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Intersections of Intervals and Balls in $\mathcal{E}^n_{\rm T}$

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The terminology and notation used in this paper are introduced in the following papers: [17], [19], [1], [4], [16], [8], [14], [2], [3], [5], [18], [13], [7], [9], [6], [15], [11], [12], and [10].

1. Preliminaries

For simplicity, we follow the rules: n denotes a natural number, a, b, r denote real numbers, x, y, z denote points of $\mathcal{E}^n_{\rm T}$, and e denotes a point of $\mathcal{E}^n$.

The following propositions are true:

(1) $x - y - z = x - z - y$.

(2) If $x + y = x + z$, then $y = z$.

(3) If n is non empty, then $x \neq x + 1.{\rm REAL}\, n$.

(4) For every set x such that $x = (1 - r) \cdot y + r \cdot z$ holds $x = y$ iff $r = 0$ or $y = z$ and $x = z$ iff $r = 1$ or $y = z$.

(5) For every finite sequence f of elements of $\mathbb{R}$ holds $|f|^{\mathbf{2}} = \sum^2 f$.

(6) For every non empty metric space M and for all points z_1, z_2, z_3 of M such that $z_1 \neq z_2$ and $z_1 \in \overline{\rm Ball}(z_3, r)$ and $z_2 \in \overline{\rm Ball}(z_3, r)$ holds $r > 0$.

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2. Subsets of $\mathcal{E}^n_{\mathrm{T}}$

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a real number. The functor $\mathrm{Ball}(x,r)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 1) $\mathrm{Ball}(x,r) = \{p; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|p-x| < r\}$.

The functor $\overline{\mathrm{Ball}}(x,r)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 2) $\overline{\mathrm{Ball}}(x,r) = \{p; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|p-x| \leqslant r\}$.

The functor $\mathrm{Sphere}(x,r)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 3) $\mathrm{Sphere}(x,r) = \{p; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|p-x| = r\}$.

We now state a number of propositions:

(7) $y \in \mathrm{Ball}(x,r)$ iff $|y-x| < r$.

(8) $y \in \overline{\mathrm{Ball}}(x,r)$ iff $|y-x| \leqslant r$.

(9) $y \in \mathrm{Sphere}(x,r)$ iff $|y-x| = r$.

(10) If $y \in \mathrm{Ball}(0_{\mathcal{E}^n_{\mathrm{T}}}, r)$, then $|y| < r$.

(11) If $y \in \overline{\mathrm{Ball}}(0_{\mathcal{E}^n_{\mathrm{T}}}, r)$, then $|y| \leqslant r$.

(12) If $y \in \mathrm{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), r)$, then $|y| = r$.

(13) If $x = e$, then $\mathrm{Ball}(e,r) = \mathrm{Ball}(x,r)$.

(14) If $x = e$, then $\overline{\mathrm{Ball}}(e,r) = \overline{\mathrm{Ball}}(x,r)$.

(15) If $x = e$, then $\mathrm{Sphere}(e,r) = \mathrm{Sphere}(x,r)$.

(16) $\mathrm{Ball}(x,r) \subseteq \overline{\mathrm{Ball}}(x,r)$.

(17) $\mathrm{Sphere}(x,r) \subseteq \overline{\mathrm{Ball}}(x,r)$.

(18) $\mathrm{Ball}(x,r) \cup \mathrm{Sphere}(x,r) = \overline{\mathrm{Ball}}(x,r)$.

(19) $\mathrm{Ball}(x,r)$ misses $\mathrm{Sphere}(x,r)$.

Let us consider n, x and let r be a non positive real number. One can check that $\mathrm{Ball}(x,r)$ is empty.

Let us consider n, x and let r be a positive real number. Note that $\mathrm{Ball}(x,r)$ is non empty.

One can prove the following propositions:

(20) If $\mathrm{Ball}(x,r)$ is non empty, then $r > 0$.

(21) If $\mathrm{Ball}(x,r)$ is empty, then $r \leqslant 0$.

Let us consider n, x and let r be a negative real number. Observe that $\overline{\mathrm{Ball}}(x,r)$ is empty.

Let us consider n, x and let r be a non negative real number. Observe that $\overline{\mathrm{Ball}}(x,r)$ is non empty.

The following three propositions are true:

(22) If $\overline{\mathrm{Ball}}(x,r)$ is non empty, then $r \geqslant 0$.

(23) If $\overline{\mathrm{Ball}}(x,r)$ is empty, then $r < 0$.

(24) If $a + b = 1$ and $|a| + |b| = 1$ and $b \neq 0$ and $x \in \overline{\mathrm{Ball}}(z, r)$ and $y \in \mathrm{Ball}(z, r)$, then $a \cdot x + b \cdot y \in \mathrm{Ball}(z, r)$.

Let us consider n, x, r. One can check the following observations:

* $\mathrm{Ball}(x, r)$ is open and Bounded,
* $\overline{\mathrm{Ball}}(x, r)$ is closed and Bounded, and
* $\mathrm{Sphere}(x, r)$ is closed and Bounded.

Let us consider n, x, r. Observe that $\mathrm{Ball}(x, r)$ is convex and $\overline{\mathrm{Ball}}(x, r)$ is convex.

Let n be a natural number and let f be a map from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. We say that f is homogeneous if and only if:

(Def. 4) For every real number r and for every point x of $\mathcal{E}^n_{\mathrm{T}}$ holds $f(r \cdot x) = r \cdot f(x)$.

We say that f is additive if and only if:

(Def. 5) For all points x, y of $\mathcal{E}^n_{\mathrm{T}}$ holds $f(x + y) = f(x) + f(y)$.

Let us consider n. One can verify that $(\mathcal{E}^n_{\mathrm{T}}) \longmapsto 0_{\mathcal{E}^n_{\mathrm{T}}}$ is homogeneous and additive.

Let us consider n. Observe that there exists a map from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$ which is homogeneous, additive, and continuous.

Let a, c be real numbers. One can check that $\mathrm{AffineMap}(a, 0, c, 0)$ is homogeneous and additive.

One can prove the following proposition

(25) For every homogeneous additive map f from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$ and for every convex subset X of $\mathcal{E}^n_{\mathrm{T}}$ holds $f^{\circ}X$ is convex.

In the sequel p, q are points of $\mathcal{E}^n_{\mathrm{T}}$.

Let n be a natural number and let p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{HL}(p, q)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 6) $\mathrm{HL}(p, q) = \{(1 - l) \cdot p + l \cdot q; l$ ranges over real numbers: $0 \leqslant l\}$.

One can prove the following proposition

(26) For every set x holds $x \in \mathrm{HL}(p, q)$ iff there exists a real number l such that $x = (1 - l) \cdot p + l \cdot q$ and $0 \leqslant l$.

Let us consider n, p, q. One can verify that $\mathrm{HL}(p, q)$ is non empty.

The following propositions are true:

(27) $p \in \mathrm{HL}(p, q)$.

(28) $q \in \mathrm{HL}(p, q)$.

(29) $\mathrm{HL}(p, p) = \{p\}$.

(30) If $x \in \mathrm{HL}(p, q)$, then $\mathrm{HL}(p, x) \subseteq \mathrm{HL}(p, q)$.

(31) If $x \in \mathrm{HL}(p, q)$ and $x \neq p$, then $\mathrm{HL}(p, q) = \mathrm{HL}(p, x)$.

(32) $\mathcal{L}(p, q) \subseteq \mathrm{HL}(p, q)$.

Let us consider n, p, q. Note that $\mathrm{HL}(p, q)$ is convex.

One can prove the following propositions:

(33) If $y \in \mathrm{Sphere}(x, r)$ and $z \in \mathrm{Ball}(x, r)$, then $\mathcal{L}(y, z) \cap \mathrm{Sphere}(x, r) = \{y\}$.

(34) If $y \in \mathrm{Sphere}(x, r)$ and $z \in \mathrm{Sphere}(x, r)$, then $\mathcal{L}(y, z) \setminus \{y, z\} \subseteq \mathrm{Ball}(x, r)$.

(35) If $y \in \mathrm{Sphere}(x, r)$ and $z \in \mathrm{Sphere}(x, r)$, then $\mathcal{L}(y, z) \cap \mathrm{Sphere}(x, r) = \{y, z\}$.

(36) If $y \in \mathrm{Sphere}(x, r)$ and $z \in \mathrm{Sphere}(x, r)$, then $\mathrm{HL}(y, z) \cap \mathrm{Sphere}(x, r) = \{y, z\}$.

(37) If $y \neq z$ and $y \in \mathrm{Ball}(x, r)$, then there exists a point e of $\mathcal{E}^n_{\mathrm{T}}$ such that $\{e\} = \mathrm{HL}(y, z) \cap \mathrm{Sphere}(x, r)$.

(38) If $y \neq z$ and $y \in \mathrm{Sphere}(x, r)$ and $z \in \overline{\mathrm{Ball}}(x, r)$, then there exists a point e of $\mathcal{E}^n_{\mathrm{T}}$ such that $e \neq y$ and $\{y, e\} = \mathrm{HL}(y, z) \cap \mathrm{Sphere}(x, r)$.

Let us consider n, x and let r be a negative real number. Observe that $\mathrm{Sphere}(x, r)$ is empty.

Let n be a non empty natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a non negative real number. Observe that $\mathrm{Sphere}(x, r)$ is non empty.

Next we state two propositions:

(39) If $\mathrm{Sphere}(x, r)$ is non empty, then $r \geqslant 0$.

(40) If n is non empty and $\mathrm{Sphere}(x, r)$ is empty, then $r < 0$.

3. Subsets of $\mathcal{E}^2_{\mathrm{T}}$

In the sequel s, t are points of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(41) $(a \cdot s + b \cdot t)_{\mathbf{1}} = a \cdot s_{\mathbf{1}} + b \cdot t_{\mathbf{1}}$.

(42) $(a \cdot s + b \cdot t)_{\mathbf{2}} = a \cdot s_{\mathbf{2}} + b \cdot t_{\mathbf{2}}$.

(43) $t \in \mathrm{Circle}(a, b, r)$ iff $|t - [a, b]| = r$.

(44) $t \in \mathrm{ClosedInsideOfCircle}(a, b, r)$ iff $|t - [a, b]| \leqslant r$.

(45) $t \in \mathrm{InsideOfCircle}(a, b, r)$ iff $|t - [a, b]| < r$.

Let a, b be real numbers and let r be a positive real number. Observe that $\mathrm{InsideOfCircle}(a, b, r)$ is non empty.

Let a, b be real numbers and let r be a non negative real number. Observe that $\mathrm{ClosedInsideOfCircle}(a, b, r)$ is non empty.

We now state a number of propositions:

(46) $\mathrm{Circle}(a, b, r) \subseteq \mathrm{ClosedInsideOfCircle}(a, b, r)$.

(47) For every point x of $\mathcal{E}^2$ such that $x = [a, b]$ holds $\overline{\mathrm{Ball}}(x, r) = \mathrm{ClosedInsideOfCircle}(a, b, r)$.

(48) For every point x of $\mathcal{E}^2$ such that $x = [a, b]$ holds $\mathrm{Ball}(x, r) = \mathrm{InsideOfCircle}(a, b, r)$.

(49) For every point x of $\mathcal{E}^2$ such that $x = [a, b]$ holds $\mathrm{Sphere}(x, r) = \mathrm{Circle}(a, b, r)$.

(50) $\mathrm{Ball}([a,b],r) = \mathrm{InsideOfCircle}(a,b,r)$.

(51) $\overline{\mathrm{Ball}}([a,b],r) = \mathrm{ClosedInsideOfCircle}(a,b,r)$.

(52) $\mathrm{Sphere}([a,b],r) = \mathrm{Circle}(a,b,r)$.

(53) $\mathrm{InsideOfCircle}(a,b,r) \subseteq \mathrm{ClosedInsideOfCircle}(a,b,r)$.

(54) $\mathrm{InsideOfCircle}(a,b,r)$ misses $\mathrm{Circle}(a,b,r)$.

(55) $\mathrm{InsideOfCircle}(a,b,r) \cup \mathrm{Circle}(a,b,r) = \mathrm{ClosedInsideOfCircle}(a,b,r)$.

(56) If $s \in \mathrm{Sphere}((0_{\mathcal{E}^2_{\mathrm{T}}}),r)$, then $(s_{\mathbf{1}})^{\mathbf{2}} + (s_{\mathbf{2}})^{\mathbf{2}} = r^{\mathbf{2}}$.

(57) If $s \neq t$ and $s \in \mathrm{ClosedInsideOfCircle}(a,b,r)$ and $t \in \mathrm{ClosedInsideOfCircle}(a,b,r)$, then $r > 0$.

(58) If $s \neq t$ and $s \in \mathrm{InsideOfCircle}(a,b,r)$, then there exists a point e of $\mathcal{E}^2_{\mathrm{T}}$ such that $\{e\} = \mathrm{HL}(s,t) \cap \mathrm{Circle}(a,b,r)$.

(59) If $s \in \mathrm{Circle}(a,b,r)$ and $t \in \mathrm{InsideOfCircle}(a,b,r)$, then $\mathcal{L}(s,t) \cap \mathrm{Circle}(a,b,r) = \{s\}$.

(60) If $s \in \mathrm{Circle}(a,b,r)$ and $t \in \mathrm{Circle}(a,b,r)$, then $\mathcal{L}(s,t) \setminus \{s,t\} \subseteq \mathrm{InsideOfCircle}(a,b,r)$.

(61) If $s \in \mathrm{Circle}(a,b,r)$ and $t \in \mathrm{Circle}(a,b,r)$, then $\mathcal{L}(s,t) \cap \mathrm{Circle}(a,b,r) = \{s,t\}$.

(62) If $s \in \mathrm{Circle}(a,b,r)$ and $t \in \mathrm{Circle}(a,b,r)$, then $\mathrm{HL}(s,t) \cap \mathrm{Circle}(a,b,r) = \{s,t\}$.

(63) If $s \neq t$ and $s \in \mathrm{Circle}(a,b,r)$ and $t \in \mathrm{ClosedInsideOfCircle}(a,b,r)$, then there exists a point e of $\mathcal{E}^2_{\mathrm{T}}$ such that $e \neq s$ and $\{s,e\} = \mathrm{HL}(s,t) \cap \mathrm{Circle}(a,b,r)$.

Let a, b, r be real numbers. Observe that $\mathrm{InsideOfCircle}(a,b,r)$ is convex and $\mathrm{ClosedInsideOfCircle}(a,b,r)$ is convex.

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