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Solving Roots of the Special Polynomial Equation with Real Coefficients

Yuzhong Ding
QingDao University of Science and Technology

Xiquan Liang
QingDao University of Science and Technology

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The papers [5], [4], [2], [3], and [1] provide the terminology and notation for this paper.

We follow the rules: x, y, a, b, c, p, q are real numbers and m, n are natural numbers.

We now state a number of propositions:

(1) If $a \neq 0$ and $\frac{b}{a} < 0$ and $\frac{c}{a} > 0$ and $\Delta(a,b,c) \geqslant 0$, then $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} > 0$ and $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} > 0$.

(2) If $a \neq 0$ and $\frac{b}{a} > 0$ and $\frac{c}{a} > 0$ and $\Delta(a,b,c) \geqslant 0$, then $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} < 0$ and $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} < 0$.

(3) If $a \neq 0$ and $\frac{c}{a} < 0$, then $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} > 0$ and $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} < 0$ or $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} < 0$ and $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} > 0$.

(4) If $a > 0$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $x^n = a$, then $x = \sqrt[n]{a}$ or $x = -\sqrt[n]{a}$.

(5) If $a \neq 0$ and $\mathrm{Poly2}(a,b,0,x) = 0$, then $x = 0$ or $x = -\frac{b}{a}$.

(6) If $a \neq 0$ and $\mathrm{Poly2}(a,0,0,x) = 0$, then $x = 0$.

(7) If $a \neq 0$ and there exists m such that $n = 2 \cdot m + 1$ and $\Delta(a,b,c) \geqslant 0$ and $\mathrm{Poly2}(a,b,c,x^n) = 0$, then $x = \sqrt[n]{\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}}$ or $x = \sqrt[n]{\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}}$.

(8) Suppose $a \neq 0$ and $\frac{b}{a} < 0$ and $\frac{c}{a} > 0$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $\Delta(a,b,c) \geqslant 0$ and $\mathrm{Poly2}(a,b,c,x^n) = 0$. Then

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$x = \sqrt[n]{\frac{-b+\sqrt{\Delta(a,b,c)}}{2\cdot a}}$ or $x = -\sqrt[n]{\frac{-b+\sqrt{\Delta(a,b,c)}}{2\cdot a}}$ or $x = \sqrt[n]{\frac{-b-\sqrt{\Delta(a,b,c)}}{2\cdot a}}$ or $x = -\sqrt[n]{\frac{-b-\sqrt{\Delta(a,b,c)}}{2\cdot a}}$.

(9) If $a \neq 0$ and there exists m such that $n = 2\cdot m+1$ and $\mathrm{Poly2}(a, b, 0, x^n) = 0$, then $x = 0$ or $x = \sqrt[n]{-\frac{b}{a}}$.

(10) If $a \neq 0$ and $\frac{b}{a} < 0$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $\mathrm{Poly2}(a, b, 0, x^n) = 0$, then $x = 0$ or $x = \sqrt[n]{-\frac{b}{a}}$ or $x = -\sqrt[n]{-\frac{b}{a}}$.

(11) $a^3 + b^3 = (a + b) \cdot ((a^2 - a \cdot b) + b^2)$ and $a^5 + b^5 = (a + b) \cdot ((((a^4 - a^3 \cdot b) + a^2 \cdot b^2) - a \cdot b^3) + b^4)$.

(12) Suppose $a \neq 0$ and $b^2 - 2 \cdot a \cdot b - 3 \cdot a^2 \geqslant 0$ and $\mathrm{Poly3}(a, b, b, a, x) = 0$. Then $x = -1$ or $x = \frac{(a-b)+\sqrt{b^2-2\cdot a\cdot b-3\cdot a^2}}{2\cdot a}$ or $x = \frac{a-b-\sqrt{b^2-2\cdot a\cdot b-3\cdot a^2}}{2\cdot a}$.

Let a, b, c, d, e, f, x be real numbers. The functor $\mathrm{Poly}_5(a, b, c, d, e, f, x)$ is defined by:

(Def. 1) $\mathrm{Poly}_5(a, b, c, d, e, f, x) = a \cdot x^5 + b \cdot x^4 + c \cdot x^3 + d \cdot x^2 + e \cdot x + f$.

We now state a number of propositions:

(13) Suppose $a \neq 0$ and $(b^2 + 2 \cdot a \cdot b + 5 \cdot a^2) - 4 \cdot a \cdot c > 0$ and $\mathrm{Poly}_5(a, b, c, c, b, a, x) = 0$. Let y_1, y_2 be real numbers. Suppose $y_1 = \frac{(a-b)+\sqrt{(b^2+2\cdot a\cdot b+5\cdot a^2)-4\cdot a\cdot c}}{2\cdot a}$ and $y_2 = \frac{a-b-\sqrt{(b^2+2\cdot a\cdot b+5\cdot a^2)-4\cdot a\cdot c}}{2\cdot a}$. Then $x = -1$ or $x = \frac{y_1+\sqrt{\Delta(1,-y_1,1)}}{2}$ or $x = \frac{y_2+\sqrt{\Delta(1,-y_2,1)}}{2}$ or $x = \frac{y_1-\sqrt{\Delta(1,-y_1,1)}}{2}$ or $x = \frac{y_2-\sqrt{\Delta(1,-y_2,1)}}{2}$.

(14) Suppose $x + y = p$ and $x \cdot y = q$ and $p^2 - 4 \cdot q \geqslant 0$. Then $x = \frac{p+\sqrt{p^2-4\cdot q}}{2}$ and $y = \frac{p-\sqrt{p^2-4\cdot q}}{2}$ or $x = \frac{p-\sqrt{p^2-4\cdot q}}{2}$ and $y = \frac{p+\sqrt{p^2-4\cdot q}}{2}$.

(15) Suppose $x^n + y^n = p$ and $x^n \cdot y^n = q$ and $p^2 - 4 \cdot q \geqslant 0$ and there exists m such that $n = 2 \cdot m + 1$. Then $x = \sqrt[n]{\frac{p+\sqrt{p^2-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p-\sqrt{p^2-4\cdot q}}{2}}$ or $x = \sqrt[n]{\frac{p-\sqrt{p^2-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p+\sqrt{p^2-4\cdot q}}{2}}$.

(16) Suppose $x^n + y^n = p$ and $x^n \cdot y^n = q$ and $p^2 - 4 \cdot q \geqslant 0$ and $p > 0$ and $q > 0$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$. Then $x = \sqrt[n]{\frac{p+\sqrt{p^2-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p-\sqrt{p^2-4\cdot q}}{2}}$ or $x = -\sqrt[n]{\frac{p+\sqrt{p^2-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p-\sqrt{p^2-4\cdot q}}{2}}$ or $x = \sqrt[n]{\frac{p+\sqrt{p^2-4\cdot q}}{2}}$ and $y = -\sqrt[n]{\frac{p-\sqrt{p^2-4\cdot q}}{2}}$ or $x = -\sqrt[n]{\frac{p+\sqrt{p^2-4\cdot q}}{2}}$ and $y = -\sqrt[n]{\frac{p-\sqrt{p^2-4\cdot q}}{2}}$ or $x = \sqrt[n]{\frac{p-\sqrt{p^2-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p+\sqrt{p^2-4\cdot q}}{2}}$ or $x = -\sqrt[n]{\frac{p-\sqrt{p^2-4\cdot q}}{2}}$ and $y = \sqrt[n]{\frac{p+\sqrt{p^2-4\cdot q}}{2}}$ or $x = \sqrt[n]{\frac{p-\sqrt{p^2-4\cdot q}}{2}}$ and $y = -\sqrt[n]{\frac{p+\sqrt{p^2-4\cdot q}}{2}}$ or $x = -\sqrt[n]{\frac{p-\sqrt{p^2-4\cdot q}}{2}}$ and $y = -\sqrt[n]{\frac{p+\sqrt{p^2-4\cdot q}}{2}}$.

(18)[1] Suppose $x^n + y^n = a$ and $x^n - y^n = b$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $a > 0$ and $a + b > 0$ and $a - b > 0$. Then

(i) $x = \sqrt[n]{\frac{a+b}{2}}$ and $y = \sqrt[n]{\frac{a-b}{2}}$, or

(ii) $x = \sqrt[n]{\frac{a+b}{2}}$ and $y = -\sqrt[n]{\frac{a-b}{2}}$, or

(iii) $x = -\sqrt[n]{\frac{a+b}{2}}$ and $y = \sqrt[n]{\frac{a-b}{2}}$, or

(iv) $x = -\sqrt[n]{\frac{a+b}{2}}$ and $y = -\sqrt[n]{\frac{a-b}{2}}$.

(19) If $a \cdot x^n + b \cdot y^n = p$ and $x \cdot y = 0$ and there exists m such that $n = 2 \cdot m + 1$ and $a \cdot b \neq 0$, then $x = 0$ and $y = \sqrt[n]{\frac{p}{b}}$ or $x = \sqrt[n]{\frac{p}{a}}$ and $y = 0$.

(20) Suppose $a \cdot x^n + b \cdot y^n = p$ and $x \cdot y = 0$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $\frac{p}{b} > 0$ and $\frac{p}{a} > 0$ and $a \cdot b \neq 0$. Then $x = 0$ and $y = \sqrt[n]{\frac{p}{b}}$ or $x = 0$ and $y = -\sqrt[n]{\frac{p}{b}}$ or $x = \sqrt[n]{\frac{p}{a}}$ and $y = 0$ or $x = -\sqrt[n]{\frac{p}{a}}$ and $y = 0$.

(21) If $a \cdot x^n = p$ and $x \cdot y = q$ and there exists m such that $n = 2 \cdot m + 1$ and $p \cdot a \neq 0$, then $x = \sqrt[n]{\frac{p}{a}}$ and $y = q \cdot \sqrt[n]{\frac{a}{p}}$.

(22) Suppose $a \cdot x^n = p$ and $x \cdot y = q$ and there exists m such that $n = 2 \cdot m$ and $m \geqslant 1$ and $\frac{p}{a} > 0$ and $a \neq 0$. Then $x = \sqrt[n]{\frac{p}{a}}$ and $y = q \cdot \sqrt[n]{\frac{a}{p}}$ or $x = -\sqrt[n]{\frac{p}{a}}$ and $y = -q \cdot \sqrt[n]{\frac{a}{p}}$.

(24)[2] For all real numbers a, x such that $a > 0$ and $a \neq 1$ and $a^x = 1$ holds $x = 0$.

(25) For all real numbers a, x such that $a > 0$ and $a \neq 1$ and $a^x = a$ holds $x = 1$.

(27)[3] For all real numbers a, b, x such that $a > 0$ and $a \neq 1$ and $x > 0$ and $\log_a x = 0$ holds $x = 1$.

(28) For all real numbers a, b, x such that $a > 0$ and $a \neq 1$ and $x > 0$ and $\log_a x = 1$ holds $x = a$.

References

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[1] The proposition (17) has been removed.

[2] The proposition (23) has been removed.

[3] The proposition (26) has been removed.

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