], [6], [13], [22], [12], and [14] provide the notation and terminology for this paper.

We use the following convention: n is a natural number, x, X, X_1 are sets, and s, r, p are real numbers.

Let S, T be 1-sorted structures. A partial function from S to T is a partial function from the carrier of S to the carrier of T.

For simplicity, we adopt the following rules: S, T denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, s_1 denotes a sequence of S, x_0, x_1, x_2 denote points of S, and Y denotes a subset of S.

Let R_1 be a real linear space and let S_1 be a sequence of R_1. The functor $-S_1$ yields a sequence of R_1 and is defined as follows:

(Def. 1) For every n holds $(-S_1)(n) = -S_1(n)$.

Next we state two propositions:

(1) For all sequences s_2, s_3 of S holds $s_2 - s_3 = s_2 + -s_3$.

(2) For every sequence s_4 of S holds $-s_4 = (-1) \cdot s_4$.

Let us consider S, T and let f be a partial function from S to T. The functor $\|f\|$ yielding a partial function from the carrier of S to $\mathbb{R}$ is defined as follows:

(Def. 2) $\mathrm{dom}\|f\| = \mathrm{dom}\, f$ and for every point c of S such that $c \in \mathrm{dom}\|f\|$ holds $\|f\|(c) = \|f_c\|$.

© 2004 University of Białystok
ISSN 1426–2630

Let us consider S, x_0. A subset of S is called a neighbourhood of x_0 if:

(Def. 3) There exists a real number g such that $0 < g$ and $\{y; y$ ranges over points of S: $\|y - x_0\| < g\} \subseteq$ it.

The following two propositions are true:

(3) For every real number g such that $0 < g$ holds $\{y; y$ ranges over points of S: $\|y - x_0\| < g\}$ is a neighbourhood of x_0.

(4) For every neighbourhood N of x_0 holds $x_0 \in N$.

Let us consider S and let X be a subset of S. We say that X is compact if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let s_1 be a sequence of S. Suppose $\operatorname{rng} s_1 \subseteq X$. Then there exists a sequence s_5 of S such that s_5 is a subsequence of s_1 and convergent and $\lim s_5 \in X$.

Let us consider S and let X be a subset of S. We say that X is closed if and only if:

(Def. 5) For every sequence s_1 of S such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent holds $\lim s_1 \in X$.

Let us consider S and let X be a subset of S. We say that X is open if and only if:

(Def. 6) X^{c} is closed.

Let us consider S, T, let us consider f, and let s_4 be a sequence of S. Let us assume that $\operatorname{rng} s_4 \subseteq \operatorname{dom} f$. The functor $f \cdot s_4$ yields a sequence of T and is defined as follows:

(Def. 7) $f \cdot s_4 = (f$ **qua** function$) \cdot (s_4)$.

Let us consider S, let f be a partial function from the carrier of S to $\mathbb{R}$, and let s_4 be a sequence of S. Let us assume that $\operatorname{rng} s_4 \subseteq \operatorname{dom} f$. The functor $f \cdot s_4$ yields a sequence of real numbers and is defined as follows:

(Def. 8) $f \cdot s_4 = (f$ **qua** function$) \cdot (s_4)$.

Let us consider S, T and let us consider f, x_0. We say that f is continuous in x_0 if and only if:

(Def. 9) $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider S, let f be a partial function from the carrier of S to $\mathbb{R}$, and let us consider x_0. We say that f is continuous in x_0 if and only if:

(Def. 10) $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

The scheme *SeqPointNormSpChoice* deals with a non empty normed structure $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

> There exists a sequence s_1 of $\mathcal{A}$ such that for every natural number n holds $\mathcal{P}[n, s_1(n)]$

provided the following condition is met:

- For every natural number n there exists a point r of $\mathcal{A}$ such that $\mathcal{P}[n, r]$.

The following propositions are true:

(5) For every sequence s_4 of S and for every partial function h from S to T such that $\operatorname{rng} s_4 \subseteq \operatorname{dom} h$ holds $s_4(n) \in \operatorname{dom} h$.

(6) For every sequence s_4 of S and for every set x holds $x \in \operatorname{rng} s_4$ iff there exists n such that $x = s_4(n)$.

(7) For all sequences s_4, s_2 of S such that s_2 is a subsequence of s_4 holds $\operatorname{rng} s_2 \subseteq \operatorname{rng} s_4$.

(8) For all f, s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and for every n holds $(f \cdot s_1)(n) = f_{s_1(n)}$.

(9) Let f be a partial function from the carrier of S to $\mathbb{R}$ and given s_1. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$, then for every n holds $(f \cdot s_1)(n) = f_{s_1(n)}$.

(10) Let h be a partial function from S to T, s_4 be a sequence of S, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} s_4 \subseteq \operatorname{dom} h$, then $(h \cdot s_4) \cdot N_1 = h \cdot (s_4 \cdot N_1)$.

(11) Let h be a partial function from the carrier of S to $\mathbb{R}$, s_4 be a sequence of S, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} s_4 \subseteq \operatorname{dom} h$, then $(h \cdot s_4) \cdot N_1 = h \cdot (s_4 \cdot N_1)$.

(12) Let h be a partial function from S to T and s_2, s_3 be sequences of S. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$ and s_3 is a subsequence of s_2, then $h \cdot s_3$ is a subsequence of $h \cdot s_2$.

(13) Let h be a partial function from the carrier of S to $\mathbb{R}$ and s_2, s_3 be sequences of S. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$ and s_3 is a subsequence of s_2, then $h \cdot s_3$ is a subsequence of $h \cdot s_2$.

(14) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(15) Let f be a partial function from the carrier of S to $\mathbb{R}$. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(16) Let given f, x_0. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_2 of f_{x_0} there exists a neighbourhood N of x_0 such that for every x_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_2$.

(17) Let given f, x_0. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_2 of f_{x_0} there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_2$.

(18) If $x_0 \in \operatorname{dom} f$ and there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$, then f is continuous in x_0.

(19) Let h_1, h_2 be partial functions from S to T and s_4 be a sequence of S. If $\operatorname{rng} s_4 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1 + h_2) \cdot s_4 = h_1 \cdot s_4 + h_2 \cdot s_4$ and $(h_1 - h_2) \cdot s_4 = h_1 \cdot s_4 - h_2 \cdot s_4$.

(20) Let h be a partial function from S to T, s_4 be a sequence of S, and r be a real number. If $\operatorname{rng} s_4 \subseteq \operatorname{dom} h$, then $(r\, h) \cdot s_4 = r \cdot (h \cdot s_4)$.

(21) Let h be a partial function from S to T and s_4 be a sequence of S. If $\operatorname{rng} s_4 \subseteq \operatorname{dom} h$, then $\|h \cdot s_4\| = \|h\| \cdot s_4$ and $-h \cdot s_4 = (-h) \cdot s_4$.

(22) If f_1 is continuous in x_0 and f_2 is continuous in x_0, then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0.

(23) If f is continuous in x_0, then $r\, f$ is continuous in x_0.

(24) If f is continuous in x_0, then $\|f\|$ is continuous in x_0 and $-f$ is continuous in x_0.

Let us consider S, T and let us consider f, X. We say that f is continuous on X if and only if:

(Def. 11) $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f{\restriction}X$ is continuous in x_0.

Let us consider S, let f be a partial function from the carrier of S to $\mathbb{R}$, and let us consider X. We say that f is continuous on X if and only if:

(Def. 12) $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f{\restriction}X$ is continuous in x_0.

One can prove the following propositions:

(25) Let given X, f. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 \in X$ holds $f \cdot s_1$ is convergent and $f_{\lim s_1} = \lim(f \cdot s_1)$.

(26) f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(27) Let f be a partial function from the carrier of S to $\mathbb{R}$. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(28) f is continuous on X iff $f{\restriction}X$ is continuous on X.

(29) Let f be a partial function from the carrier of S to $\mathbb{R}$. Then f is continuous on X if and only if $f{\restriction}X$ is continuous on X.

(30) If f is continuous on X and $X_1 \subseteq X$, then f is continuous on X_1.

(31) If $x_0 \in \operatorname{dom} f$, then f is continuous on $\{x_0\}$.

(32) For all X, f_1, f_2 such that f_1 is continuous on X and f_2 is continuous on X holds $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X.

(33) Let given X, X_1, f_1, f_2. Suppose f_1 is continuous on X and f_2 is continuous on X_1. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

(34) For all r, X, f such that f is continuous on X holds $r\,f$ is continuous on X.

(35) If f is continuous on X, then $\|f\|$ is continuous on X and $-f$ is continuous on X.

(36) Suppose f is total and for all x_1, x_2 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists x_0 such that f is continuous in x_0. Then f is continuous on the carrier of S.

(37) For every f such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds $\operatorname{rng} f$ is compact.

(38) Let f be a partial function from the carrier of S to $\mathbb{R}$. If $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$, then $\operatorname{rng} f$ is compact.

(39) If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y, then $f^{\circ}Y$ is compact.

(40) Let f be a partial function from the carrier of S to $\mathbb{R}$. Suppose $\operatorname{dom} f \neq \emptyset$ and $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$. Then there exist x_1, x_2 such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $f_{x_1} = \sup \operatorname{rng} f$ and $f_{x_2} = \inf \operatorname{rng} f$.

(41) Let given f. Suppose $\operatorname{dom} f \neq \emptyset$ and $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$. Then there exist x_1, x_2 such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $\|f\|_{x_1} = \sup \operatorname{rng}\|f\|$ and $\|f\|_{x_2} = \inf \operatorname{rng}\|f\|$.

(42) $\|f\|{\restriction}X = \|f{\restriction}X\|$.

(43) Let given f, Y. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y. Then there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $\|f\|_{x_1} = \sup(\|f\|^{\circ}Y)$ and $\|f\|_{x_2} = \inf(\|f\|^{\circ}Y)$.

(44) Let f be a partial function from the carrier of S to $\mathbb{R}$ and given Y. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y. Then there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $f_{x_1} = \sup(f^{\circ}Y)$

and $f_{x_2} = \inf(f^{\circ}Y)$.

Let us consider S, T and let us consider X, f. We say that f is Lipschitzian on X if and only if:

(Def. 13) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ holds $\|f_{x_1} - f_{x_2}\| \leqslant r \cdot \|x_1 - x_2\|$.

Let us consider S, let us consider X, and let f be a partial function from the carrier of S to $\mathbb{R}$. We say that f is Lipschitzian on X if and only if:

(Def. 14) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ holds $|f_{x_1} - f_{x_2}| \leqslant r \cdot \|x_1 - x_2\|$.

The following propositions are true:

(45) If f is Lipschitzian on X and $X_1 \subseteq X$, then f is Lipschitzian on X_1.

(46) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1, then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(47) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1, then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(48) If f is Lipschitzian on X, then $p\,f$ is Lipschitzian on X.

(49) If f is Lipschitzian on X, then $-f$ is Lipschitzian on X and $\|f\|$ is Lipschitzian on X.

(50) If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is Lipschitzian on X.

(51) id_Y is Lipschitzian on Y.

(52) If f is Lipschitzian on X, then f is continuous on X.

(53) Let f be a partial function from the carrier of S to $\mathbb{R}$. If f is Lipschitzian on X, then f is continuous on X.

(54) For every f such that there exists a point r of T such that $\operatorname{rng} f = \{r\}$ holds f is continuous on $\operatorname{dom} f$.

(55) If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is continuous on X.

(56) For every partial function f from S to S such that for every x_0 such that $x_0 \in \operatorname{dom} f$ holds $f_{x_0} = x_0$ holds f is continuous on $\operatorname{dom} f$.

(57) For every partial function f from S to S such that $f = \mathrm{id}_{\operatorname{dom} f}$ holds f is continuous on $\operatorname{dom} f$.

(58) For every partial function f from S to S such that $Y \subseteq \operatorname{dom} f$ and $f{\restriction}Y = \mathrm{id}_Y$ holds f is continuous on Y.

(59) Let f be a partial function from S to S, r be a real number, and p be a point of S. Suppose $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f_{x_0} = r \cdot x_0 + p$. Then f is continuous on X.

(60) Let f be a partial function from the carrier of S to $\mathbb{R}$. If for every x_0 such that $x_0 \in \operatorname{dom} f$ holds $f_{x_0} = \|x_0\|$, then f is continuous on $\operatorname{dom} f$.

(61) Let f be a partial function from the carrier of S to $\mathbb{R}$. If $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f_{x_0} = \|x_0\|$, then f is continuous on X.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[6] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[7] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[8] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[9] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.
[10] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[11] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[12] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[13] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[14] Yasunari Shidama. The series on Banach algebra. *Formalized Mathematics*, 12(**2**):131–138, 2004.
[15] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[18] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[21] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[22] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(**2**):171–175, 1992.

Received April 6, 2004