FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Partial Sum of Some Series

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Summary. Solving the partial sum of some often used series.

MML Identifier: `SERIES_2`.

The articles [2], [1], [4], [3], [5], [7], and [6] provide the notation and terminology for this paper.

In this paper n is a natural number and s is a sequence of real numbers.

Next we state a number of propositions:

(1) $|(-1)^n| = 1$.

(2) $(n+1)^3 = n^3 + 3 \cdot n^2 + 3 \cdot n + 1$ and $(n+1)^4 = n^4 + 4 \cdot n^3 + 6 \cdot n^2 + 4 \cdot n + 1$ and $(n+1)^5 = n^5 + 5 \cdot n^4 + 10 \cdot n^3 + 10 \cdot n^2 + 5 \cdot n + 1$.

(3) If for every n holds $s(n) = n$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) = \frac{n \cdot (n+1)}{2}$.

(4) If for every n holds $s(n) = 2 \cdot n$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) = n \cdot (n+1)$.

(5) If for every n holds $s(n) = 2 \cdot n + 1$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) = (n+1)^2$.

(6) If for every n holds $s(n) = n \cdot (n+1)$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) = \frac{n \cdot (n+1) \cdot (n+2)}{3}$.

(7) If for every n holds $s(n) = n \cdot (n+1) \cdot (n+2)$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) = \frac{n \cdot (n+1) \cdot (n+2) \cdot (n+3)}{4}$.

(8) If for every n holds $s(n) = n \cdot (n+1) \cdot (n+2) \cdot (n+3)$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) = \frac{n \cdot (n+1) \cdot (n+2) \cdot (n+3) \cdot (n+4)}{5}$.

(9) If for every n holds $s(n) = \frac{1}{n \cdot (n+1)}$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) = 1 - \frac{1}{n+1}$.

ISSN 1426–2630

(10) If for every n holds $s(n) = \frac{1}{n\cdot(n+1)\cdot(n+2)}$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1}{4} - \frac{1}{2\cdot(n+1)\cdot(n+2)}$.
(11) If for every n holds $s(n) = \frac{1}{n\cdot(n+1)\cdot(n+2)\cdot(n+3)}$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1}{18} - \frac{1}{3\cdot(n+1)\cdot(n+2)\cdot(n+3)}$.
(12) If for every n holds $s(n) = n^2$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}$.
(13) If for every n holds $s(n) = (-1)^{n+1}\cdot n^2$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{(-1)^{n+1}\cdot n\cdot(n+1)}{2}$.
(14) If for every n such that $n \geq 1$ holds $s(n) = (2\cdot n - 1)^2$ and $s(0) = 0$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{n\cdot(4\cdot n^2-1)}{3}$.
(15) If for every n holds $s(n) = n^3$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{n^2\cdot(n+1)^2}{4}$.
(16) If for every n such that $n \geq 1$ holds $s(n) = (2\cdot n-1)^3$ and $s(0) = 0$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = n^2\cdot(2\cdot n^2 - 1)$.
(17) If for every n holds $s(n) = n^4$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{n\cdot(n+1)\cdot(2\cdot n+1)\cdot((3\cdot n^2+3\cdot n)-1)}{30}$.
(18) If for every n holds $s(n) = (-1)^{n+1}\cdot n^4$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{(-1)^{n+1}\cdot n\cdot(n+1)\cdot((n^2+n)-1)}{2}$.
(19) If for every n holds $s(n) = n^5$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{n^2\cdot(n+1)^2\cdot((2\cdot n^2+2\cdot n)-1)}{12}$.
(20) If for every n holds $s(n) = n^6$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{n\cdot(n+1)\cdot(2\cdot n+1)\cdot(((3\cdot n^4+6\cdot n^3)-3\cdot n)+1)}{42}$.
(21) If for every n holds $s(n) = n^7$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{n^2\cdot(n+1)^2\cdot(((3\cdot n^4+6\cdot n^3)-n^2-4\cdot n)+2)}{24}$.
(22) If for every n holds $s(n) = n\cdot(n+1)^2$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{n\cdot(n+1)\cdot(n+2)\cdot(3\cdot n+5)}{12}$.
(23) If for every n holds $s(n) = n\cdot(n+1)^2\cdot(n+2)$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{n\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(2\cdot n+3)}{10}$.
(24) If for every n holds $s(n) = n\cdot(n+1)\cdot 2^n$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = 2^{n+1}\cdot((n^2-n)+2) - 4$.
(25) Suppose that for every n such that $n \geq 2$ holds $s(n) = \frac{1}{(n-1)\cdot(n+1)}$ and $s(0) = 0$ and $s(1) = 0$. Let given n. If $n \geq 2$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{3}{4} - \frac{1}{2\cdot n} - \frac{1}{2\cdot(n+1)}$.
(26) If for every n such that $n \geq 1$ holds $s(n) = \frac{1}{(2\cdot n-1)\cdot(2\cdot n+1)}$ and $s(0) = 0$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{n}{2\cdot n+1}$.
(27) If for every n such that $n \geq 1$ holds $s(n) = \frac{1}{(3\cdot n-2)\cdot(3\cdot n+1)}$ and $s(0) = 0$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{n}{3\cdot n+1}$.

(28) Suppose that for every n such that $n \geq 1$ holds $s(n) = \frac{1}{(2\cdot n-1)\cdot(2\cdot n+1)\cdot(2\cdot n+3)}$ and $s(0) = 0$. Let given n. If $n \geq 1$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1}{12} - \frac{1}{4\cdot(2\cdot n+1)\cdot(2\cdot n+3)}$.

(29) Suppose that for every n such that $n \geq 1$ holds $s(n) = \frac{1}{(3\cdot n-2)\cdot(3\cdot n+1)\cdot(3\cdot n+4)}$ and $s(0) = 0$. Let given n. If $n \geq 1$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1}{24} - \frac{1}{6\cdot(3\cdot n+1)\cdot(3\cdot n+4)}$.

(30) Suppose that for every n such that $n \geq 1$ holds $s(n) = \frac{2\cdot n-1}{n\cdot(n+1)\cdot(n+2)}$ and $s(0) = 0$. Let given n. If $n \geq 1$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = (\frac{3}{4} - \frac{2}{n+2}) + \frac{1}{2\cdot(n+1)\cdot(n+2)}$.

(31) Suppose that for every n such that $n \geq 1$ holds $s(n) = \frac{n+2}{n\cdot(n+1)\cdot(n+3)}$ and $s(0) = 0$. Let given n. If $n \geq 1$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{29}{36} - \frac{1}{n+3} - \frac{3}{2\cdot(n+2)\cdot(n+3)} - \frac{4}{3\cdot(n+1)\cdot(n+2)\cdot(n+3)}$.

(32) If for every n holds $s(n) = \frac{(n+1)\cdot 2^n}{(n+2)\cdot(n+3)}$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{2^{n+1}}{n+3} - \frac{1}{2}$.

(33) Suppose that for every n such that $n \geq 1$ holds $s(n) = \frac{n^2\cdot 4^n}{(n+1)\cdot(n+2)}$ and $s(0) = 0$. Let given n. If $n \geq 1$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{2}{3} + \frac{(n-1)\cdot 4^{n+1}}{3\cdot(n+2)}$.

(34) If for every n such that $n \geq 1$ holds $s(n) = \frac{n+2}{n\cdot(n+1)\cdot 2^n}$ and $s(0) = 0$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = 1 - \frac{1}{(n+1)\cdot 2^n}$.

(35) Suppose that for every n such that $n \geq 1$ holds $s(n) = \frac{2\cdot n+3}{n\cdot(n+1)\cdot 3^n}$ and $s(0) = 0$. Let given n. If $n \geq 1$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = 1 - \frac{1}{(n+1)\cdot 3^n}$.

(36) If for every n holds $s(n) = \frac{(-1)^n\cdot 2^{n+1}}{(2^{n+1}+(-1)^{n+1})\cdot(2^{n+2}+(-1)^{n+2})}$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1}{3} + \frac{(-1)^{n+2}}{3\cdot(2^{n+2}+(-1)^{n+2})}$.

(37) If for every n holds $s(n) = n! \cdot n$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = (n+1)! - 1$.

(38) If for every n holds $s(n) = \frac{n}{(n+1)!}$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = 1 - \frac{1}{(n+1)!}$.

(39) If for every n such that $n \geq 1$ holds $s(n) = \frac{(n^2+n)-1}{(n+2)!}$ and $s(0) = 0$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1}{2} - \frac{n+1}{(n+2)!}$.

(40) If for every n such that $n \geq 1$ holds $s(n) = \frac{n\cdot 2^n}{(n+2)!}$ and $s(0) = 0$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = 1 - \frac{2^{n+1}}{(n+2)!}$.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Substitution in First-Order Formulas: Elementary Properties[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349-360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The present article introduces the basic concepts of substitution of a variable for a variable in a first-order formula. The contents of this article correspond to Chapter III par. 8, Definition 8.1, 8.2 of Ebbinghaus, Flum, Thomas.

MML Identifier: `SUBSTUT1`.

The terminology and notation used here are introduced in the following articles: [15], [7], [17], [18], [4], [12], [1], [14], [2], [11], [8], [6], [3], [9], [19], [5], [10], [13], and [16].

1. Preliminaries

For simplicity, we follow the rules: a, b are sets, i, k are natural numbers, x, y are bound variables, P is a k-ary predicate symbol, l_1 is a variables list of k, l_2 is a finite sequence of elements of Var, and p is a formula.

The functor vSUB is defined by:

[1]This research was carried out within the project "Wissensformate" and was financially supported by the Mathematical Institute of the University of Bonn (http://www.-wissensformate.uni-bonn.de). Preparation of the Mizar code was part of the first author's graduate work under the supervision of the second author. The authors thank Jip Veldman for his work on the final version of this article.

ISSN 1426–2630

(Def. 1) $\mathrm{vSUB} = \mathrm{BoundVar} \dot{\rightarrow} \mathrm{BoundVar}$.

One can check that vSUB is non empty.

A CQC-substitution is an element of vSUB.

Let us note that vSUB is functional.

In the sequel S_1 is a CQC-substitution.

Let us consider S_1. The functor ${}^{@}S_1$ yielding a partial function from BoundVar to BoundVar is defined as follows:

(Def. 2) ${}^{@}S_1 = S_1$.

Next we state the proposition

(1) If $a \in \mathrm{dom}\, S_1$, then $S_1(a) \in \mathrm{BoundVar}$.

Let l be a finite sequence of elements of Var and let us consider S_1. The functor $\text{CQC-subst}(l, S_1)$ yields a finite sequence of elements of Var and is defined as follows:

(Def. 3) $\mathrm{len}\, \text{CQC-subst}(l, S_1) = \mathrm{len}\, l$ and for every k such that $1 \leq k$ and $k \leq \mathrm{len}\, l$ holds if $l(k) \in \mathrm{dom}\, S_1$, then $(\text{CQC-subst}(l, S_1))(k) = S_1(l(k))$ and if $l(k) \notin \mathrm{dom}\, S_1$, then $(\text{CQC-subst}(l, S_1))(k) = l(k)$.

Let l be a finite sequence of elements of BoundVar. The functor ${}^{@}l$ yielding a finite sequence of elements of Var is defined by:

(Def. 4) ${}^{@}l = l$.

Let l be a finite sequence of elements of BoundVar and let us consider S_1. The functor $\text{CQC-subst}(l, S_1)$ yields a finite sequence of elements of BoundVar and is defined as follows:

(Def. 5) $\text{CQC-subst}(l, S_1) = \text{CQC-subst}({}^{@}l, S_1)$.

Let us consider S_1 and let X be a set. Then $S_1 \restriction X$ is a CQC-substitution.

One can verify that there exists a CQC-substitution which is finite.

Let us consider x, p, S_1. The functor $\mathrm{RestrictSub}(x, p, S_1)$ yielding a finite CQC-substitution is defined by:

(Def. 6) $\mathrm{RestrictSub}(x, p, S_1) = S_1 \restriction \{y : y \in \mathrm{snb}(p) \wedge y \text{ is an element of } \mathrm{dom}\, S_1 \wedge y \neq x \ \wedge \ y \neq S_1(y)\}$.

Let us consider l_2. The functor $\mathrm{BoundVars}(l_2)$ yielding an element of 2^{BoundVar} is defined as follows:

(Def. 7) $\mathrm{BoundVars}(l_2) = \{l_2(k) : 1 \leq k \ \wedge \ k \leq \mathrm{len}\, l_2 \ \wedge \ l_2(k) \in \mathrm{BoundVar}\}$.

Let us consider p. The functor $\mathrm{BoundVars}(p)$ yielding an element of 2^{BoundVar} is defined by the condition (Def. 8).

(Def. 8) There exists a function F from WFF into 2^{BoundVar} such that

(i) $\mathrm{BoundVars}(p) = F(p)$, and

(ii) for every element p of WFF and for all elements d_1, d_2 of 2^{BoundVar} holds if $p = \mathrm{VERUM}$, then $F(p) = \emptyset_{\mathrm{BoundVar}}$ and if p is atomic, then $F(p) = \mathrm{BoundVars}(\mathrm{Args}(p))$ and if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = d_1$ and if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = d_1 \cup d_2$ and if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = d_1 \cup \{\mathrm{Bound}(p)\}$.

One can prove the following propositions:

(2) $\mathrm{BoundVars}(\mathrm{VERUM}) = \emptyset$.

(3) For every formula p such that p is atomic holds $\mathrm{BoundVars}(p) = \mathrm{BoundVars}(\mathrm{Args}(p))$.

(4) For every formula p such that p is negative holds $\mathrm{BoundVars}(p) = \mathrm{BoundVars}(\mathrm{Arg}(p))$.

(5) For every formula p such that p is conjunctive holds $\mathrm{BoundVars}(p) = \mathrm{BoundVars}(\mathrm{LeftArg}(p)) \cup \mathrm{BoundVars}(\mathrm{RightArg}(p))$.

(6) For every formula p such that p is universal holds $\mathrm{BoundVars}(p) = \mathrm{BoundVars}(\mathrm{Scope}(p)) \cup \{\mathrm{Bound}(p)\}$.

Let us consider p. One can check that $\mathrm{BoundVars}(p)$ is finite.

Let us consider p. The functor $\mathrm{DomBoundVars}(p)$ yielding a finite subset of $\mathbb{N}$ is defined as follows:

(Def. 9) $\mathrm{DomBoundVars}(p) = \{i : \mathrm{x}_i \in \mathrm{BoundVars}(p)\}$.

In the sequel f_1 denotes a finite CQC-substitution.

Let us consider f_1. The functor $\mathrm{Sub\text{-}Var}(f_1)$ yields a finite subset of $\mathbb{N}$ and is defined as follows:

(Def. 10) $\mathrm{Sub\text{-}Var}(f_1) = \{i : \mathrm{x}_i \in \mathrm{rng}\, f_1\}$.

Let us consider p, f_1. The functor $\mathrm{NSub}(p, f_1)$ yields a non empty subset of $\mathbb{N}$ and is defined as follows:

(Def. 11) $\mathrm{NSub}(p, f_1) = \mathbb{N} \setminus (\mathrm{DomBoundVars}(p) \cup \mathrm{Sub\text{-}Var}(f_1))$.

Let us consider f_1, p. The functor $\mathrm{upVar}(f_1, p)$ yielding a natural number is defined as follows:

(Def. 12) $\mathrm{upVar}(f_1, p) = \min \mathrm{NSub}(p, f_1)$.

Let us consider x, p, f_1. Let us assume that there exists S_1 such that $f_1 = \mathrm{RestrictSub}(x, \forall_x p, S_1)$. The functor $\mathrm{ExpandSub}(x, p, f_1)$ yielding a CQC-substitution is defined by:

(Def. 13) $\mathrm{ExpandSub}(x, p, f_1) = \begin{cases} f_1 \cup \{\langle x,\, \mathrm{x}_{\mathrm{upVar}(f_1, p)}\rangle\}, & \text{if } x \in \mathrm{rng}\, f_1, \\ f_1 \cup \{\langle x,\, x\rangle\}, & \text{otherwise.} \end{cases}$

Let us consider p, S_1, b. The predicate $b = \mathrm{PQSub}(p, S_1)$ is defined as follows:

(Def. 14) If p is universal, then $b = \mathrm{ExpandSub}(\mathrm{Bound}(p), \mathrm{Scope}(p), \mathrm{RestrictSub}(\mathrm{Bound}(p), p, S_1))$ and if p is not universal, then $b = \emptyset$.

The function QSub is defined as follows:

(Def. 15) $a \in \mathrm{QSub}$ iff there exist p, S_1, b such that $a = \langle\langle p,\, S_1\rangle,\, b\rangle$ and $b = \mathrm{PQSub}(p, S_1)$.

2. Definition and Properties of the Formula – Substitution – Construction

In the sequel e denotes an element of vSUB.

We now state the proposition

(7)(i) $[\!:$ WFF, vSUB $:\!]$ is a subset of $[\!:[\!:\mathbb{N}, \mathbb{N}:\!]^*$, vSUB $:\!]$,

(ii) for every natural number k and for every k-ary predicate symbol p and for every list of variables l_1 of the length k and for every element e of vSUB holds $\langle\langle p\rangle \frown l_1, e\rangle \in [\!:$ WFF, vSUB $:\!]$,

(iii) for every element e of vSUB holds $\langle\langle\langle 0, 0\rangle\rangle, e\rangle \in [\!:$ WFF, vSUB $:\!]$,

(iv) for every finite sequence p of elements of $[\!:\mathbb{N}, \mathbb{N}:\!]$ and for every element e of vSUB such that $\langle p, e\rangle \in [\!:$ WFF, vSUB $:\!]$ holds $\langle\langle\langle 1, 0\rangle\rangle \frown p, e\rangle \in [\!:$ WFF, vSUB $:\!]$,

(v) for all finite sequences p, q of elements of $[\!:\mathbb{N}, \mathbb{N}:\!]$ and for every element e of vSUB such that $\langle p, e\rangle \in [\!:$ WFF, vSUB $:\!]$ and $\langle q, e\rangle \in [\!:$ WFF, vSUB $:\!]$ holds $\langle\langle\langle 2, 0\rangle\rangle \frown p \frown q, e\rangle \in [\!:$ WFF, vSUB $:\!]$, and

(vi) for every bound variable x and for every finite sequence p of elements of $[\!:\mathbb{N}, \mathbb{N}:\!]$ and for every element e of vSUB such that $\langle p, \mathrm{QSub}(\langle\langle\langle 3, 0\rangle\rangle \frown \langle x\rangle \frown p, e\rangle)\rangle \in [\!:$ WFF, vSUB $:\!]$ holds $\langle\langle\langle 3, 0\rangle\rangle \frown \langle x\rangle \frown p, e\rangle \in [\!:$ WFF, vSUB $:\!]$.

Let I_1 be a set. We say that I_1 is QC-Sub-closed if and only if the conditions (Def. 16) are satisfied.

(Def. 16)(i) I_1 is a subset of $[\!:[\!:\mathbb{N}, \mathbb{N}:\!]^*$, vSUB $:\!]$,

(ii) for every natural number k and for every k-ary predicate symbol p and for every list of variables l_1 of the length k and for every element e of vSUB holds $\langle\langle p\rangle \frown l_1, e\rangle \in I_1$,

(iii) for every element e of vSUB holds $\langle\langle\langle 0, 0\rangle\rangle, e\rangle \in I_1$,

(iv) for every finite sequence p of elements of $[\!:\mathbb{N}, \mathbb{N}:\!]$ and for every element e of vSUB such that $\langle p, e\rangle \in I_1$ holds $\langle\langle\langle 1, 0\rangle\rangle \frown p, e\rangle \in I_1$,

(v) for all finite sequences p, q of elements of $[\!:\mathbb{N}, \mathbb{N}:\!]$ and for every element e of vSUB such that $\langle p, e\rangle \in I_1$ and $\langle q, e\rangle \in I_1$ holds $\langle\langle\langle 2, 0\rangle\rangle \frown p \frown q, e\rangle \in I_1$, and

(vi) for every bound variable x and for every finite sequence p of elements of $[\!:\mathbb{N}, \mathbb{N}:\!]$ and for every element e of vSUB such that $\langle p, \mathrm{QSub}(\langle\langle\langle 3, 0\rangle\rangle \frown \langle x\rangle \frown p, e\rangle)\rangle \in I_1$ holds $\langle\langle\langle 3, 0\rangle\rangle \frown \langle x\rangle \frown p, e\rangle \in I_1$.

Let us mention that there exists a set which is QC-Sub-closed and non empty.

The non empty set QC-Sub-WFF is defined as follows:

(Def. 17) QC-Sub-WFF is QC-Sub-closed and for every non empty set D such that D is QC-Sub-closed holds QC-Sub-WFF $\subseteq D$.

In the sequel S, S', S_2, S_3, S'_1, S'_2 are elements of QC-Sub-WFF.

Next we state the proposition

(8) There exist p, e such that $S = \langle p, e\rangle$.

Let us note that QC-Sub-WFF is QC-Sub-closed.

Let P be a predicate symbol, let l be a finite sequence of elements of Var, and let us consider e. Let us assume that $\mathrm{Arity}(P) = \mathrm{len}\, l$. The functor $\mathrm{SubP}(P, l, e)$ yields an element of QC-Sub-WFF and is defined as follows:

(Def. 18) $\mathrm{SubP}(P, l, e) = \langle P[l], e \rangle$.

We now state the proposition

(9) Let k be a natural number, P be a k-ary predicate symbol, and l_1 be a list of variables of the length k. Then $\mathrm{SubP}(P, l_1, e) = \langle P[l_1], e \rangle$.

Let us consider S. We say that S is sub-verum if and only if:

(Def. 19) There exists e such that $S = \langle \mathrm{VERUM}, e \rangle$.

Let us consider S. Then $S_{\mathbf{1}}$ is an element of WFF. Then $S_{\mathbf{2}}$ is an element of vSUB.

The following proposition is true

(10) $S = \langle S_{\mathbf{1}}, S_{\mathbf{2}} \rangle$.

Let us consider S. The functor $\mathrm{SubNot}(S)$ yields an element of QC-Sub-WFF and is defined as follows:

(Def. 20) $\mathrm{SubNot}(S) = \langle \neg(S_{\mathbf{1}}), S_{\mathbf{2}} \rangle$.

Let us consider S, S'. Let us assume that $S_{\mathbf{2}} = S'_{\mathbf{2}}$. The functor $\mathrm{SubAnd}(S, S')$ yields an element of QC-Sub-WFF and is defined by:

(Def. 21) $\mathrm{SubAnd}(S, S') = \langle S_{\mathbf{1}} \wedge S'_{\mathbf{1}}, S_{\mathbf{2}} \rangle$.

In the sequel B denotes an element of [: QC-Sub-WFF, BoundVar :].

Let us consider B. Then $B_{\mathbf{1}}$ is an element of QC-Sub-WFF. Then $B_{\mathbf{2}}$ is an element of BoundVar.

Let us consider B. We say that B is quantifiable if and only if:

(Def. 22) There exists e such that $(B_{\mathbf{1}})_{\mathbf{2}} = \mathrm{QSub}(\langle \forall_{B_{\mathbf{2}}}((B_{\mathbf{1}})_{\mathbf{1}}), e \rangle)$.

Let us consider B. Let us assume that B is quantifiable. An element of vSUB is called a second q.-component of B if:

(Def. 23) $(B_{\mathbf{1}})_{\mathbf{2}} = \mathrm{QSub}(\langle \forall_{B_{\mathbf{2}}}((B_{\mathbf{1}})_{\mathbf{1}}), \mathrm{it} \rangle)$.

In the sequel S_4 is a second q.-component of B.

Let us consider B, S_4. Let us assume that B is quantifiable. The functor $\mathrm{SubAll}(B, S_4)$ yields an element of QC-Sub-WFF and is defined by:

(Def. 24) $\mathrm{SubAll}(B, S_4) = \langle \forall_{B_{\mathbf{2}}}((B_{\mathbf{1}})_{\mathbf{1}}), S_4 \rangle$.

Let us consider S, x. Then $\langle S, x \rangle$ is an element of [: QC-Sub-WFF, BoundVar :].

The scheme *SubQCInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every element S of QC-Sub-WFF holds $\mathcal{P}[S]$

provided the following conditions are satisfied:

- Let k be a natural number, P be a k-ary predicate symbol, l_1 be a list of variables of the length k, and e be an element of vSUB. Then $\mathcal{P}[\mathrm{SubP}(P, l_1, e)]$,

- For every element S of QC-Sub-WFF such that S is sub-verum holds $\mathcal{P}[S]$,
- For every element S of QC-Sub-WFF such that $\mathcal{P}[S]$ holds $\mathcal{P}[\mathrm{SubNot}(S)]$,
- For all elements S, S' of QC-Sub-WFF such that $S_{\mathbf{2}} = S'_{\mathbf{2}}$ and $\mathcal{P}[S]$ and $\mathcal{P}[S']$ holds $\mathcal{P}[\mathrm{SubAnd}(S, S')]$, and
- Let x be a bound variable, S be an element of QC-Sub-WFF, and S_4 be a second q.-component of $\langle S, x\rangle$. If $\langle S, x\rangle$ is quantifiable and $\mathcal{P}[S]$, then $\mathcal{P}[\mathrm{SubAll}(\langle S, x\rangle, S_4)]$.

Let us consider S. We say that S is sub-atomic if and only if the condition (Def. 25) is satisfied.

(Def. 25) There exists a natural number k and there exists a k-ary predicate symbol P and there exists a list of variables l_1 of the length k and there exists an element e of vSUB such that $S = \mathrm{SubP}(P, l_1, e)$.

One can prove the following proposition

(11) If S is sub-atomic, then $S_{\mathbf{1}}$ is atomic.

Let k be a natural number, let P be a k-ary predicate symbol, let l_1 be a list of variables of the length k, and let e be an element of vSUB. One can verify that $\mathrm{SubP}(P, l_1, e)$ is sub-atomic.

Let us consider S. We say that S is sub-negative if and only if:

(Def. 26) There exists S' such that $S = \mathrm{SubNot}(S')$.

We say that S is sub-conjunctive if and only if:

(Def. 27) There exist S_2, S_3 such that $S = \mathrm{SubAnd}(S_2, S_3)$ and $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$.

Let A be a set. We say that A is sub-universal if and only if:

(Def. 28) There exist B, S_4 such that $A = \mathrm{SubAll}(B, S_4)$ and B is quantifiable.

Next we state the proposition

(12) Every S is either sub-verum, sub-atomic, sub-negative, sub-conjunctive, or sub-universal.

Let us consider S. Let us assume that S is sub-atomic. The functor SubArguments(S) yields a finite sequence of elements of Var and is defined by the condition (Def. 29).

(Def. 29) There exists a natural number k and there exists a k-ary predicate symbol P and there exists a list of variables l_1 of the length k and there exists an element e of vSUB such that $\mathrm{SubArguments}(S) = l_1$ and $S = \mathrm{SubP}(P, l_1, e)$.

Let us consider S. Let us assume that S is sub-negative. The functor SubArgument(S) yields an element of QC-Sub-WFF and is defined as follows:

(Def. 30) $S = \mathrm{SubNot}(\mathrm{SubArgument}(S))$.

Let us consider S. Let us assume that S is sub-conjunctive. The functor SubLeftArgument(S) yields an element of QC-Sub-WFF and is defined by:

(Def. 31) There exists S' such that $S = \mathrm{SubAnd}(\mathrm{SubLeftArgument}(S), S')$ and $(\mathrm{SubLeftArgument}(S))_{\mathbf{2}} = S'_{\mathbf{2}}$.

Let us consider S. Let us assume that S is sub-conjunctive. The functor SubRightArgument(S) yielding an element of QC-Sub-WFF is defined as follows:

(Def. 32) There exists S' such that $S = \mathrm{SubAnd}(S', \mathrm{SubRightArgument}(S))$ and $S'_{\mathbf{2}} = (\mathrm{SubRightArgument}(S))_{\mathbf{2}}$.

Let A be a set. Let us assume that A is sub-universal. The functor SubBound(A) yields a bound variable and is defined as follows:

(Def. 33) There exist B, S_4 such that $A = \mathrm{SubAll}(B, S_4)$ and $B_{\mathbf{2}} = \mathrm{SubBound}(A)$ and B is quantifiable.

Let A be a set. Let us assume that A is sub-universal. The functor SubScope(A) yielding an element of QC-Sub-WFF is defined as follows:

(Def. 34) There exist B, S_4 such that $A = \mathrm{SubAll}(B, S_4)$ and $B_{\mathbf{1}} = \mathrm{SubScope}(A)$ and B is quantifiable.

Let us consider S. One can verify that SubNot(S) is sub-negative.

The following propositions are true:

(13) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{SubAnd}(S_2, S_3)$ is sub-conjunctive.

(14) If B is quantifiable, then $\mathrm{SubAll}(B, S_4)$ is sub-universal.

(15) If $\mathrm{SubNot}(S) = \mathrm{SubNot}(S')$, then $S = S'$.

(16) $\mathrm{SubArgument}(\mathrm{SubNot}(S)) = S$.

(17) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$ and $(S'_1)_{\mathbf{2}} = (S'_2)_{\mathbf{2}}$ and $\mathrm{SubAnd}(S_2, S_3) = \mathrm{SubAnd}(S'_1, S'_2)$, then $S_2 = S'_1$ and $S_3 = S'_2$.

(18) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{SubLeftArgument}(\mathrm{SubAnd}(S_2, S_3)) = S_2$.

(19) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{SubRightArgument}(\mathrm{SubAnd}(S_2, S_3)) = S_3$.

(20) Let B_1, B_2 be elements of $[\!:$ QC-Sub-WFF, BoundVar $:\!]$, S_5 be a second q.-component of B_1, and S_6 be a second q.-component of B_2. If B_1 is quantifiable and B_2 is quantifiable and $\mathrm{SubAll}(B_1, S_5) = \mathrm{SubAll}(B_2, S_6)$, then $B_1 = B_2$.

(21) If B is quantifiable, then $\mathrm{SubScope}(\mathrm{SubAll}(B, S_4)) = B_{\mathbf{1}}$.

The scheme *SubQCInd2* concerns a unary predicate $\mathcal{P}$, and states that:

For every element S of QC-Sub-WFF holds $\mathcal{P}[S]$

provided the following requirement is met:

- Let S be an element of QC-Sub-WFF. Then
  - (i) if S is sub-atomic, then $\mathcal{P}[S]$,
  - (ii) if S is sub-verum, then $\mathcal{P}[S]$,
  - (iii) if S is sub-negative and $\mathcal{P}[\mathrm{SubArgument}(S)]$, then $\mathcal{P}[S]$,
  - (iv) if S is sub-conjunctive and $\mathcal{P}[\mathrm{SubLeftArgument}(S)]$ and $\mathcal{P}[\mathrm{SubRightArgument}(S)]$, then $\mathcal{P}[S]$, and

(v) if S is sub-universal and $\mathcal{P}[\mathrm{SubScope}(S)]$, then $\mathcal{P}[S]$.

One can prove the following propositions:

(22) If S is sub-negative, then $\mathrm{len}({}^{@}((\mathrm{SubArgument}(S))_{\mathbf{1}})) < \mathrm{len}({}^{@}(S_{\mathbf{1}}))$.

(23) If S is sub-conjunctive, then $\mathrm{len}({}^{@}((\mathrm{SubLeftArgument}(S))_{\mathbf{1}})) < \mathrm{len}({}^{@}(S_{\mathbf{1}}))$ and $\mathrm{len}({}^{@}((\mathrm{SubRightArgument}(S))_{\mathbf{1}})) < \mathrm{len}({}^{@}(S_{\mathbf{1}}))$.

(24) If S is sub-universal, then $\mathrm{len}({}^{@}((\mathrm{SubScope}(S))_{\mathbf{1}})) < \mathrm{len}({}^{@}(S_{\mathbf{1}}))$.

(25)(i) If S is sub-verum, then $({}^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} = 0$,

(ii) if S is sub-atomic, then there exists a natural number k such that $({}^{@}(S_{\mathbf{1}}))(1)$ is a k-ary predicate symbol,

(iii) if S is sub-negative, then $({}^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} = 1$,

(iv) if S is sub-conjunctive, then $({}^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} = 2$, and

(v) if S is sub-universal, then $({}^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} = 3$.

(26) If S is sub-atomic, then $({}^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} \neq 0$ and $({}^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} \neq 1$ and $({}^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} \neq 2$ and $({}^{@}(S_{\mathbf{1}}))(1)_{\mathbf{1}} \neq 3$.

(27) There exists no S which satisfies any of the following conditions:

(i) it is sub-atomic and sub-negative,
(ii) it is sub-atomic and sub-conjunctive,
(iii) it is sub-atomic and sub-universal,
(iv) it is sub-negative and sub-conjunctive,
(v) it is sub-negative and sub-universal,
(vi) it is sub-conjunctive and sub-universal,
(vii) it is sub-verum and sub-atomic,
(viii) it is sub-verum and sub-negative,
(ix) it is sub-verum and sub-conjunctive,
(x) it is sub-verum and sub-universal.

Now we present two schemes. The scheme *SubFuncEx* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and states that:

There exists a function F from QC-Sub-WFF into $\mathcal{A}$ such that for every element S of QC-Sub-WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds

(i) if S is sub-verum, then $F(S) = \mathcal{B}$,
(ii) if S is sub-atomic, then $F(S) = \mathcal{F}(S)$,
(iii) if S is sub-negative and $d_1 = F(\mathrm{SubArgument}(S))$, then $F(S) = \mathcal{G}(d_1)$,
(iv) if S is sub-conjunctive and $d_1 = F(\mathrm{SubLeftArgument}(S))$ and $d_2 = F(\mathrm{SubRightArgument}(S))$, then $F(S) = \mathcal{H}(d_1, d_2)$, and
(v) if S is sub-universal and $d_1 = F(\mathrm{SubScope}(S))$, then $F(S) = \mathcal{I}(S, d_1)$

for all values of the parameters.

The scheme *SubQCFuncUniq* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$ from QC-Sub-WFF into $\mathcal{A}$, a function $\mathcal{C}$ from QC-Sub-WFF into $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and states that:

$$\mathcal{B} = \mathcal{C}$$

provided the parameters satisfy the following conditions:

- Let S be an element of QC-Sub-WFF and d_1, d_2 be elements of $\mathcal{A}$. Then
  (i) if S is sub-verum, then $\mathcal{B}(S) = \mathcal{D}$,
  (ii) if S is sub-atomic, then $\mathcal{B}(S) = \mathcal{F}(S)$,
  (iii) if S is sub-negative and $d_1 = \mathcal{B}(\mathrm{SubArgument}(S))$, then $\mathcal{B}(S) = \mathcal{G}(d_1)$,
  (iv) if S is sub-conjunctive and $d_1 = \mathcal{B}(\mathrm{SubLeftArgument}(S))$ and $d_2 = \mathcal{B}(\mathrm{SubRightArgument}(S))$, then $\mathcal{B}(S) = \mathcal{H}(d_1, d_2)$, and
  (v) if S is sub-universal and $d_1 = \mathcal{B}(\mathrm{SubScope}(S))$, then $\mathcal{B}(S) = \mathcal{I}(S, d_1)$,
  and
- Let S be an element of QC-Sub-WFF and d_1, d_2 be elements of $\mathcal{A}$. Then
  (i) if S is sub-verum, then $\mathcal{C}(S) = \mathcal{D}$,
  (ii) if S is sub-atomic, then $\mathcal{C}(S) = \mathcal{F}(S)$,
  (iii) if S is sub-negative and $d_1 = \mathcal{C}(\mathrm{SubArgument}(S))$, then $\mathcal{C}(S) = \mathcal{G}(d_1)$,
  (iv) if S is sub-conjunctive and $d_1 = \mathcal{C}(\mathrm{SubLeftArgument}(S))$ and $d_2 = \mathcal{C}(\mathrm{SubRightArgument}(S))$, then $\mathcal{C}(S) = \mathcal{H}(d_1, d_2)$, and
  (v) if S is sub-universal and $d_1 = \mathcal{C}(\mathrm{SubScope}(S))$, then $\mathcal{C}(S) = \mathcal{I}(S, d_1)$.

Let us consider S. The functor ${}^{@}S$ yielding an element of $[\!:\mathrm{WFF}, \mathrm{vSUB}:\!]$ is defined as follows:

(Def. 35) ${}^{@}S = S$.

In the sequel Z denotes an element of $[\!:\mathrm{WFF}, \mathrm{vSUB}:\!]$.

Let us consider Z. Then $Z_{\mathbf{1}}$ is an element of WFF. Then $Z_{\mathbf{2}}$ is a CQC-substitution.

Let us consider Z. The functor S-Bound(Z) yields a bound variable and is defined by:

(Def. 36) $$\text{S-Bound}(Z) = \begin{cases} \mathrm{x}_{\mathrm{upVar}(\mathrm{RestrictSub}(\mathrm{Bound}(Z_{\mathbf{1}}), Z_{\mathbf{1}}, Z_{\mathbf{2}}), \mathrm{Scope}(Z_{\mathbf{1}}))}, \\ \quad \text{if } \mathrm{Bound}(Z_{\mathbf{1}}) \in \mathrm{rng}\,\mathrm{RestrictSub}(\mathrm{Bound}(Z_{\mathbf{1}}), Z_{\mathbf{1}}, Z_{\mathbf{2}}), \\ \mathrm{Bound}(Z_{\mathbf{1}}), \text{ otherwise.} \end{cases}$$

Let us consider S, p. The functor Quant(S, p) yielding an element of WFF is defined by:

(Def. 37) $\mathrm{Quant}(S, p) = \forall_{\text{S-Bound}(@S)} p$.

3. Definition and Properties of Substitution

Let S be an element of QC-Sub-WFF. The functor CQCSub(S) yielding an element of WFF is defined by the condition (Def. 38).

(Def. 38) There exists a function F from QC-Sub-WFF into WFF such that

(i) $\mathrm{CQCSub}(S) = F(S)$, and

(ii) for every element S' of QC-Sub-WFF holds if S' is sub-verum, then $F(S') = \mathrm{VERUM}$ and if S' is sub-atomic, then $F(S') = \mathrm{PredSym}(S'_{\mathbf{1}})[\text{CQC-subst}(\mathrm{SubArguments}(S'), S'_{\mathbf{2}})]$ and if S' is sub-negative, then $F(S') = \neg F(\mathrm{SubArgument}(S'))$ and if S' is sub-conjunctive, then $F(S') = F(\mathrm{SubLeftArgument}(S')) \wedge F(\mathrm{SubRightArgument}(S'))$ and if S' is sub-universal, then $F(S') = \mathrm{Quant}(S', F(\mathrm{SubScope}(S')))$.

We now state several propositions:

(28) If S is sub-negative, then $\mathrm{CQCSub}(S) = \neg\,\mathrm{CQCSub}(\mathrm{SubArgument}(S))$.

(29) $\mathrm{CQCSub}(\mathrm{SubNot}(S)) = \neg\,\mathrm{CQCSub}(S)$.

(30) If S is sub-conjunctive, then $\mathrm{CQCSub}(S) = \mathrm{CQCSub}(\mathrm{SubLeftArgument}(S)) \wedge \mathrm{CQCSub}(\mathrm{SubRightArgument}(S))$.

(31) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{CQCSub}(\mathrm{SubAnd}(S_2, S_3)) = \mathrm{CQCSub}(S_2) \wedge \mathrm{CQCSub}(S_3)$.

(32) If S is sub-universal, then $\mathrm{CQCSub}(S) = \mathrm{Quant}(S, \mathrm{CQCSub}(\mathrm{SubScope}(S)))$.

The subset CQC-Sub-WFF of QC-Sub-WFF is defined by:

(Def. 39) CQC-Sub-WFF $= \{S : S_{\mathbf{1}}$ is an element of CQC-WFF$\}$.

Let us observe that CQC-Sub-WFF is non empty.

Next we state several propositions:

(33) If S is sub-verum, then CQCSub(S) is an element of CQC-WFF.

(34) Let h be a finite sequence. Then h is a variables list of k if and only if h is a finite sequence of elements of BoundVar and $\mathrm{len}\, h = k$.

(35) CQCSub(SubP(P, l_1, e)) is an element of CQC-WFF.

(36) If CQCSub(S) is an element of CQC-WFF, then CQCSub(SubNot(S)) is an element of CQC-WFF.

(37) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$ and CQCSub(S_2) is an element of CQC-WFF and CQCSub(S_3) is an element of CQC-WFF, then CQCSub(SubAnd(S_2, S_3)) is an element of CQC-WFF.

In the sequel x_1 denotes a second q.-component of $\langle S, x\rangle$.

We now state the proposition

(38) If CQCSub(S) is an element of CQC-WFF and $\langle S, x\rangle$ is quantifiable, then CQCSub(SubAll($\langle S, x\rangle, x_1$)) is an element of CQC-WFF.

In the sequel S is an element of CQC-Sub-WFF.

The scheme *SubCQCInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every S holds $\mathcal{P}[S]$

provided the following requirement is met:

- Let S, S' be elements of CQC-Sub-WFF, x be a bound variable, S_4 be a second q.-component of $\langle S, x \rangle$, k be a natural number, l_1 be a variables list of k, P be a k-ary predicate symbol, and e be an element of vSUB. Then
  - (i) $\mathcal{P}[\mathrm{SubP}(P, l_1, e)]$,
  - (ii) if S is sub-verum, then $\mathcal{P}[S]$,
  - (iii) if $\mathcal{P}[S]$, then $\mathcal{P}[\mathrm{SubNot}(S)]$,
  - (iv) if $S_{\mathbf{2}} = S'_{\mathbf{2}}$ and $\mathcal{P}[S]$ and $\mathcal{P}[S']$, then $\mathcal{P}[\mathrm{SubAnd}(S, S')]$, and
  - (v) if $\langle S, x \rangle$ is quantifiable and $\mathcal{P}[S]$, then $\mathcal{P}[\mathrm{SubAll}(\langle S, x \rangle, S_4)]$.

Let us consider S. Then $\mathrm{CQCSub}(S)$ is an element of CQC-WFF.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Coincidence Lemma and Substitution Lemma[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349–360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The present article establishes further concepts of substitution of a variable for a variable in a first-order formula. The main result is the substitution lemma. The contents of this article correspond to Chapter III par. 5, 5.1 Coincidence Lemma and Chapter III par. 8, 8.3 Substitution Lemma of Ebbinghaus, Flum, Thomas.

MML Identifier: `SUBLEMMA`.

The articles [13], [7], [15], [1], [4], [9], [8], [10], [3], [18], [6], [16], [19], [5], [12], [17], [11], [14], and [2] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: a, b are sets, i, k are natural numbers, p, q are elements of CQC-WFF, x, y are bound variables, A is a non empty set, J is an interpretation of A, v, w are elements of $\boldsymbol{V}(A)$, P, P' are

[1]This research was carried out within the project "Wissensformate" and was financially supported by the Mathematical Institute of the University of Bonn (http://www.-wissensformate.uni-bonn.de). Preparation of the Mizar code was part of the first author's graduate work under the supervision of the second author. The authors thank Jip Veldman for his work on the final version of this article.

ISSN 1426–2630

k-ary predicate symbols, l_1, l'_1 are variables lists of k, l_2 is a finite sequence of elements of Var, S_1, S'_1 are CQC-substitutions, and S, S_2, S_3 are elements of CQC-Sub-WFF.

Next we state two propositions:

(1) For all functions f, g, h, h_1, h_2 such that $\operatorname{dom} h_1 \subseteq \operatorname{dom} h$ and $\operatorname{dom} h_2 \subseteq \operatorname{dom} h$ holds $f{+}\!\cdot g{+}\!\cdot h = f{+}\!\cdot h_1{+}\!\cdot(g{+}\!\cdot h_2){+}\!\cdot h$.

(2) For every function v_1 such that $x \in \operatorname{dom} v_1$ holds $v_1{\restriction}(\operatorname{dom} v_1 \setminus \{x\}){+}\!\cdot(x \longmapsto v_1(x)) = v_1$.

Let us consider A. A value substitution of A is a partial function from BoundVar to A.

In the sequel v_2, v_1, v_3 are value substitutions of A.

Let us consider A, v, v_2. The functor $v(v_2)$ yields an element of $\boldsymbol{V}(A)$ and is defined by:

(Def. 1) $v(v_2) = v{+}\!\cdot v_2$.

Let us consider S. Then $S_{\mathbf{1}}$ is an element of CQC-WFF.

Let us consider S, A, v. The functor $\operatorname{ValS}(v, S)$ yielding a value substitution of A is defined by:

(Def. 2) $\operatorname{ValS}(v, S) = ({}^{@}(S_{\mathbf{2}})) \cdot v$.

The following proposition is true

(3) If S is sub-verum, then $\operatorname{CQCSub}(S) = \mathrm{VERUM}$.

Let us consider S, A, v, J. The predicate $J, v \models S$ is defined as follows:

(Def. 3) $J, v \models S_{\mathbf{1}}$.

The following propositions are true:

(4) If S is sub-verum, then for every v holds $J, v \models \operatorname{CQCSub}(S)$ iff $J, v(\operatorname{ValS}(v, S)) \models S$.

(5) If $i \in \operatorname{dom} l_1$, then $l_1(i)$ is a bound variable.

(6) If S is sub-atomic, then $\operatorname{CQCSub}(S) = \operatorname{PredSym}(S_{\mathbf{1}})[\text{CQC-Subst}(\operatorname{SubArguments}(S), S_{\mathbf{2}})]$.

(7) If $\operatorname{SubArguments}(\operatorname{SubP}(P, l_1, S_1)) = \operatorname{SubArguments}(\operatorname{SubP}(P', l'_1, S'_1))$, then $l_1 = l'_1$.

(8) $\operatorname{SubArguments}(\operatorname{SubP}(P, l_1, S_1)) = l_1$.

Let us consider k, P, l_1, S_1. Then $\operatorname{SubP}(P, l_1, S_1)$ is an element of CQC-Sub-WFF.

We now state three propositions:

(9) $\operatorname{CQCSub}(\operatorname{SubP}(P, l_1, S_1)) = P[\text{CQC-Subst}(l_1, S_1)]$.

(10) $P[\text{CQC-Subst}(l_1, S_1)]$ is an element of CQC-WFF.

(11) $\text{CQC-Subst}(l_1, S_1)$ is a variables list of k.

Let us consider k, l_1, S_1. Then $\text{CQC-Subst}(l_1, S_1)$ is a variables list of k.

One can prove the following propositions:

(12) If $x \notin \mathrm{dom}(S_{\mathbf{2}})$, then $v(\mathrm{ValS}(v, S))(x) = v(x)$.

(13) If $x \in \mathrm{dom}(S_{\mathbf{2}})$, then $v(\mathrm{ValS}(v, S))(x) = (\mathrm{ValS}(v, S))(x)$.

(14) $v(\mathrm{ValS}(v, \mathrm{SubP}(P, l_1, S_1))) * l_1 = v * \text{CQC-Subst}(l_1, S_1)$.

(15) $(\mathrm{SubP}(P, l_1, S_1))_{\mathbf{1}} = P[l_1]$.

(16) For every v holds $J, v \models \mathrm{CQCSub}(\mathrm{SubP}(P, l_1, S_1))$ iff $J, v(\mathrm{ValS}(v, \mathrm{SubP}(P, l_1, S_1))) \models \mathrm{SubP}(P, l_1, S_1)$.

(17) $(\mathrm{SubNot}(S))_{\mathbf{1}} = \neg(S_{\mathbf{1}})$ and $(\mathrm{SubNot}(S))_{\mathbf{2}} = S_{\mathbf{2}}$.

Let us consider S. Then $\mathrm{SubNot}(S)$ is an element of CQC-Sub-WFF.

We now state three propositions:

(18) $J, v(\mathrm{ValS}(v, S)) \not\models S$ iff $J, v(\mathrm{ValS}(v, S)) \models \mathrm{SubNot}(S)$.

(19) $\mathrm{ValS}(v, S) = \mathrm{ValS}(v, \mathrm{SubNot}(S))$.

(20) If for every v holds $J, v \models \mathrm{CQCSub}(S)$ iff $J, v(\mathrm{ValS}(v, S)) \models S$, then for every v holds $J, v \models \mathrm{CQCSub}(\mathrm{SubNot}(S))$ iff $J, v(\mathrm{ValS}(v, \mathrm{SubNot}(S))) \models \mathrm{SubNot}(S)$.

Let us consider S_2, S_3. Let us assume that $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$. The functor $\mathrm{CQCSubAnd}(S_2, S_3)$ yielding an element of CQC-Sub-WFF is defined as follows:

(Def. 4) $\mathrm{CQCSubAnd}(S_2, S_3) = \mathrm{SubAnd}(S_2, S_3)$.

Next we state several propositions:

(21) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $(\mathrm{CQCSubAnd}(S_2, S_3))_{\mathbf{1}} = (S_2)_{\mathbf{1}} \wedge (S_3)_{\mathbf{1}}$ and $(\mathrm{CQCSubAnd}(S_2, S_3))_{\mathbf{2}} = (S_2)_{\mathbf{2}}$.

(22) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $(\mathrm{CQCSubAnd}(S_2, S_3))_{\mathbf{2}} = (S_2)_{\mathbf{2}}$.

(23) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{ValS}(v, S_2) = \mathrm{ValS}(v, \mathrm{CQCSubAnd}(S_2, S_3))$ and $\mathrm{ValS}(v, S_3) = \mathrm{ValS}(v, \mathrm{CQCSubAnd}(S_2, S_3))$.

(24) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{CQCSub}(\mathrm{CQCSubAnd}(S_2, S_3)) = \mathrm{CQCSub}(S_2) \wedge \mathrm{CQCSub}(S_3)$.

(25) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $J, v(\mathrm{ValS}(v, S_2)) \models S_2$ and $J, v(\mathrm{ValS}(v, S_3)) \models S_3$ iff $J, v(\mathrm{ValS}(v, \mathrm{CQCSubAnd}(S_2, S_3))) \models \mathrm{CQCSubAnd}(S_2, S_3)$.

(26) Suppose $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$ and for every v holds $J, v \models \mathrm{CQCSub}(S_2)$ iff $J, v(\mathrm{ValS}(v, S_2)) \models S_2$ and for every v holds $J, v \models \mathrm{CQCSub}(S_3)$ iff $J, v(\mathrm{ValS}(v, S_3)) \models S_3$. Let given v. Then $J, v \models \mathrm{CQCSub}(\mathrm{CQCSubAnd}(S_2, S_3))$ if and only if $J, v(\mathrm{ValS}(v, \mathrm{CQCSubAnd}(S_2, S_3))) \models \mathrm{CQCSubAnd}(S_2, S_3)$.

In the sequel B is an element of $[\!:$ QC-Sub-WFF, BoundVar $:\!]$ and S_4 is a second q.-component of B.

The following proposition is true

(27) If B is quantifiable, then $(\mathrm{SubAll}(B, S_4))_{\mathbf{1}} = \forall_{B_{\mathbf{2}}}((B_{\mathbf{1}})_{\mathbf{1}})$ and $(\mathrm{SubAll}(B, S_4))_{\mathbf{2}} = S_4$.

Let B be an element of $[\!:$ QC-Sub-WFF, BoundVar $:\!]$. We say that B is CQC-WFF-like if and only if:

(Def. 5) $B_{\mathbf{1}} \in$ CQC-Sub-WFF.

Let us observe that there exists an element of [: QC-Sub-WFF, BoundVar :] which is CQC-WFF-like.

Let us consider S, x. Then $\langle S, x\rangle$ is a CQC-WFF-like element of [: QC-Sub-WFF, BoundVar :].

In the sequel B denotes a CQC-WFF-like element of [: QC-Sub-WFF, BoundVar :], x_1 denotes a second q.-component of $\langle S, x\rangle$, and S_4 denotes a second q.-component of B.

Let us consider B. Then $B_{\mathbf{1}}$ is an element of CQC-Sub-WFF.

Let us consider B, S_4. Let us assume that B is quantifiable. The functor CQCSubAll(B, S_4) yields an element of CQC-Sub-WFF and is defined as follows:

(Def. 6) CQCSubAll$(B, S_4) =$ SubAll(B, S_4).

We now state the proposition

(28) If B is quantifiable, then CQCSubAll(B, S_4) is sub-universal.

Let us consider S. Let us assume that S is sub-universal. The functor CQCSubScope(S) yielding an element of CQC-Sub-WFF is defined as follows:

(Def. 7) CQCSubScope$(S) =$ SubScope(S).

Let us consider S_2, p. Let us assume that S_2 is sub-universal and $p =$ CQCSub(CQCSubScope(S_2)). The functor CQCQuant(S_2, p) yielding an element of CQC-WFF is defined as follows:

(Def. 8) CQCQuant$(S_2, p) =$ Quant(S_2, p).

The following two propositions are true:

(29) If S is sub-universal, then CQCSub$(S) =$ CQCQuant$(S,$ CQCSub(CQCSubScope$(S)))$.

(30) If B is quantifiable, then CQCSubScope(CQCSubAll$(B, S_4)) = B_{\mathbf{1}}$.

2. The Substitution Lemma

The following propositions are true:

(31) If $\langle S, x\rangle$ is quantifiable, then CQCSubScope(CQCSubAll$(\langle S, x\rangle, x_1)) = S$ and CQCQuant(CQCSubAll$(\langle S, x\rangle, x_1)$, CQCSub(CQCSubScope (CQCSubAll$(\langle S, x\rangle, x_1)$))) $=$ CQCQuant(CQCSubAll$(\langle S, x\rangle, x_1)$, CQCSub$(S)$).

(32) If $\langle S, x\rangle$ is quantifiable, then CQCQuant(CQCSubAll$(\langle S, x\rangle, x_1)$, CQCSub$(S)) = \forall_{\text{S-Bound}({}^{@}\text{CQCSubAll}(\langle S, x\rangle, x_1))}$ CQCSub(S).

(33) If $x \in \operatorname{dom}(S_{\mathbf{2}})$, then $v(({}^{@}(S_{\mathbf{2}}))(x)) = v(\mathrm{ValS}(v, S))(x)$.

(34) If $x \in \operatorname{dom}({}^{@}(S_{\mathbf{2}}))$, then $({}^{@}(S_{\mathbf{2}}))(x)$ is a bound variable.

(35) [: WFF, vSUB :] $\subseteq \operatorname{dom}$ QSub.

In the sequel B_1 denotes an element of $[:\text{QC-Sub-WFF}, \text{BoundVar}:]$ and S_5 denotes a second q.-component of B_1.

We now state a number of propositions:

(36) If B is quantifiable and B_1 is quantifiable and $\mathrm{SubAll}(B, S_4) = \mathrm{SubAll}(B_1, S_5)$, then $B_{\mathbf{2}} = (B_1)_{\mathbf{2}}$ and $S_4 = S_5$.

(37) If B is quantifiable and B_1 is quantifiable and $\mathrm{CQCSubAll}(B, S_4) = \mathrm{SubAll}(B_1, S_5)$, then $B_{\mathbf{2}} = (B_1)_{\mathbf{2}}$ and $S_4 = S_5$.

(38) If $\langle S, x\rangle$ is quantifiable, then $\mathrm{SubBound}(\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)) = x$.

(39) If $\langle S, x\rangle$ is quantifiable and $x \in \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$, then $\text{S-Bound}({}^{@}\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)) \notin \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$ and $\text{S-Bound}({}^{@}\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)) \notin \mathrm{BoundVars}(S_{\mathbf{1}})$.

(40) If $\langle S, x\rangle$ is quantifiable and $x \notin \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$, then $\text{S-Bound}({}^{@}\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)) \notin \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$.

(41) If $\langle S, x\rangle$ is quantifiable, then $\text{S-Bound}({}^{@}\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)) \notin \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$.

(42) If $\langle S, x\rangle$ is quantifiable, then $S_{\mathbf{2}} = \mathrm{ExpandSub}(x, S_{\mathbf{1}}, \mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1))$.

(43) $\mathrm{snb}(\mathrm{VERUM}) \subseteq \mathrm{BoundVars}(\mathrm{VERUM})$.

(44) $\mathrm{snb}(P[l_1]) \subseteq \mathrm{BoundVars}(P[l_1])$.

(45) If $\mathrm{snb}(p) \subseteq \mathrm{BoundVars}(p)$, then $\mathrm{snb}(\neg p) \subseteq \mathrm{BoundVars}(\neg p)$.

(46) If $\mathrm{snb}(p) \subseteq \mathrm{BoundVars}(p)$ and $\mathrm{snb}(q) \subseteq \mathrm{BoundVars}(q)$, then $\mathrm{snb}(p \wedge q) \subseteq \mathrm{BoundVars}(p \wedge q)$.

(47) If $\mathrm{snb}(p) \subseteq \mathrm{BoundVars}(p)$, then $\mathrm{snb}(\forall_x p) \subseteq \mathrm{BoundVars}(\forall_x p)$.

(48) For every p holds $\mathrm{snb}(p) \subseteq \mathrm{BoundVars}(p)$.

Let us consider A, let a be an element of A, and let us consider x. The functor $x \restriction a$ yields a value substitution of A and is defined as follows:

(Def. 9) $x \restriction a = x \longmapsto a$.

In the sequel a denotes an element of A.

The following propositions are true:

(49) If $x \neq b$, then $v(x \restriction a)(b) = v(b)$.

(50) If $x = y$, then $v(x \restriction a)(y) = a$.

(51) $J, v \models \forall_x p$ iff for every a holds $J, v(x \restriction a) \models p$.

Let us consider S, x, x_1, A, v. The functor $\mathrm{NExVal}(v, S, x, x_1)$ yielding a value substitution of A is defined as follows:

(Def. 10) $\mathrm{NExVal}(v, S, x, x_1) = ({}^{@}\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)) \cdot v$.

Let us consider A and let v, w be value substitutions of A. Then $v{+\cdot}w$ is a value substitution of A.

One can prove the following propositions:

(52) If $\langle S, x\rangle$ is quantifiable and $x \in \operatorname{rng}\operatorname{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$, then S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$)) $= \mathrm{x}_{\mathrm{upVar}(\mathrm{RestrictSub}(x,\forall_x(S_{\mathbf{1}}),x_1),S_{\mathbf{1}})}$.

(53) If $\langle S, x\rangle$ is quantifiable and $x \notin \operatorname{rng}\operatorname{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$, then S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$)) $= x$.

(54) If $\langle S, x\rangle$ is quantifiable, then for every a holds ValS(v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a), S$) = NExVal(v(S-Bound ($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a), S, x, x_1)+\cdot x\restriction a$ and dom RestrictSub($x, \forall_x(S_{\mathbf{1}}), x_1$) misses $\{x\}$.

(55) Suppose $\langle S, x\rangle$ is quantifiable. Then for every a holds J, v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a$)(ValS(v(S-Bound($^{@}$CQCSubAll ($\langle S, x\rangle, x_1$))$\restriction a), S$)) $\models S$ if and only if for every a holds J, v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a$)(NExVal(v(S-Bound ($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a), S, x, x_1)+\cdot x\restriction a) \models S$.

(56) If $\langle S, x\rangle$ is quantifiable, then for every a holds NExVal(v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a), S, x, x_1) =$ NExVal(v, S, x, x_1).

(57) Suppose $\langle S, x\rangle$ is quantifiable. Then for every a holds J, v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a$)(NExVal(v(S-Bound ($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a), S, x, x_1)+\cdot x\restriction a) \models S$ if and only if for every a holds J, v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a$)(NExVal(v, S, x, x_1) $+\cdot x\restriction a) \models S$.

3. The Coincidence Lemma

The following propositions are true:

(58) If $\operatorname{rng} l_2 \subseteq$ BoundVar, then $\operatorname{snb}(l_2) = \operatorname{rng} l_2$.

(59) $\operatorname{dom} v =$ BoundVar and $\operatorname{dom}(x\restriction a) = \{x\}$.

(60) $v * l_1 = l_1 \cdot (v\restriction \operatorname{snb}(l_1))$.

(61) For all v, w such that $v\restriction \operatorname{snb}(P[l_1]) = w\restriction \operatorname{snb}(P[l_1])$ holds $J, v \models P[l_1]$ iff $J, w \models P[l_1]$.

(62) Suppose that for all v, w such that $v\restriction \operatorname{snb}(p) = w\restriction \operatorname{snb}(p)$ holds $J, v \models p$ iff $J, w \models p$. Let given v, w. If $v\restriction \operatorname{snb}(\neg p) = w\restriction \operatorname{snb}(\neg p)$, then $J, v \models \neg p$ iff $J, w \models \neg p$.

(63) Suppose that
 (i) for all v, w such that $v\restriction \operatorname{snb}(p) = w\restriction \operatorname{snb}(p)$ holds $J, v \models p$ iff $J, w \models p$, and
 (ii) for all v, w such that $v\restriction \operatorname{snb}(q) = w\restriction \operatorname{snb}(q)$ holds $J, v \models q$ iff $J, w \models q$.

Let given v, w. If $v\restriction \operatorname{snb}(p \wedge q) = w\restriction \operatorname{snb}(p \wedge q)$, then $J, v \models p \wedge q$ iff $J, w \models p \wedge q$.

(64) For every set X such that $X \subseteq$ BoundVar holds $\mathrm{dom}(v{\restriction}X) = \mathrm{dom}(v(x{\restriction}a){\restriction}X)$ and $\mathrm{dom}(v{\restriction}X) = X$.

(65) If $v{\restriction}\mathrm{snb}(p) = w{\restriction}\mathrm{snb}(p)$, then $v(x{\restriction}a){\restriction}\mathrm{snb}(p) = w(x{\restriction}a){\restriction}\mathrm{snb}(p)$.

(66) $\mathrm{snb}(p) \subseteq \mathrm{snb}(\forall_x p) \cup \{x\}$.

(67) If $v{\restriction}(\mathrm{snb}(p) \setminus \{x\}) = w{\restriction}(\mathrm{snb}(p) \setminus \{x\})$, then $v(x{\restriction}a){\restriction}\mathrm{snb}(p) = w(x{\restriction}a){\restriction}\mathrm{snb}(p)$.

(68) Suppose that for all v, w such that $v{\restriction}\mathrm{snb}(p) = w{\restriction}\mathrm{snb}(p)$ holds $J, v \models p$ iff $J, w \models p$. Let given v, w. If $v{\restriction}\mathrm{snb}(\forall_x p) = w{\restriction}\mathrm{snb}(\forall_x p)$, then $J, v \models \forall_x p$ iff $J, w \models \forall_x p$.

(69) For all v, w such that $v{\restriction}\mathrm{snb}(\mathrm{VERUM}) = w{\restriction}\mathrm{snb}(\mathrm{VERUM})$ holds $J, v \models$ VERUM iff $J, w \models$ VERUM.

(70) For every p and for all v, w such that $v{\restriction}\mathrm{snb}(p) = w{\restriction}\mathrm{snb}(p)$ holds $J, v \models p$ iff $J, w \models p$.

(71) If $\langle S, x\rangle$ is quantifiable, then $v(\text{S-Bound}({}^{@}\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)){\restriction}a)(\mathrm{NExVal}(v, S, x, x_1){+\cdot}x{\restriction}a){\restriction}\mathrm{snb}(S_{\mathbf{1}}) = v(\mathrm{NExVal}(v, S, x, x_1){+\cdot}x{\restriction}a){\restriction}\mathrm{snb}(S_{\mathbf{1}})$.

(72) If $\langle S, x\rangle$ is quantifiable, then for every a holds $J, v(\text{S-Bound}({}^{@}\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)){\restriction}a)(\mathrm{NExVal}(v, S, x, x_1){+\cdot}x{\restriction}a) \models S$ iff for every a holds $J, v(\mathrm{NExVal}(v, S, x, x_1){+\cdot}x{\restriction}a) \models S$.

(73) $\mathrm{dom}\,\mathrm{NExVal}(v, S, x, x_1) = \mathrm{dom}\,\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$.

(74) If $\langle S, x\rangle$ is quantifiable, then $v(\mathrm{NExVal}(v, S, x, x_1){+\cdot}x{\restriction}a) = v(\mathrm{NExVal}(v, S, x, x_1))(x{\restriction}a)$.

(75) If $\langle S, x\rangle$ is quantifiable, then for every a holds $J, v(\mathrm{NExVal}(v, S, x, x_1){+\cdot}x{\restriction}a) \models S$ iff for every a holds $J, v(\mathrm{NExVal}(v, S, x, x_1))(x{\restriction}a) \models S$.

(76) For every a holds $J, v(\mathrm{NExVal}(v, S, x, x_1))(x{\restriction}a) \models S$ iff for every a holds $J, v(\mathrm{NExVal}(v, S, x, x_1))(x{\restriction}a) \models S_{\mathbf{1}}$.

(77) Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(\mathrm{VERUM})$ and for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_3$. Then $J, v(v_2) \models$ VERUM if and only if $J, v(v_2{+\cdot}v_1{+\cdot}v_3) \models$ VERUM.

(78) Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(l_1)$ and for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_3$. Then $v(v_2) * l_1 = v(v_2{+\cdot}v_1{+\cdot}v_3) * l_1$.

(79) Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(P[l_1])$ and for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_3$. Then $J, v(v_2) \models P[l_1]$ if and only if $J, v(v_2{+\cdot}v_1{+\cdot}v_3) \models P[l_1]$.

(80) Suppose that for all v, v_2, v_1, v_3 such that for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(p)$ and for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) =$

$v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_3$ holds $J, v(v_2) \models p$ iff $J, v(v_2 +\cdot v_1 +\cdot v_3) \models p$. Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(\neg p)$ and for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_3$. Then $J, v(v_2) \models \neg p$ if and only if $J, v(v_2 +\cdot v_1 +\cdot v_3) \models \neg p$.

(81) Suppose that

(i) for all v, v_2, v_1, v_3 such that for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(p)$ and for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_3$ holds $J, v(v_2) \models p$ iff $J, v(v_2 +\cdot v_1 +\cdot v_3) \models p$, and

(ii) for all v, v_2, v_1, v_3 such that for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(q)$ and for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_3$ holds $J, v(v_2) \models q$ iff $J, v(v_2 +\cdot v_1 +\cdot v_3) \models q$.

Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(p \wedge q)$ and for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_3$. Then $J, v(v_2) \models p \wedge q$ if and only if $J, v(v_2 +\cdot v_1 +\cdot v_3) \models p \wedge q$.

(82) If for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(\forall_x p)$, then for every y such that $y \in \mathrm{dom}\, v_1 \setminus \{x\}$ holds $y \notin \mathrm{snb}(p)$.

(83) Let v_1 be a function. Suppose for every y such that $y \in \mathrm{dom}\, v_1$ holds $v_1(y) = v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_1$. Let given y. If $y \in \mathrm{dom}\, v_1 \setminus \{x\}$, then $(v_1 \restriction (\mathrm{dom}\, v_1 \setminus \{x\}))(y) = v(v_2)(y)$.

(84) Suppose that for all v, v_2, v_1, v_3 such that for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(p)$ and for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_3$ holds $J, v(v_2) \models p$ iff $J, v(v_2 +\cdot v_1 +\cdot v_3) \models p$. Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(\forall_x p)$ and for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_3$. Then $J, v(v_2) \models \forall_x p$ if and only if $J, v(v_2 +\cdot v_1 +\cdot v_3) \models \forall_x p$.

(85) Let given p and given v, v_2, v_1, v_3. Suppose for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(p)$ and for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$ and $\mathrm{dom}\, v_2$ misses $\mathrm{dom}\, v_3$. Then $J, v(v_2) \models p$ if and only if $J, v(v_2 +\cdot v_1 +\cdot v_3) \models p$.

Let us consider p. The functor $\mathrm{RSub1}\, p$ yields a set and is defined by:

(Def. 11) $b \in \mathrm{RSub1}\, p$ iff there exists x such that $x = b$ and $x \notin \mathrm{snb}(p)$.

Let us consider p, S_1. The functor $\mathrm{RSub2}(p, S_1)$ yielding a set is defined as follows:

(Def. 12) $b \in \mathrm{RSub2}(p, S_1)$ iff there exists x such that $x = b$ and $x \in \mathrm{snb}(p)$ and $x = ({}^{@}S_1)(x)$.

Next we state several propositions:

(86) $\mathrm{dom}(({}^{@}S_1) \restriction \mathrm{RSub1}\, p)$ misses $\mathrm{dom}(({}^{@}S_1) \restriction \mathrm{RSub2}(p, S_1))$.

(87) $^{@}\mathrm{RestrictSub}(x, \forall_x p, S_1) =$
$(^{@}S_1) \setminus ((^{@}S_1) \restriction \mathrm{RSub1}\,\forall_x p {+\cdot} (^{@}S_1) \restriction \mathrm{RSub2}(\forall_x p, S_1))$.

(88) $\mathrm{dom}(^{@}\mathrm{RestrictSub}(x, p, S_1))$ misses
$\mathrm{dom}((^{@}S_1) \restriction \mathrm{RSub1}\,p) \cup \mathrm{dom}((^{@}S_1) \restriction \mathrm{RSub2}(p, S_1))$.

(89) If $\langle S, x\rangle$ is quantifiable, then $^{@}((\mathrm{CQCSubAll}(\langle S, x\rangle, x_1))_{\mathbf{2}}) = (^{@}\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)) {+\cdot} (^{@}x_1) \restriction \mathrm{RSub1}\,\forall_x(S_{\mathbf{1}}) {+\cdot} (^{@}x_1) \restriction \mathrm{RSub2}$ $(\forall_x(S_{\mathbf{1}}), x_1)$.

(90) Suppose $\langle S, x\rangle$ is quantifiable. Then there exist v_1, v_3 such that
  (i) for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(\forall_x(S_{\mathbf{1}}))$,
  (ii) for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$,
  (iii) $\mathrm{dom}\,\mathrm{NExVal}(v, S, x, x_1)$ misses $\mathrm{dom}\, v_3$, and
  (iv) $v(\mathrm{ValS}(v, \mathrm{CQCSubAll}(\langle S, x\rangle, x_1))) = v(\mathrm{NExVal}(v, S, x, x_1) {+\cdot} v_1 {+\cdot} v_3)$.

(91) If $\langle S, x\rangle$ is quantifiable, then for every v holds $J, v(\mathrm{NExVal}(v, S, x, x_1)) \models \forall_x(S_{\mathbf{1}})$ iff $J, v(\mathrm{ValS}(v, \mathrm{CQCSubAll}(\langle S, x\rangle, x_1))) \models \mathrm{CQCSubAll}(\langle S, x\rangle, x_1)$.

(92) Suppose $\langle S, x\rangle$ is quantifiable and for every v holds $J, v \models \mathrm{CQCSub}(S)$ iff $J, v(\mathrm{ValS}(v, S)) \models S$. Let given v. Then $J, v \models \mathrm{CQCSub}(\mathrm{CQCSubAll}(\langle S, x\rangle, x_1))$ if and only if $J, v(\mathrm{ValS}(v, \mathrm{CQCSubAll}(\langle S, x\rangle, x_1))) \models \mathrm{CQCSubAll}(\langle S, x\rangle, x_1)$.

The scheme *SubCQCInd1* concerns a unary predicate $\mathcal{P}$, and states that:

For every S holds $\mathcal{P}[S]$

provided the following condition is met:

- Let S, S' be elements of CQC-Sub-WFF, x be a bound variable, S_4 be a second q.-component of $\langle S, x\rangle$, k be a natural number, l_1 be a variables list of k, P be a k-ary predicate symbol, and e be an element of vSUB. Then
  (i) $\mathcal{P}[\mathrm{SubP}(P, l_1, e)]$,
  (ii) if S is sub-verum, then $\mathcal{P}[S]$,
  (iii) if $\mathcal{P}[S]$, then $\mathcal{P}[\mathrm{SubNot}(S)]$,
  (iv) if $S_{\mathbf{2}} = S'_{\mathbf{2}}$ and $\mathcal{P}[S]$ and $\mathcal{P}[S']$, then $\mathcal{P}[\mathrm{CQCSubAnd}(S, S')]$, and
  (v) if $\langle S, x\rangle$ is quantifiable and $\mathcal{P}[S]$, then $\mathcal{P}[\mathrm{CQCSubAll}(\langle S, x\rangle, S_4)]$.

Next we state the proposition

(93) For all S, v holds $J, v \models \mathrm{CQCSub}(S)$ iff $J, v(\mathrm{ValS}(v, S)) \models S$.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Substitution in First-Order Formulas. Part II. The Construction of First-Order Formulas[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349-360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The present article establishes that every substitution can be applied to every formula as in Chapter III par. 8, Definition 8.1, 8.2 of Ebbinghaus, Flum, Thomas. After that, it is observed that substitution doesn't change the number of quantifiers of a formula. Then further details about substitution and some results about the construction of formulas are proven.

MML Identifier: `SUBSTUT2`.

The papers [15], [10], [17], [3], [7], [13], [1], [11], [2], [6], [18], [9], [8], [12], [14], [16], [5], and [4] provide the terminology and notation for this paper.

[1]This research was carried out within the project "Wissensformate" and was financially supported by the Mathematical Institute of the University of Bonn (http://www.-wissensformate.uni-bonn.de). Preparation of the Mizar code was part of the first author's graduate work under the supervision of the second author. The authors thank Jip Veldman for his work on the final version of this article.

ISSN 1426–2630

1. Further Properties of Substitution

For simplicity, we adopt the following convention: i, k, n denote natural numbers, p, q, r, s denote elements of CQC-WFF, x, y denote bound variables, P denotes a k-ary predicate symbol, l, l_1 denote variables lists of k, S_1 denotes a CQC-substitution, and S, S_2 denote elements of CQC-Sub-WFF.

Next we state several propositions:

(1) For every S_1 there exists S such that $S_{\mathbf{1}} = \text{VERUM}$ and $S_{\mathbf{2}} = S_1$.

(2) For every S_1 there exists S such that $S_{\mathbf{1}} = P[l_1]$ and $S_{\mathbf{2}} = S_1$.

(3) Let k, l be natural numbers. Suppose P is a k-ary predicate symbol and a l-ary predicate symbol. Then $k = l$.

(4) If for every S_1 there exists S such that $S_{\mathbf{1}} = p$ and $S_{\mathbf{2}} = S_1$, then for every S_1 there exists S such that $S_{\mathbf{1}} = \neg p$ and $S_{\mathbf{2}} = S_1$.

(5) Suppose for every S_1 there exists S such that $S_{\mathbf{1}} = p$ and $S_{\mathbf{2}} = S_1$ and for every S_1 there exists S such that $S_{\mathbf{1}} = q$ and $S_{\mathbf{2}} = S_1$. Let given S_1. Then there exists S such that $S_{\mathbf{1}} = p \wedge q$ and $S_{\mathbf{2}} = S_1$.

Let us consider p, S_1. Then $\langle p, S_1 \rangle$ is an element of $[\!:\text{WFF}, \text{vSUB}:\!]$.

We now state several propositions:

(6) $\operatorname{dom}\text{RestrictSub}(x, \forall_x p, S_1)$ misses $\{x\}$.

(7) If $x \in \operatorname{rng}\text{RestrictSub}(x, \forall_x p, S_1)$, then $\text{S-Bound}(\langle \forall_x p, S_1 \rangle) = \text{x}_{\text{upVar}(\text{RestrictSub}(x, \forall_x p, S_1), p)}$.

(8) If $x \notin \operatorname{rng}\text{RestrictSub}(x, \forall_x p, S_1)$, then $\text{S-Bound}(\langle \forall_x p, S_1 \rangle) = x$.

(9) $\text{ExpandSub}(x, p, \text{RestrictSub}(x, \forall_x p, S_1)) = ({}^{@}\text{RestrictSub}(x, \forall_x p, S_1)) + \cdot x \restriction \text{S-Bound}(\langle \forall_x p, S_1 \rangle)$.

(10) If $S_{\mathbf{2}} = ({}^{@}\text{RestrictSub}(x, \forall_x p, S_1)) + \cdot x \restriction \text{S-Bound}(\langle \forall_x p, S_1 \rangle)$ and $S_{\mathbf{1}} = p$, then $\langle S, x \rangle$ is quantifiable and there exists S_2 such that $S_2 = \langle \forall_x p, S_1 \rangle$.

(11) If for every S_1 there exists S such that $S_{\mathbf{1}} = p$ and $S_{\mathbf{2}} = S_1$, then for every S_1 there exists S such that $S_{\mathbf{1}} = \forall_x p$ and $S_{\mathbf{2}} = S_1$.

(12) For all p, S_1 there exists S such that $S_{\mathbf{1}} = p$ and $S_{\mathbf{2}} = S_1$.

Let us consider p, S_1. Then $\langle p, S_1 \rangle$ is an element of CQC-Sub-WFF.

Let us consider x, y. The functor $\text{Sbst}(x, y)$ yielding a CQC-substitution is defined by:

(Def. 1) $\text{Sbst}(x, y) = x \dot{\longmapsto} y$.

2. Facts about Substitution and Quantifiers of a Formula

Let us consider p, x, y. The functor $p(x, y)$ yields an element of CQC-WFF and is defined as follows:

(Def. 2) $p(x, y) = \text{CQCSub}(\langle p, \text{Sbst}(x, y) \rangle)$.

In this article we present several logical schemes. The scheme *CQCInd1* concerns a unary predicate $\mathcal{P}$, and states that:

For every p holds $\mathcal{P}[p]$

provided the parameters meet the following conditions:

- For every p such that the number of quantifiers in $p = 0$ holds $\mathcal{P}[p]$, and
- Let given k. Suppose that for every p such that the number of quantifiers in $p = k$ holds $\mathcal{P}[p]$. Let given p. If the number of quantifiers in $p = k + 1$, then $\mathcal{P}[p]$.

The scheme *CQCInd2* concerns a unary predicate $\mathcal{P}$, and states that:

For every p holds $\mathcal{P}[p]$

provided the following conditions are met:

- For every p such that the number of quantifiers in $p \leq 0$ holds $\mathcal{P}[p]$, and
- Let given k. Suppose that for every p such that the number of quantifiers in $p \leq k$ holds $\mathcal{P}[p]$. Let given p. If the number of quantifiers in $p \leq k + 1$, then $\mathcal{P}[p]$.

We now state three propositions:

(13) $\mathrm{VERUM}(x, y) = \mathrm{VERUM}$.

(14) $P[l](x, y) = P[\text{CQC-Subst}(l, \mathrm{Sbst}(x, y))]$ and the number of quantifiers in $P[l]$ = the number of quantifiers in $P[l](x, y)$.

(15) The number of quantifiers in $P[l]$ = the number of quantifiers in $\mathrm{CQCSub}(\langle P[l], S_1\rangle)$.

Let S be an element of QC-Sub-WFF. Then $S_{\mathbf{2}}$ is a CQC-substitution.

Next we state several propositions:

(16) $\langle \neg p, S_1\rangle = \mathrm{SubNot}(\langle p, S_1\rangle)$.

(17)(i) $(\neg p)(x, y) = \neg p(x, y)$, and

(ii) if the number of quantifiers in p = the number of quantifiers in $p(x, y)$, then the number of quantifiers in $\neg p$ = the number of quantifiers in $(\neg p)(x, y)$.

(18) Suppose that for every S_1 holds the number of quantifiers in p = the number of quantifiers in $\mathrm{CQCSub}(\langle p, S_1\rangle)$. Let given S_1. Then the number of quantifiers in $\neg p$ = the number of quantifiers in $\mathrm{CQCSub}(\langle \neg p, S_1\rangle)$.

(19) $\langle p \wedge q, S_1\rangle = \mathrm{CQCSubAnd}(\langle p, S_1\rangle, \langle q, S_1\rangle)$.

(20)(i) $(p \wedge q)(x, y) = p(x, y) \wedge q(x, y)$, and

(ii) if the number of quantifiers in p = the number of quantifiers in $p(x, y)$ and the number of quantifiers in q = the number of quantifiers in $q(x, y)$, then the number of quantifiers in $p \wedge q$ = the number of quantifiers in $(p \wedge q)(x, y)$.

(21) Suppose that

(i) for every S_1 holds the number of quantifiers in p = the number of quantifiers in CQCSub($\langle p, S_1\rangle$), and

(ii) for every S_1 holds the number of quantifiers in q = the number of quantifiers in CQCSub($\langle q, S_1\rangle$).

Let given S_1. Then the number of quantifiers in $p \wedge q$ = the number of quantifiers in CQCSub($\langle p \wedge q, S_1\rangle$).

The function CFQ from CQC-Sub-WFF into vSUB is defined as follows:

(Def. 3) CFQ = QSub $\restriction$ CQC-Sub-WFF .

Let us consider p, x, S_1. The functor QScope(p, x, S_1) yielding a CQC-WFF-like element of [: QC-Sub-WFF, BoundVar :] is defined by:

(Def. 4) QScope(p, x, S_1) = $\langle\langle p$, CFQ($\langle \forall_x p, S_1\rangle$)$\rangle$, $x\rangle$.

Let us consider p, x, S_1. The functor Qsc(p, x, S_1) yielding a second q.-component of QScope(p, x, S_1) is defined by:

(Def. 5) Qsc(p, x, S_1) = S_1.

The following propositions are true:

(22) $\langle \forall_x p, S_1\rangle$ = CQCSubAll(QScope(p, x, S_1), Qsc(p, x, S_1)) and QScope(p, x, S_1) is quantifiable.

(23) Suppose that for every S_1 holds the number of quantifiers in p = the number of quantifiers in CQCSub($\langle p, S_1\rangle$). Let given S_1. Then the number of quantifiers in $\forall_x p$ = the number of quantifiers in CQCSub($\langle \forall_x p, S_1\rangle$).

(24) The number of quantifiers in VERUM = the number of quantifiers in CQCSub($\langle$ VERUM, $S_1\rangle$).

(25) For all p, S_1 holds the number of quantifiers in p = the number of quantifiers in CQCSub($\langle p, S_1\rangle$).

(26) If p is atomic, then there exist k, P, l_1 such that $p = P[l_1]$.

The scheme *CQCInd3* concerns a unary predicate $\mathcal{P}$, and states that:

For every p such that the number of quantifiers in p = 0 holds $\mathcal{P}[p]$

provided the following condition is satisfied:

- Let given r, s, x, k, l be a variables list of k, and P be a k-ary predicate symbol. Then $\mathcal{P}$[VERUM] and $\mathcal{P}[P[l]]$ and if $\mathcal{P}[r]$, then $\mathcal{P}[\neg r]$ and if $\mathcal{P}[r]$ and $\mathcal{P}[s]$, then $\mathcal{P}[r \wedge s]$.

3. Results about the Construction of Formulas

In the sequel F_1, F_2, F_3 denote formulae and L denotes a finite sequence.

Let G, H be formulae. Let us assume that G is a subformula of H. A finite sequence is called a path from G to H if it satisfies the conditions (Def. 6).

(Def. 6)(i) $1 \leq \operatorname{len} \mathrm{it}$,

(ii) $\mathrm{it}(1) = G$,

(iii) $\mathrm{it}(\operatorname{len} \mathrm{it}) = H$, and

(iv) for every k such that $1 \leq k$ and $k < \operatorname{len} \mathrm{it}$ there exist elements G_1, H_1 of WFF such that $\mathrm{it}(k) = G_1$ and $\mathrm{it}(k+1) = H_1$ and G_1 is an immediate constituent of H_1.

The following propositions are true:

(27) Let L be a path from F_1 to F_2. Suppose F_1 is a subformula of F_2 and $1 \leq i$ and $i \leq \operatorname{len} L$. Then there exists F_3 such that $F_3 = L(i)$ and F_3 is a subformula of F_2.

(28) For every path L from F_1 to p such that F_1 is a subformula of p and $1 \leq i$ and $i \leq \operatorname{len} L$ holds $L(i)$ is an element of CQC-WFF.

(29) Let L be a path from q to p. Suppose the number of quantifiers in $p \leq n$ and q is a subformula of p and $1 \leq i$ and $i \leq \operatorname{len} L$. Then there exists r such that $r = L(i)$ and the number of quantifiers in $r \leq n$.

(30) If the number of quantifiers in $p = n$ and q is a subformula of p, then the number of quantifiers in $q \leq n$.

(31) For all n, p such that for every q such that q is a subformula of p holds the number of quantifiers in $q = n$ holds $n = 0$.

(32) Let given p. Suppose that for every q such that q is a subformula of p and for all x, r holds $q \neq \forall_x r$. Then the number of quantifiers in $p = 0$.

(33) Let given p. Suppose that for every q such that q is a subformula of p holds the number of quantifiers in $q \neq 1$. Then the number of quantifiers in $p = 0$.

(34) Suppose $1 \leq$ the number of quantifiers in p. Then there exists q such that q is a subformula of p and the number of quantifiers in $q = 1$.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

A Sequent Calculus for First-Order Logic[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349–360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The present article introduces a sequent calculus for first-order logic. The correctness of this calculus is shown and some important inferences are derived. The contents of this article correspond to Chapter IV of Ebbinghaus, Flum, Thomas.

MML Identifier: CALCUL_1.

The notation and terminology used here are introduced in the following papers: [18], [11], [20], [4], [9], [14], [15], [3], [1], [2], [8], [23], [12], [21], [13], [24], [10], [17], [22], [16], [19], [6], [7], and [5].

1. Preliminaries

For simplicity, we adopt the following rules: a, b, c, d denote sets, i, j, m, n denote natural numbers, p, q, r denote elements of CQC-WFF, x, y denote bound variables, X denotes a subset of CQC-WFF, A denotes a non empty set,

[1]This research was carried out within the project "Wissensformate" and was financially supported by the Mathematical Institute of the University of Bonn (http://www.-wissensformate.uni-bonn.de). Preparation of the Mizar code was part of the first author's graduate work under the supervision of the second author. The authors thank Jip Veldman for his work on the final version of this article.

ISSN 1426–2630

J denotes an interpretation of A, v, w denote elements of $\boldsymbol{V}(A)$, S_1 denotes a CQC-substitution, and f, g denote finite sequences of elements of CQC-WFF.

Let g be a finite sequence and let N be a set. Observe that $g{\restriction}N$ is finite subsequence-like.

Let D be a non empty set and let f be a finite sequence of elements of D. The functor $\mathrm{Ant}(f)$ yields a finite sequence of elements of D and is defined as follows:

(Def. 1)(i) For every i such that $\mathrm{len}\, f = i+1$ holds $\mathrm{Ant}(f) = f{\restriction}\mathrm{Seg}\, i$ if $\mathrm{len}\, f > 0$,

(ii) $\mathrm{Ant}(f) = \emptyset$, otherwise.

Let D be a non empty set and let f be a finite sequence of elements of D. Let us assume that $\mathrm{len}\, f > 0$. The functor $\mathrm{Suc}(f)$ yielding an element of D is defined as follows:

(Def. 2) $\mathrm{Suc}(f) = f(\mathrm{len}\, f)$.

Let D be a non empty set, let p be an element of D, and let f be a finite sequence of elements of D. We say that p is a tail of f if and only if:

(Def. 3) There exists i such that $i \in \mathrm{dom}\, f$ and $f(i) = p$.

Let us consider f, g. We say that f is a subsequence of g if and only if:

(Def. 4) There exists a subset N of $\mathbb{N}$ such that $f \subseteq \mathrm{Seq}(g{\restriction}N)$.

We now state several propositions:

(1) If f is a subsequence of g, then $\mathrm{rng}\, f \subseteq \mathrm{rng}\, g$ and there exists a subset N of $\mathbb{N}$ such that $\mathrm{rng}\, f \subseteq \mathrm{rng}(g{\restriction}N)$.

(2) If $\mathrm{len}\, f > 0$, then $\mathrm{len}\,\mathrm{Ant}(f) + 1 = \mathrm{len}\, f$ and $\mathrm{len}\,\mathrm{Ant}(f) < \mathrm{len}\, f$.

(3) If $\mathrm{len}\, f > 0$, then $f = (\mathrm{Ant}(f)) \frown \langle \mathrm{Suc}(f) \rangle$ and $\mathrm{rng}\, f = \mathrm{rng}\,\mathrm{Ant}(f) \cup \{\mathrm{Suc}(f)\}$.

(4) If $\mathrm{len}\, f > 1$, then $\mathrm{len}\,\mathrm{Ant}(f) > 0$.

(5) $\mathrm{Suc}(f \frown \langle p \rangle) = p$ and $\mathrm{Ant}(f \frown \langle p \rangle) = f$.

In the sequel f_1, f_2 are finite sequences.

We now state several propositions:

(6) $\mathrm{len}\, f_1 \leq \mathrm{len}(f_1 \frown f_2)$ and $\mathrm{len}\, f_2 \leq \mathrm{len}(f_1 \frown f_2)$ and if $f_1 \neq \emptyset$, then $1 \leq \mathrm{len}\, f_1$ and $\mathrm{len}\, f_2 < \mathrm{len}(f_2 \frown f_1)$.

(7) $\mathrm{Seq}((f \frown g){\restriction}\mathrm{dom}\, f) = (f \frown g){\restriction}\mathrm{dom}\, f$.

(8) f is a subsequence of $f \frown g$.

(9) $1 < \mathrm{len}(f_1 \frown \langle b \rangle \frown \langle c \rangle)$.

(10) $1 \leq \mathrm{len}(f_1 \frown \langle b \rangle)$ and $\mathrm{len}(f_1 \frown \langle b \rangle) \in \mathrm{dom}(f_1 \frown \langle b \rangle)$.

(11) If $0 < m$, then $\mathrm{len}\,\mathrm{Sgm}(\mathrm{Seg}\, n \cup \{n+m\}) = n+1$.

(12) If $0 < m$, then $\mathrm{dom}\,\mathrm{Sgm}(\mathrm{Seg}\, n \cup \{n+m\}) = \mathrm{Seg}(n+1)$.

(13) If $0 < \mathrm{len}\, f$, then f is a subsequence of $(\mathrm{Ant}(f)) \frown g \frown \langle \mathrm{Suc}(f) \rangle$.

(14) $1 \in \mathrm{dom}\langle c, d\rangle$ and $2 \in \mathrm{dom}\langle c, d\rangle$ and $(f ^\frown \langle c, d\rangle)(\mathrm{len}\, f + 1) = c$ and $(f ^\frown \langle c, d\rangle)(\mathrm{len}\, f + 2) = d$.

2. A Sequent Calculus

Let us consider f. The functor $\mathrm{snb}(f)$ yielding an element of 2^{BoundVar} is defined by:

(Def. 5) $a \in \mathrm{snb}(f)$ iff there exist i, p such that $i \in \mathrm{dom}\, f$ and $p = f(i)$ and $a \in \mathrm{snb}(p)$.

The set of CQC-WFF-sequences is defined as follows:

(Def. 6) $a \in$ the set of CQC-WFF-sequences iff a is a finite sequence of elements of CQC-WFF.

In the sequel P_1, P_2 denote finite sequences of elements of [: the set of CQC-WFF-sequences, $\mathbb{K}$:].

Let us consider P_1 and let n be a natural number. We say that step n in P_1 is correct if and only if:

(Def. 7)(i) There exists f such that $\mathrm{Suc}(f)$ is a tail of $\mathrm{Ant}(f)$ and $P_1(n)_{\mathbf{1}} = f$ if $P_1(n)_{\mathbf{2}} = 0$,

(ii) there exists f such that $P_1(n)_{\mathbf{1}} = f ^\frown \langle \mathrm{VERUM}\rangle$ if $P_1(n)_{\mathbf{2}} = 1$,

(iii) there exist i, f, g such that $1 \leq i$ and $i < n$ and $\mathrm{Ant}(f)$ is a subsequence of $\mathrm{Ant}(g)$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $P_1(i)_{\mathbf{1}} = f$ and $P_1(n)_{\mathbf{1}} = g$ if $P_1(n)_{\mathbf{2}} = 2$,

(iv) there exist i, j, f, g such that $1 \leq i$ and $i < n$ and $1 \leq j$ and $j < i$ and $\mathrm{len}\, f > 1$ and $\mathrm{len}\, g > 1$ and $\mathrm{Ant}(\mathrm{Ant}(f)) = \mathrm{Ant}(\mathrm{Ant}(g))$ and $\neg \mathrm{Suc}(\mathrm{Ant}(f)) = \mathrm{Suc}(\mathrm{Ant}(g))$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $f = P_1(j)_{\mathbf{1}}$ and $g = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(\mathrm{Ant}(f))) ^\frown \langle \mathrm{Suc}(f)\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 3$,

(v) there exist i, j, f, g, p such that $1 \leq i$ and $i < n$ and $1 \leq j$ and $j < i$ and $\mathrm{len}\, f > 1$ and $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\mathrm{Suc}(\mathrm{Ant}(f)) = \neg p$ and $\neg \mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $f = P_1(j)_{\mathbf{1}}$ and $g = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(\mathrm{Ant}(f))) ^\frown \langle p\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 4$,

(vi) there exist i, j, f, g such that $1 \leq i$ and $i < n$ and $1 \leq j$ and $j < i$ and $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $f = P_1(j)_{\mathbf{1}}$ and $g = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(f)) ^\frown \langle \mathrm{Suc}(f) \wedge \mathrm{Suc}(g)\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 5$,

(vii) there exist i, f, p, q such that $1 \leq i$ and $i < n$ and $p \wedge q = \mathrm{Suc}(f)$ and $f = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(f)) ^\frown \langle p\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 6$,

(viii) there exist i, f, p, q such that $1 \leq i$ and $i < n$ and $p \wedge q = \mathrm{Suc}(f)$ and $f = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(f)) ^\frown \langle q\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 7$,

(ix) there exist i, f, p, x, y such that $1 \leq i$ and $i < n$ and $\mathrm{Suc}(f) = \forall_x p$ and $f = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(f)) ^\frown \langle p(x, y)\rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 8$,

(x) there exist i, f, p, x, y such that $1 \leq i$ and $i < n$ and $\mathrm{Suc}(f) = p(x, y)$ and $y \notin \mathrm{snb}(\mathrm{Ant}(f))$ and $y \notin \mathrm{snb}(\forall_x p)$ and $f = P_1(i)_{\mathbf{1}}$ and $(\mathrm{Ant}(f)) \frown \langle \forall_x p \rangle = P_1(n)_{\mathbf{1}}$ if $P_1(n)_{\mathbf{2}} = 9$.

Let us consider P_1. We say that P_1 is a formal proof if and only if:

(Def. 8) $P_1 \neq \emptyset$ and for every n such that $1 \leq n$ and $n \leq \mathrm{len}\, P_1$ holds step n in P_1 is correct.

Let us consider f. The predicate $\vdash f$ is defined by:

(Def. 9) There exists P_1 such that P_1 is a formal proof and $f = P_1(\mathrm{len}\, P_1)_{\mathbf{1}}$.

Let us consider p, X. We say that p is formally provable from X if and only if:

(Def. 10) There exists f such that $\mathrm{rng}\, \mathrm{Ant}(f) \subseteq X$ and $\mathrm{Suc}(f) = p$ and $\vdash f$.

Let us consider X, let us consider A, let us consider J, and let us consider v. The predicate $J, v \models X$ is defined as follows:

(Def. 11) If $p \in X$, then $J, v \models p$.

Let us consider X, p. The predicate $X \models p$ is defined as follows:

(Def. 12) If $J, v \models X$, then $J, v \models p$.

Let us consider p. The predicate $\vDash p$ is defined as follows:

(Def. 13) $\emptyset_{\text{CQC-WFF}} \models p$.

Let us consider f, A, J, v. The predicate $J, v \models f$ is defined as follows:

(Def. 14) $J, v \models \mathrm{rng}\, f$.

Let us consider f, p. The predicate $f \models p$ is defined by:

(Def. 15) If $J, v \models f$, then $J, v \models p$.

One can prove the following propositions:

(15) If $\mathrm{Suc}(f)$ is a tail of $\mathrm{Ant}(f)$, then $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$.

(16) If $\mathrm{Ant}(f)$ is a subsequence of $\mathrm{Ant}(g)$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$, then $\mathrm{Ant}(g) \models \mathrm{Suc}(g)$.

(17) If $\mathrm{len}\, f > 0$, then $J, v \models \mathrm{Ant}(f)$ and $J, v \models \mathrm{Suc}(f)$ iff $J, v \models f$.

(18) If $\mathrm{len}\, f > 1$ and $\mathrm{len}\, g > 1$ and $\mathrm{Ant}(\mathrm{Ant}(f)) = \mathrm{Ant}(\mathrm{Ant}(g))$ and $\neg \mathrm{Suc}(\mathrm{Ant}(f)) = \mathrm{Suc}(\mathrm{Ant}(g))$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$ and $\mathrm{Ant}(g) \models \mathrm{Suc}(g)$, then $\mathrm{Ant}(\mathrm{Ant}(f)) \models \mathrm{Suc}(f)$.

(19) If $\mathrm{len}\, f > 1$ and $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\neg p = \mathrm{Suc}(\mathrm{Ant}(f))$ and $\neg \mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$ and $\mathrm{Ant}(g) \models \mathrm{Suc}(g)$, then $\mathrm{Ant}(\mathrm{Ant}(f)) \models p$.

(20) If $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$ and $\mathrm{Ant}(g) \models \mathrm{Suc}(g)$, then $\mathrm{Ant}(f) \models \mathrm{Suc}(f) \wedge \mathrm{Suc}(g)$.

(21) If $\mathrm{Suc}(f) = p \wedge q$ and $\mathrm{Ant}(f) \models p \wedge q$, then $\mathrm{Ant}(f) \models p$.

(22) If $\mathrm{Suc}(f) = p \wedge q$ and $\mathrm{Ant}(f) \models p \wedge q$, then $\mathrm{Ant}(f) \models q$.

(23) $J, v \models \langle p, S_1 \rangle$ iff $J, v \models p$.

In the sequel a is an element of A.

We now state several propositions:

(24) $J, v \models p(x,\, y)$ iff there exists a such that $v(y) = a$ and $J, v(x{\restriction}a) \models p$.

(25) If $\mathrm{Suc}(f) = \forall_x p$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$, then for every y holds $\mathrm{Ant}(f) \models p(x,\, y)$.

(26) For every set X such that $X \subseteq \mathrm{BoundVar}$ holds if $x \notin X$, then $v(x{\restriction}a){\restriction}X = v{\restriction}X$.

(27) For all v, w such that $v{\restriction}\mathrm{snb}(f) = w{\restriction}\mathrm{snb}(f)$ holds $J, v \models f$ iff $J, w \models f$.

(28) If $y \notin \mathrm{snb}(\forall_x p)$, then $v(y{\restriction}a)(x{\restriction}a){\restriction}\mathrm{snb}(p) = v(x{\restriction}a){\restriction}\mathrm{snb}(p)$.

(29) If $\mathrm{Suc}(f) = p(x,\, y)$ and $\mathrm{Ant}(f) \models \mathrm{Suc}(f)$ and $y \notin \mathrm{snb}(\mathrm{Ant}(f))$ and $y \notin \mathrm{snb}(\forall_x p)$, then $\mathrm{Ant}(f) \models \forall_x p$.

(30) $\mathrm{Ant}(f \mathbin{^\frown} \langle \mathrm{VERUM} \rangle) \models \mathrm{Suc}(f \mathbin{^\frown} \langle \mathrm{VERUM} \rangle)$.

(31) Suppose $1 \leq n$ and $n \leq \mathrm{len}\, P_1$. Then $P_1(n)_{\mathbf{2}} = 0$ or $P_1(n)_{\mathbf{2}} = 1$ or $P_1(n)_{\mathbf{2}} = 2$ or $P_1(n)_{\mathbf{2}} = 3$ or $P_1(n)_{\mathbf{2}} = 4$ or $P_1(n)_{\mathbf{2}} = 5$ or $P_1(n)_{\mathbf{2}} = 6$ or $P_1(n)_{\mathbf{2}} = 7$ or $P_1(n)_{\mathbf{2}} = 8$ or $P_1(n)_{\mathbf{2}} = 9$.

(32) If p is formally provable from X, then $X \models p$.

3. Derived Rules

Next we state a number of propositions:

(33) If $\mathrm{Suc}(f)$ is a tail of $\mathrm{Ant}(f)$, then $\vdash f$.

(34) If $1 \leq n$ and $n \leq \mathrm{len}\, P_1$, then step n in P_1 is correct iff step n in $P_1 \mathbin{^\frown} P_2$ is correct.

(35) If $1 \leq n$ and $n \leq \mathrm{len}\, P_2$ and step n in P_2 is correct, then step $n + \mathrm{len}\, P_1$ in $P_1 \mathbin{^\frown} P_2$ is correct.

(36) If $\mathrm{Ant}(f)$ is a subsequence of $\mathrm{Ant}(g)$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\vdash f$, then $\vdash g$.

(37) If $1 < \mathrm{len}\, f$ and $1 < \mathrm{len}\, g$ and $\mathrm{Ant}(\mathrm{Ant}(f)) = \mathrm{Ant}(\mathrm{Ant}(g))$ and $\neg\, \mathrm{Suc}(\mathrm{Ant}(f)) = \mathrm{Suc}(\mathrm{Ant}(g))$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\vdash f$ and $\vdash g$, then $\vdash (\mathrm{Ant}(\mathrm{Ant}(f))) \mathbin{^\frown} \langle \mathrm{Suc}(f) \rangle$.

(38) If $\mathrm{len}\, f > 1$ and $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\mathrm{Suc}(\mathrm{Ant}(f)) = \neg p$ and $\neg\, \mathrm{Suc}(f) = \mathrm{Suc}(g)$ and $\vdash f$ and $\vdash g$, then $\vdash (\mathrm{Ant}(\mathrm{Ant}(f))) \mathbin{^\frown} \langle p \rangle$.

(39) If $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\vdash f$ and $\vdash g$, then $\vdash (\mathrm{Ant}(f)) \mathbin{^\frown} \langle \mathrm{Suc}(f) \wedge \mathrm{Suc}(g) \rangle$.

(40) If $p \wedge q = \mathrm{Suc}(f)$ and $\vdash f$, then $\vdash (\mathrm{Ant}(f)) \mathbin{^\frown} \langle p \rangle$.

(41) If $p \wedge q = \mathrm{Suc}(f)$ and $\vdash f$, then $\vdash (\mathrm{Ant}(f)) \mathbin{^\frown} \langle q \rangle$.

(42) If $\mathrm{Suc}(f) = \forall_x p$ and $\vdash f$, then $\vdash (\mathrm{Ant}(f)) \mathbin{^\frown} \langle p(x,\, y) \rangle$.

(43) If $\mathrm{Suc}(f) = p(x,\, y)$ and $y \notin \mathrm{snb}(\mathrm{Ant}(f))$ and $y \notin \mathrm{snb}(\forall_x p)$ and $\vdash f$, then $\vdash (\mathrm{Ant}(f)) \mathbin{^\frown} \langle \forall_x p \rangle$.

(44) If $\vdash f$ and $\vdash (\mathrm{Ant}(f)) \frown \langle \neg\, \mathrm{Suc}(f) \rangle$, then $\vdash (\mathrm{Ant}(f)) \frown \langle p \rangle$.

(45) If $1 \leq \mathrm{len}\, f$ and $\vdash f$ and $\vdash f \frown \langle p \rangle$, then $\vdash (\mathrm{Ant}(f)) \frown \langle p \rangle$.

(46) If $\vdash f \frown \langle p \rangle \frown \langle q \rangle$, then $\vdash f \frown \langle \neg q \rangle \frown \langle \neg p \rangle$.

(47) If $\vdash f \frown \langle \neg p \rangle \frown \langle \neg q \rangle$, then $\vdash f \frown \langle q \rangle \frown \langle p \rangle$.

(48) If $\vdash f \frown \langle \neg p \rangle \frown \langle q \rangle$, then $\vdash f \frown \langle \neg q \rangle \frown \langle p \rangle$.

(49) If $\vdash f \frown \langle p \rangle \frown \langle \neg q \rangle$, then $\vdash f \frown \langle q \rangle \frown \langle \neg p \rangle$.

(50) If $\vdash f \frown \langle p \rangle \frown \langle r \rangle$ and $\vdash f \frown \langle q \rangle \frown \langle r \rangle$, then $\vdash f \frown \langle p \vee q \rangle \frown \langle r \rangle$.

(51) If $\vdash f \frown \langle p \rangle$, then $\vdash f \frown \langle p \vee q \rangle$.

(52) If $\vdash f \frown \langle q \rangle$, then $\vdash f \frown \langle p \vee q \rangle$.

(53) If $\vdash f \frown \langle p \rangle \frown \langle r \rangle$ and $\vdash f \frown \langle q \rangle \frown \langle r \rangle$, then $\vdash f \frown \langle p \vee q \rangle \frown \langle r \rangle$.

(54) If $\vdash f \frown \langle p \rangle$, then $\vdash f \frown \langle \neg\neg p \rangle$.

(55) If $\vdash f \frown \langle \neg\neg p \rangle$, then $\vdash f \frown \langle p \rangle$.

(56) If $\vdash f \frown \langle p \Rightarrow q \rangle$ and $\vdash f \frown \langle p \rangle$, then $\vdash f \frown \langle q \rangle$.

(57) $(\neg p)(x, y) = \neg p(x, y)$.

(58) If there exists y such that $\vdash f \frown \langle p(x, y) \rangle$, then $\vdash f \frown \langle \exists_x p \rangle$.

(59) $\mathrm{snb}(f \frown g) = \mathrm{snb}(f) \cup \mathrm{snb}(g)$.

(60) $\mathrm{snb}(\langle p \rangle) = \mathrm{snb}(p)$.

(61) If $\vdash f \frown \langle p(x, y) \rangle \frown \langle q \rangle$ and $y \notin \mathrm{snb}(f \frown \langle \exists_x p \rangle \frown \langle q \rangle)$, then $\vdash f \frown \langle \exists_x p \rangle \frown \langle q \rangle$.

(62) $\mathrm{snb}(f) = \bigcup\{\mathrm{snb}(p) : \bigvee_i\ (i \in \mathrm{dom}\, f\ \wedge\ p = f(i))\}$.

(63) $\mathrm{snb}(f)$ is finite.

(64) $\overline{\overline{\mathrm{BoundVar}}} = \aleph_0$ and BoundVar is not finite.

(65) There exists x such that $x \notin \mathrm{snb}(f)$.

(66) If $\vdash f \frown \langle \forall_x p \rangle$, then $\vdash f \frown \langle \forall_x \neg\neg p \rangle$.

(67) If $\vdash f \frown \langle \forall_x \neg\neg p \rangle$, then $\vdash f \frown \langle \forall_x p \rangle$.

(68) $\vdash f \frown \langle \forall_x p \rangle$ iff $\vdash f \frown \langle \neg \exists_x \neg p \rangle$.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Consequences of the Sequent Calculus[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349-360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The first main result of the present article is that the derivablility of a sequent doesn't depend on the ordering of the antecedent. The second main result says: if a sequent is derivable, then the formulas in the antecendent only need to occur once.

MML Identifier: `CALCUL_2`.

The articles [15], [16], [3], [14], [4], [1], [2], [17], [10], [6], [8], [13], [12], [9], [18], [11], [5], and [7] provide the terminology and notation for this paper.

1. f is a Subsequence of g^f

For simplicity, we adopt the following convention: p, q denote elements of CQC-WFF, k, m, n, i denote natural numbers, f, g denote finite sequences of elements of CQC-WFF, and a, b, b_1, b_2, c denote natural numbers.

Let m, n be natural numbers. The functor $\mathrm{seq}(m, n)$ yielding a set is defined as follows:

[1]This research was carried out within the project "Wissensformate" and was financially supported by the Mathematical Institute of the University of Bonn (http://www.-wissensformate.uni-bonn.de). Preparation of the Mizar code was part of the first author's graduate work under the supervision of the second author. The authors thank Jip Veldman for his work on the final version of this article.

ISSN 1426–2630

(Def. 1) $\operatorname{seq}(m, n) = \{k : 1 + m \leq k \ \wedge \ k \leq n + m\}$.

Let m, n be natural numbers. Then $\operatorname{seq}(m, n)$ is a subset of $\mathbb{N}$.

One can prove the following propositions:

(1) $c \in \operatorname{seq}(a, b)$ iff $1 + a \leq c$ and $c \leq b + a$.

(2) $\operatorname{seq}(a, 0) = \emptyset$.

(3) $b = 0$ or $b + a \in \operatorname{seq}(a, b)$.

(4) $b_1 \leq b_2$ iff $\operatorname{seq}(a, b_1) \subseteq \operatorname{seq}(a, b_2)$.

(5) $\operatorname{seq}(a, b) \cup \{a + b + 1\} = \operatorname{seq}(a, b + 1)$.

(6) $\operatorname{seq}(m, n) \approx n$.

Let us consider m, n. Observe that $\operatorname{seq}(m, n)$ is finite.

Let us consider f. Observe that $\operatorname{len} f$ is finite.

Next we state a number of propositions:

(7) $\operatorname{seq}(m, n) \subseteq \operatorname{Seg}(m + n)$.

(8) $\operatorname{Seg} n$ misses $\operatorname{seq}(n, m)$.

(9) For all finite sequences f, g holds $\operatorname{Seg} \operatorname{len}(f \frown g) = \operatorname{Seg} \operatorname{len} f \cup \operatorname{seq}(\operatorname{len} f, \operatorname{len} g)$.

(10) $\operatorname{len} \operatorname{Sgm} \operatorname{seq}(\operatorname{len} g, \operatorname{len} f) = \operatorname{len} f$.

(11) $\operatorname{dom} \operatorname{Sgm} \operatorname{seq}(\operatorname{len} g, \operatorname{len} f) = \operatorname{dom} f$.

(12) $\operatorname{rng} \operatorname{Sgm} \operatorname{seq}(\operatorname{len} g, \operatorname{len} f) = \operatorname{seq}(\operatorname{len} g, \operatorname{len} f)$.

(13) If $i \in \operatorname{dom} \operatorname{Sgm} \operatorname{seq}(\operatorname{len} g, \operatorname{len} f)$, then $(\operatorname{Sgm} \operatorname{seq}(\operatorname{len} g, \operatorname{len} f))(i) = \operatorname{len} g + i$.

(14) $\operatorname{seq}(\operatorname{len} g, \operatorname{len} f) \subseteq \operatorname{dom}(g \frown f)$.

(15) $\operatorname{dom}((g \frown f) \restriction \operatorname{seq}(\operatorname{len} g, \operatorname{len} f)) = \operatorname{seq}(\operatorname{len} g, \operatorname{len} f)$.

(16) $\operatorname{Seq}((g \frown f) \restriction \operatorname{seq}(\operatorname{len} g, \operatorname{len} f)) = \operatorname{Sgm} \operatorname{seq}(\operatorname{len} g, \operatorname{len} f) \cdot (g \frown f)$.

(17) $\operatorname{dom} \operatorname{Seq}((g \frown f) \restriction \operatorname{seq}(\operatorname{len} g, \operatorname{len} f)) = \operatorname{dom} f$.

(18) f is a subsequence of $g \frown f$.

Let D be a non empty set, let f be a finite sequence of elements of D, and let P be a permutation of $\operatorname{dom} f$. The functor $\operatorname{Per}(f, P)$ yielding a finite sequence of elements of D is defined as follows:

(Def. 2) $\operatorname{Per}(f, P) = P \cdot f$.

In the sequel P denotes a permutation of $\operatorname{dom} f$.

The following propositions are true:

(19) $\operatorname{dom} \operatorname{Per}(f, P) = \operatorname{dom} f$.

(20) If $\vdash f \frown \langle p \rangle$, then $\vdash g \frown f \frown \langle p \rangle$.

2. The Ordering of the Antecedent is Irrelevant

Let us consider f. The functor $\mathrm{Begin}(f)$ yielding an element of CQC-WFF is defined by:

(Def. 3) $\mathrm{Begin}(f) = \begin{cases} f(1), & \text{if } 1 \leq \mathrm{len}\, f, \\ \mathrm{VERUM}, & \text{otherwise.} \end{cases}$

Let us consider f. Let us assume that $1 \leq \mathrm{len}\, f$. The functor $\mathrm{Impl}(f)$ yields an element of CQC-WFF and is defined by the condition (Def. 4).

(Def. 4) There exists a finite sequence F of elements of CQC-WFF such that

(i) $\mathrm{Impl}(f) = F(\mathrm{len}\, f)$,

(ii) $\mathrm{len}\, F = \mathrm{len}\, f$,

(iii) $F(1) = \mathrm{Begin}(f)$ or $\mathrm{len}\, f = 0$, and

(iv) for every n such that $1 \leq n$ and $n < \mathrm{len}\, f$ there exist p, q such that $p = f(n+1)$ and $q = F(n)$ and $F(n+1) = p \Rightarrow q$.

We now state a number of propositions:

(21) $\vdash f \frown \langle p \rangle \frown \langle p \rangle$.

(22) If $\vdash f \frown \langle p \wedge q \rangle$, then $\vdash f \frown \langle p \rangle$.

(23) If $\vdash f \frown \langle p \wedge q \rangle$, then $\vdash f \frown \langle q \rangle$.

(24) If $\vdash f \frown \langle p \rangle$ and $\vdash f \frown \langle p \rangle \frown \langle q \rangle$, then $\vdash f \frown \langle q \rangle$.

(25) If $\vdash f \frown \langle p \rangle$ and $\vdash f \frown \langle \neg p \rangle$, then $\vdash f \frown \langle q \rangle$.

(26) If $\vdash f \frown \langle p \rangle \frown \langle q \rangle$ and $\vdash f \frown \langle \neg p \rangle \frown \langle q \rangle$, then $\vdash f \frown \langle q \rangle$.

(27) If $\vdash f \frown \langle p \rangle \frown \langle q \rangle$, then $\vdash f \frown \langle p \Rightarrow q \rangle$.

(28) If $1 \leq \mathrm{len}\, g$ and $\vdash f \frown g$, then $\vdash f \frown \langle \mathrm{Impl}(\mathrm{Rev}(g)) \rangle$.

(29) If $\vdash (\mathrm{Per}(f, P)) \frown \langle \mathrm{Impl}(\mathrm{Rev}(f \frown \langle p \rangle)) \rangle$, then $\vdash (\mathrm{Per}(f, P)) \frown \langle p \rangle$.

(30) If $\vdash f \frown \langle p \rangle$, then $\vdash (\mathrm{Per}(f, P)) \frown \langle p \rangle$.

3. Multiple Occurrence in the Antecedent is Irrelevant

Let us consider n and let c be a set. We introduce $\mathrm{IdFinS}(c, n)$ as a synonym of $n \mapsto c$.

We now state the proposition

(31) For every set c such that $1 \leq n$ holds $\mathrm{rng}\, \mathrm{IdFinS}(c, n) = \mathrm{rng} \langle c \rangle$.

Let D be a non empty set, let n be a natural number, and let p be an element of D. Then $\mathrm{IdFinS}(p, n)$ is a finite sequence of elements of D.

The following proposition is true

(32) If $1 \leq n$ and $\vdash f \frown \mathrm{IdFinS}(p, n) \frown \langle q \rangle$, then $\vdash f \frown \langle p \rangle \frown \langle q \rangle$.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Equivalences of Inconsistency and Henkin Models[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349–360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag, New York Inc. The present article establishes some equivalences of inconsistency. It is proved that a countable union of consistent sets is consistent. Then the concept of a Henkin model is introduced. The contents of this article correspond to Chapter IV, par. 7 and Chapter V, par. 1 of Ebbinghaus, Flum, Thomas.

MML Identifier: `HENMODEL`.

The articles [17], [9], [19], [5], [22], [7], [2], [4], [13], [6], [11], [20], [10], [23], [8], [16], [1], [21], [12], [15], [18], [14], and [3] provide the notation and terminology for this paper.

1. Preliminaries and Equivalences of Inconsistency

For simplicity, we use the following convention: a denotes a set, X, Y denote subsets of CQC-WFF, k, m, n denote natural numbers, p, q denote elements of

[1]This research was carried out within the project "Wissensformate" and was financially supported by the Mathematical Institute of the University of Bonn (http://www.-wissensformate.uni-bonn.de). Preparation of the Mizar code was part of the first author's graduate work under the supervision of the second author. The authors thank Jip Veldman for his work on the final version of this article.

ISSN 1426–2630

CQC-WFF, P denotes a k-ary predicate symbol, l_1 denotes a variables list of k, and f, g denote finite sequences of elements of CQC-WFF.

Let D be a non empty set and let X be a subset of 2^D. Then $\bigcup X$ is a subset of D.

In the sequel A is a non empty finite subset of $\mathbb{N}$.

The following two propositions are true:

(1) Let f be a function from n into A. Suppose there exists m such that $\operatorname{succ} m = n$ and f is one-to-one and $\operatorname{rng} f = A$ and for all n, m such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $n < m$ holds $f(n) \in f(m)$. Then $f(\bigcup n) = \bigcup \operatorname{rng} f$.

(2) $\bigcup A \in A$ and for every a such that $a \in A$ holds $a \in \bigcup A$ or $a = \bigcup A$.

Let A be a set. The functor $\min^* A$ yielding a natural number is defined by:

(Def. 1)(i) $\min^* A \in A$ and for every k such that $k \in A$ holds $\min^* A \leq k$ if A is a non empty subset of $\mathbb{N}$,

(ii) $\min^* A = 0$, otherwise.

In the sequel C denotes a non empty set.

Next we state the proposition

(3) Let f be a function from $\mathbb{N}$ into C and X be a finite set. Suppose for all n, m such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $n < m$ holds $f(n) \subseteq f(m)$ and $X \subseteq \bigcup \operatorname{rng} f$. Then there exists k such that $X \subseteq f(k)$.

Let us consider X, p. The predicate $X \vdash p$ is defined as follows:

(Def. 2) There exists f such that $\operatorname{rng} f \subseteq X$ and $\vdash f \frown \langle p \rangle$.

Let us consider X. We say that X is consistent if and only if:

(Def. 3) For every p holds $X \nvdash p$ or $X \nvdash \neg p$.

Let us consider X. We introduce X is inconsistent as an antonym of X is consistent.

Let f be a finite sequence of elements of CQC-WFF. We say that f is consistent if and only if:

(Def. 4) For every p holds $\nvdash f \frown \langle p \rangle$ or $\nvdash f \frown \langle \neg p \rangle$.

Let f be a finite sequence of elements of CQC-WFF. We introduce f is inconsistent as an antonym of f is consistent.

Next we state several propositions:

(4) If X is consistent and $\operatorname{rng} g \subseteq X$, then g is consistent.

(5) If $\vdash f \frown \langle p \rangle$, then $\vdash f \frown g \frown \langle p \rangle$.

(6) X is inconsistent iff for every p holds $X \vdash p$.

(7) If X is inconsistent, then there exists Y such that $Y \subseteq X$ and Y is finite and inconsistent.

(8) If $X \cup \{p\} \vdash q$, then there exists g such that $\operatorname{rng} g \subseteq X$ and $\vdash g \frown \langle p \rangle \frown \langle q \rangle$.

(9) $X \vdash p$ iff $X \cup \{\neg p\}$ is inconsistent.

(10) $X \vdash \neg p$ iff $X \cup \{p\}$ is inconsistent.

2. Unions of Consistent Sets

We now state the proposition

(11) Let f be a function from $\mathbb{N}$ into $2^{\text{CQC-WFF}}$. Suppose that for all n, m such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $n < m$ holds $f(n)$ is consistent and $f(n) \subseteq f(m)$. Then $\bigcup \operatorname{rng} f$ is consistent.

3. Construction of a Henkin Model

In the sequel A is a non empty set, v is an element of $\boldsymbol{V}(A)$, and J is an interpretation of A.

We now state two propositions:

(12) If X is inconsistent, then for all J, v holds $J, v \not\models X$.

(13) $\{\text{VERUM}\}$ is consistent.

Let us observe that there exists a subset of CQC-WFF which is consistent.

In the sequel C_1 denotes a consistent subset of CQC-WFF.

The non empty set HCar is defined by:

(Def. 5) $\text{HCar} = \text{BoundVar}$.

Let P be an element of PredSym and let l_1 be a variables list of $\text{Arity}(P)$. Then $P[l_1]$ is an element of CQC-WFF.

Let us consider C_1. An interpretation of HCar is said to be a Henkin interpretation of C_1 if it satisfies the condition (Def. 6).

(Def. 6) Let P be an element of PredSym and r be an element of $\text{Rel}(\text{HCar})$. Suppose $\text{it}(P) = r$. Let given a. Then $a \in r$ if and only if there exists a variables list l_1 of $\text{Arity}(P)$ such that $a = l_1$ and $C_1 \vdash P[l_1]$.

The element valH of $\boldsymbol{V}(\text{HCar})$ is defined as follows:

(Def. 7) $\text{valH} = \text{id}_{\text{BoundVar}}$.

4. Some Properties of the Henkin Model

In the sequel J_1 is a Henkin interpretation of C_1.

We now state four propositions:

(14) $\text{valH} * l_1 = l_1$.

(15) $\vdash f \frown \langle \text{VERUM} \rangle$.

(16) $J_1, \text{valH} \models \text{VERUM}$ iff $C_1 \vdash \text{VERUM}$.

(17) $J_1, \text{valH} \models P[l_1]$ iff $C_1 \vdash P[l_1]$.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Gödel's Completeness Theorem[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349–360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The present article contains the proof of a simplified completeness theorem for a countable relational language without equality.

MML Identifier: GOEDELCP.

The notation and terminology used in this paper are introduced in the following articles: [19], [13], [21], [2], [4], [11], [16], [1], [17], [10], [23], [14], [22], [24], [12], [15], [18], [20], [3], [8], [5], [9], [7], and [6].

1. Henkin's Theorem

For simplicity, we adopt the following convention: X, Y denote subsets of CQC-WFF, n denotes a natural number, p, q denote elements of CQC-WFF, x, y denote bound variables, A denotes a non empty set, J denotes an interpretation of A, v denotes an element of $\boldsymbol{V}(A)$, f_1 denotes a finite sequence of

[1]This research was carried out within the project "Wissensformate" and was financially supported by the Mathematical Institute of the University of Bonn (http://www.-wissensformate.uni-bonn.de). Preparation of the Mizar code was part of the first author's graduate work under the supervision of the second author. The authors thank Jip Veldman for his work on the final version of this article.

ISSN 1426–2630

elements of CQC-WFF, C_1, C_2, C_3 denote consistent subsets of CQC-WFF, J_1 denotes a Henkin interpretation of C_1, and a denotes an element of A.

Let us consider X. We say that X is negation faithful if and only if:

(Def. 1) $X \vdash p$ or $X \vdash \neg p$.

Let us consider X. We say that X has examples if and only if:

(Def. 2) For all x, p there exists y such that $X \vdash \neg \exists_x p \vee p(x, y)$.

One can prove the following propositions:

(1) If C_1 is negation faithful, then $C_1 \vdash p$ iff $C_1 \nvdash \neg p$.

(2) For every finite sequence f of elements of CQC-WFF such that $\vdash f \frown \langle \neg p \vee q \rangle$ and $\vdash f \frown \langle p \rangle$ holds $\vdash f \frown \langle q \rangle$.

(3) If X has examples, then $X \vdash \exists_x p$ iff there exists y such that $X \vdash p(x, y)$.

(4) Suppose if C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$. Suppose C_1 is negation faithful and has examples. Then $J_1, \mathrm{valH} \models \neg p$ if and only if $C_1 \vdash \neg p$.

(5) If $\vdash f_1 \frown \langle p \rangle$ and $\vdash f_1 \frown \langle q \rangle$, then $\vdash f_1 \frown \langle p \wedge q \rangle$.

(6) $X \vdash p$ and $X \vdash q$ iff $X \vdash p \wedge q$.

(7) Suppose that

(i) if C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$, and

(ii) if C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models q$ iff $C_1 \vdash q$.

Suppose C_1 is negation faithful and has examples. Then $J_1, \mathrm{valH} \models p \wedge q$ if and only if $C_1 \vdash p \wedge q$.

(8) Let given p. Suppose the number of quantifiers in $p \leq 0$. If C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$.

(9) $J, v \models \exists_x p$ iff there exists a such that $J, v(x{\restriction}a) \models p$.

(10) $J_1, \mathrm{valH} \models \exists_x p$ iff there exists y such that $J_1, \mathrm{valH} \models p(x, y)$.

(11) $J, v \models \neg \exists_x \neg p$ iff $J, v \models \forall_x p$.

(12) $X \vdash \neg \exists_x \neg p$ iff $X \vdash \forall_x p$.

(13) The number of quantifiers in $\exists_x p =$ (the number of quantifiers in p) $+ 1$.

(14) The number of quantifiers in $p =$ the number of quantifiers in $p(x, y)$.

In the sequel a denotes a set.

The following three propositions are true:

(15) Let given p. Suppose the number of quantifiers in $p = 1$. If C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$.

(16) Let given n. Suppose that for every p such that the number of quantifiers in $p \leq n$ holds if C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$. Let given p. Suppose the number of quantifiers in $p \leq n + 1$. If C_1 is negation faithful and has examples, then $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$.

(17) For every p such that C_1 is negation faithful and has examples holds $J_1, \mathrm{valH} \models p$ iff $C_1 \vdash p$.

2. Satisfiability of Consistent Sets of Formulas with Finitely Many Free Variables

The following proposition is true

(18) WFF is countable.

The subset ExCl of CQC-WFF is defined by:

(Def. 3) $a \in \mathrm{ExCl}$ iff there exist x, p such that $a = \exists_x p$.

The following propositions are true:

(19) CQC-WFF is countable.

(20) ExCl is non empty and ExCl is countable.

Let p be an element of WFF. Let us assume that p is existential. The functor ExBound(p) yielding a bound variable is defined as follows:

(Def. 4) There exists an element q of WFF such that $p = \exists_{\mathrm{ExBound}(p)} q$.

Let p be an element of CQC-WFF. Let us assume that p is existential. The functor ExScope(p) yielding an element of CQC-WFF is defined by:

(Def. 5) There exists x such that $p = \exists_x \mathrm{ExScope}(p)$.

Let F be a function from $\mathbb{N}$ into CQC-WFF and let a be a natural number. The bound in $F(a)$ yields a bound variable and is defined as follows:

(Def. 6) If $p = F(a)$, then the bound in $F(a) = \mathrm{ExBound}(p)$.

Let F be a function from $\mathbb{N}$ into CQC-WFF and let a be a natural number. The scope of $F(a)$ yields an element of CQC-WFF and is defined by:

(Def. 7) If $p = F(a)$, then the scope of $F(a) = \mathrm{ExScope}(p)$.

Let us consider X. The functor $\mathrm{snb}(X)$ yields an element of 2^{BoundVar} and is defined by:

(Def. 8) $\mathrm{snb}(X) = \bigcup\{\mathrm{snb}(p) : p \in X\}$.

Next we state a number of propositions:

(21) If $p \in X$, then $X \vdash p$.

(22) $\mathrm{ExBound}(\exists_x p) = x$ and $\mathrm{ExScope}(\exists_x p) = p$.

(23) $X \vdash \mathrm{VERUM}$.

(24) $X \vdash \neg\mathrm{VERUM}$ iff X is inconsistent.

(25) For all finite sequences f, g of elements of CQC-WFF such that $0 < \mathrm{len}\, f$ and $\vdash f \frown \langle p \rangle$ holds $\vdash (\mathrm{Ant}(f)) \frown g \frown \langle \mathrm{Suc}(f) \rangle \frown \langle p \rangle$.

(26) $\mathrm{snb}(\{p\}) = \mathrm{snb}(p)$.

(27) $\mathrm{snb}(X \cup Y) = \mathrm{snb}(X) \cup \mathrm{snb}(Y)$.

(28) For every element A of 2^{BoundVar} such that A is finite there exists x such that $x \notin A$.

(29) If $X \subseteq Y$, then $\mathrm{snb}(X) \subseteq \mathrm{snb}(Y)$.

(30) For every finite sequence f of elements of CQC-WFF holds $\mathrm{snb}(\mathrm{rng}\, f) = \mathrm{snb}(f)$.

(31) If $\mathrm{snb}(C_1)$ is finite, then there exists C_2 such that $C_1 \subseteq C_2$ and C_2 has examples.

(32) If $X \vdash p$ and $X \subseteq Y$, then $Y \vdash p$.

(33) If C_1 has examples, then there exists C_2 such that $C_1 \subseteq C_2$ and C_2 is negation faithful and has examples.

In the sequel J_2 denotes a Henkin interpretation of C_3, J denotes an interpretation of A, and v denotes an element of $\boldsymbol{V}(A)$.

We now state the proposition

(34) If $\mathrm{snb}(C_1)$ is finite, then there exist C_3, J_2 such that $J_2, \mathrm{valH} \models C_1$.

3. Gödel's Completeness Theorem

We now state four propositions:

(35) If $J, v \models X$ and $Y \subseteq X$, then $J, v \models Y$.

(36) If $\mathrm{snb}(X)$ is finite, then $\mathrm{snb}(X \cup \{p\})$ is finite.

(37) If $X \models p$, then $J, v \not\models X \cup \{\neg p\}$.

(38) If $\mathrm{snb}(X)$ is finite and $X \models p$, then $X \vdash p$.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Propositional Calculus for Boolean Valued Functions. Part VIII

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Summary. In this paper, we proved some elementary propositional calculus formulae for Boolean valued functions.

MML Identifier: `BVFUNC26`.

The articles [5], [6], [8], [7], [9], [1], [4], [3], and [2] provide the notation and terminology for this paper.

In this paper Y denotes a non empty set and a, b, c denote elements of $Boolean^Y$.

Let p, q be boolean-valued functions. The functor $p \,{}'\mathrm{nand}'\, q$ yielding a function is defined as follows:

(Def. 1) $\mathrm{dom}(p \,{}'\mathrm{nand}'\, q) = \mathrm{dom}\, p \cap \mathrm{dom}\, q$ and for every set x such that $x \in \mathrm{dom}(p \,{}'\mathrm{nand}'\, q)$ holds $(p \,{}'\mathrm{nand}'\, q)(x) = p(x) \,{}'\mathrm{nand}'\, q(x)$.

Let us observe that the functor $p \,{}'\mathrm{nand}'\, q$ is commutative. The functor $p \,{}'\mathrm{nor}'\, q$ yielding a function is defined as follows:

(Def. 2) $\mathrm{dom}(p \,{}'\mathrm{nor}'\, q) = \mathrm{dom}\, p \cap \mathrm{dom}\, q$ and for every set x such that $x \in \mathrm{dom}(p \,{}'\mathrm{nor}'\, q)$ holds $(p \,{}'\mathrm{nor}'\, q)(x) = p(x) \,{}'\mathrm{nor}'\, q(x)$.

Let us note that the functor $p \,{}'\mathrm{nor}'\, q$ is commutative.

Let p, q be boolean-valued functions. Note that $p \,{}'\mathrm{nand}'\, q$ is boolean-valued and $p \,{}'\mathrm{nor}'\, q$ is boolean-valued.

Let A be a non empty set and let p, q be elements of $Boolean^A$. Then $p \,{}'\mathrm{nand}'\, q$ is an element of $Boolean^A$ and it can be characterized by the condition:

(Def. 3) For every element x of A holds $(p \,{}'\mathrm{nand}'\, q)(x) = p(x) \,{}'\mathrm{nand}'\, q(x)$.

ISSN 1426–2630

Then p 'nor' q is an element of $\mathit{Boolean}^A$ and it can be characterized by the condition:

(Def. 4) For every element x of A holds $(p \text{ 'nor' } q)(x) = p(x) \text{ 'nor' } q(x)$.

Let us consider Y and let a, b be elements of BVF(Y). Then a 'nand' b is an element of BVF(Y). Then a 'nor' b is an element of BVF(Y).

We now state a number of propositions:

(1) $a \text{ 'nand' } b = \neg(a \wedge b)$.

(2) $a \text{ 'nor' } b = \neg(a \vee b)$.

(3) $\mathit{true}(Y) \text{ 'nand' } a = \neg a$.

(4) $\mathit{false}(Y) \text{ 'nand' } a = \mathit{true}(Y)$.

(5) $\mathit{false}(Y) \text{ 'nand' } \mathit{false}(Y) = \mathit{true}(Y)$ and $\mathit{false}(Y) \text{ 'nand' } \mathit{true}(Y) = \mathit{true}(Y)$ and $\mathit{true}(Y) \text{ 'nand' } \mathit{true}(Y) = \mathit{false}(Y)$.

(6) $a \text{ 'nand' } a = \neg a$ and $\neg(a \text{ 'nand' } a) = a$.

(7) $\neg(a \text{ 'nand' } b) = a \wedge b$.

(8) $a \text{ 'nand' } \neg a = \mathit{true}(Y)$ and $\neg(a \text{ 'nand' } \neg a) = \mathit{false}(Y)$.

(9) $a \text{ 'nand' } b \wedge c = \neg(a \wedge b \wedge c)$.

(10) $a \text{ 'nand' } b \wedge c = a \wedge b \text{ 'nand' } c$.

(11) $a \text{ 'nand' } (b \vee c) = \neg(a \wedge b) \wedge \neg(a \wedge c)$.

(12) $a \text{ 'nand' } (b \oplus c) = a \wedge b \Leftrightarrow a \wedge c$.

(13) $a \text{ 'nand' } (b \text{ 'nand' } c) = \neg a \vee b \wedge c$ and $a \text{ 'nand' } (b \text{ 'nand' } c) = a \Rightarrow b \wedge c$.

(14) $a \text{ 'nand' } (b \text{ 'nor' } c) = \neg a \vee b \vee c$ and $a \text{ 'nand' } (b \text{ 'nor' } c) = a \Rightarrow b \vee c$.

(15) $a \text{ 'nand' } (b \Leftrightarrow c) = a \Rightarrow b \oplus c$.

(16) $a \text{ 'nand' } a \wedge b = a \text{ 'nand' } b$.

(17) $a \text{ 'nand' } (a \vee b) = \neg a \wedge \neg(a \wedge b)$.

(18) $a \text{ 'nand' } (a \Leftrightarrow b) = a \Rightarrow a \oplus b$.

(19) $a \text{ 'nand' } (a \text{ 'nand' } b) = \neg a \vee b$ and $a \text{ 'nand' } (a \text{ 'nand' } b) = a \Rightarrow b$.

(20) $a \text{ 'nand' } (a \text{ 'nor' } b) = \mathit{true}(Y)$.

(21) $a \text{ 'nand' } (a \Leftrightarrow b) = \neg a \vee \neg b$.

(22) $a \wedge b = a \text{ 'nand' } b \text{ 'nand' } (a \text{ 'nand' } b)$.

(23) $a \text{ 'nand' } b \text{ 'nand' } (a \text{ 'nand' } c) = a \wedge (b \vee c)$.

(24) $a \text{ 'nand' } (b \Rightarrow c) = (\neg a \vee b) \wedge \neg(a \wedge c)$.

(25) $a \text{ 'nand' } (a \Rightarrow b) = \neg(a \wedge b)$.

(26) $\mathit{true}(Y) \text{ 'nor' } a = \mathit{false}(Y)$.

(27) $\mathit{false}(Y) \text{ 'nor' } a = \neg a$.

(28) $\mathit{false}(Y) \text{ 'nor' } \mathit{false}(Y) = \mathit{true}(Y)$ and $\mathit{false}(Y) \text{ 'nor' } \mathit{true}(Y) = \mathit{false}(Y)$ and $\mathit{true}(Y) \text{ 'nor' } \mathit{true}(Y) = \mathit{false}(Y)$.

(29) $a \text{ 'nor' } a = \neg a$ and $\neg(a \text{ 'nor' } a) = a$.

(30) $\neg(a \ \text{'nor'}\ b) = a \vee b$.

(31) $a \ \text{'nor'}\ \neg a = \mathit{false}(Y)$ and $\neg(a \ \text{'nor'}\ \neg a) = \mathit{true}(Y)$.

(32) $\neg a \wedge (a \oplus b) = \neg a \wedge b$.

(33) $a \ \text{'nor'}\ b \wedge c = \neg(a \vee b) \vee \neg(a \vee c)$.

(34) $a \ \text{'nor'}\ (b \vee c) = \neg(a \vee b \vee c)$.

(35) $a \ \text{'nor'}\ (b \Leftrightarrow c) = \neg a \wedge (b \oplus c)$.

(36) $a \ \text{'nor'}\ (b \Rightarrow c) = \neg a \wedge b \wedge \neg c$.

(37) $a \ \text{'nor'}\ (b \ \text{'nand'}\ c) = \neg a \wedge b \wedge c$.

(38) $a \ \text{'nor'}\ (b \ \text{'nor'}\ c) = \neg a \wedge (b \vee c)$.

(39) $a \ \text{'nor'}\ a \wedge b = \neg(a \wedge (a \vee b))$.

(40) $a \ \text{'nor'}\ (a \vee b) = \neg(a \vee b)$.

(41) $a \ \text{'nor'}\ (a \Leftrightarrow b) = \neg a \wedge b$.

(42) $a \ \text{'nor'}\ (a \Rightarrow b) = \mathit{false}(Y)$.

(43) $a \ \text{'nor'}\ (a \ \text{'nand'}\ b) = \mathit{false}(Y)$.

(44) $a \ \text{'nor'}\ (a \ \text{'nor'}\ b) = \neg a \wedge b$.

(45) $\mathit{false}(Y) \Leftrightarrow \mathit{false}(Y) = \mathit{true}(Y)$.

(46) $\mathit{false}(Y) \Leftrightarrow \mathit{true}(Y) = \mathit{false}(Y)$.

(47) $\mathit{true}(Y) \Leftrightarrow \mathit{true}(Y) = \mathit{true}(Y)$.

(48) $a \Leftrightarrow a = \mathit{true}(Y)$ and $\neg(a \Leftrightarrow a) = \mathit{false}(Y)$.

(49) $a \Leftrightarrow a \vee b = a \vee \neg b$.

(50) $a \wedge (b \ \text{'nand'}\ c) = a \wedge \neg b \vee a \wedge \neg c$.

(51) $a \vee (b \ \text{'nand'}\ c) = a \vee \neg b \vee \neg c$.

(52) $a \oplus (b \ \text{'nand'}\ c) = \neg a \wedge \neg(b \wedge c) \vee a \wedge b \wedge c$.

(53) $a \Leftrightarrow b \ \text{'nand'}\ c = a \wedge \neg(b \wedge c) \vee \neg a \wedge b \wedge c$.

(54) $a \Rightarrow b \ \text{'nand'}\ c = \neg(a \wedge b \wedge c)$.

(55) $a \ \text{'nor'}\ (b \ \text{'nand'}\ c) = \neg(a \vee \neg b \vee \neg c)$.

(56) $a \wedge (a \ \text{'nand'}\ b) = a \wedge \neg b$.

(57) $a \vee (a \ \text{'nand'}\ b) = \mathit{true}(Y)$.

(58) $a \oplus (a \ \text{'nand'}\ b) = \neg a \vee b$.

(59) $a \Leftrightarrow a \ \text{'nand'}\ b = a \wedge \neg b$.

(60) $a \Rightarrow a \ \text{'nand'}\ b = \neg(a \wedge b)$.

(61) $a \ \text{'nor'}\ (a \ \text{'nand'}\ b) = \mathit{false}(Y)$.

(62) $a \wedge (b \ \text{'nor'}\ c) = a \wedge \neg b \wedge \neg c$.

(63) $a \vee (b \ \text{'nor'}\ c) = (a \vee \neg b) \wedge (a \vee \neg c)$.

(64) $a \oplus (b \ \text{'nor'}\ c) = (a \vee \neg(b \vee c)) \wedge (\neg a \vee b \vee c)$.

(65) $a \Leftrightarrow b \ \text{'nor'}\ c = (a \vee b \vee c) \wedge (\neg a \vee \neg(b \vee c))$.

(66) $a \Rightarrow b \ \text{'nor'}\ c = \neg(a \wedge (b \vee c))$.

(67) $a\ '\mathrm{nand}'\ (b\ '\mathrm{nor}'\ c) = \neg a \vee b \vee c$.

(68) $a \wedge (a\ '\mathrm{nor}'\ b) = \mathit{false}(Y)$.

(69) $a \vee (a\ '\mathrm{nor}'\ b) = a \vee \neg b$.

(70) $a \oplus (a\ '\mathrm{nor}'\ b) = a \vee \neg b$.

(71) $a \Leftrightarrow a\ '\mathrm{nor}'\ b = \neg a \wedge b$.

(72) $a \Rightarrow a\ '\mathrm{nor}'\ b = \neg(a \vee a \wedge b)$.

(73) $a\ '\mathrm{nand}'\ (a\ '\mathrm{nor}'\ b) = \mathit{true}(Y)$.

Acknowledgments

This research was partially supported by the research funds of the Matsumoto University.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Hölder's Inequality and Minkowski's Inequality

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Summary. In this article, Hölder's inequality and Minkowski's inequality are proved. These equalities are basic ones of functional analysis.

MML Identifier: `HOLDER_1`.

The papers [12], [13], [14], [3], [1], [11], [4], [2], [7], [5], [6], [10], [8], and [9] provide the notation and terminology for this paper.

1. Hölder's Inequality

In this paper a, b, p, q are real numbers.

Let x be a real number. One can verify that $[x,+\infty[$ is non empty.

Next we state several propositions:

(1) For all real numbers p, q such that $0 < p$ and $0 < q$ and for every real number a such that $0 \leq a$ holds $a^p \cdot a^q = a^{p+q}$.

(2) For all real numbers p, q such that $0 < p$ and $0 < q$ and for every real number a such that $0 \leq a$ holds $(a^p)^q = a^{p \cdot q}$.

(3) For every real number p such that $0 < p$ and for all real numbers a, b such that $0 \leq a$ and $a \leq b$ holds $a^p \leq b^p$.

(4) If $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < a$ and $0 < b$, then $a \cdot b \leq \frac{a_{\mathbb{R}}^p}{p} + \frac{b_{\mathbb{R}}^q}{q}$ and $a \cdot b = \frac{a_{\mathbb{R}}^p}{p} + \frac{b_{\mathbb{R}}^q}{q}$ iff $a_{\mathbb{R}}^p = b_{\mathbb{R}}^q$.

(5) If $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $0 \leq a$ and $0 \leq b$, then $a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$ and $a \cdot b = \frac{a^p}{p} + \frac{b^q}{q}$ iff $a^p = b^q$.

ISSN 1426–2630

2. Minkowski's Inequality

Next we state several propositions:

(6) Let p, q be real numbers. Suppose $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let a, b, a_1, b_1, a_2 be sequences of real numbers. Suppose that for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_1(n) = |b(n)|^q$ and $a_2(n) = |a(n) \cdot b(n)|$. Let n be a natural number. Then $(\sum_{\alpha=0}^{\kappa}(a_2)(\alpha))_{\kappa\in\mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa}(a_1)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}} \cdot (\sum_{\alpha=0}^{\kappa}(b_1)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{q}}$.

(7) Let p be a real number. Suppose $1 < p$. Let a, b, a_1, b_2, a_2 be sequences of real numbers. Suppose that for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_2(n) = |b(n)|^p$ and $a_2(n) = |a(n) + b(n)|^p$. Let n be a natural number. Then $(\sum_{\alpha=0}^{\kappa}(a_2)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}} \leq (\sum_{\alpha=0}^{\kappa}(a_1)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}} + (\sum_{\alpha=0}^{\kappa}(b_2)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}}$.

(8) Let a, b be sequences of real numbers. Suppose for every natural number n holds $a(n) \leq b(n)$ and b is convergent and a is non-decreasing. Then a is convergent and $\lim a \leq \lim b$.

(9) Let a, b, c be sequences of real numbers. Suppose for every natural number n holds $a(n) \leq b(n) + c(n)$ and b is convergent and c is convergent and a is non-decreasing. Then a is convergent and $\lim a \leq \lim b + \lim c$.

(10) Let p be a real number. Suppose $0 < p$. Let a, a_1 be sequences of real numbers. Suppose a is convergent and for every natural number n holds $0 \leq a(n)$ and for every natural number n holds $a_1(n) = a(n)^p$. Then a_1 is convergent and $\lim a_1 = (\lim a)^p$.

(11) Let p be a real number. Suppose $0 < p$. Let a, a_1 be sequences of real numbers. Suppose a is summable and for every natural number n holds $0 \leq a(n)$ and for every natural number n holds $a_1(n) = (\sum_{\alpha=0}^{\kappa} a(\alpha))_{\kappa\in\mathbb{N}}(n)^p$. Then a_1 is convergent and $\lim a_1 = (\sum a)^p$ and a_1 is non-decreasing and for every natural number n holds $a_1(n) \leq (\sum a)^p$.

(12) Let p, q be real numbers. Suppose $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let a, b, a_1, b_1, a_2 be sequences of real numbers. Suppose for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_1(n) = |b(n)|^q$ and $a_2(n) = |a(n) \cdot b(n)|$ and a_1 is summable and b_1 is summable. Then a_2 is summable and $\sum a_2 \leq (\sum a_1)^{\frac{1}{p}} \cdot (\sum b_1)^{\frac{1}{q}}$.

(13) Let p be a real number. Suppose $1 < p$. Let a, b, a_1, b_2, a_2 be sequences of real numbers. Suppose that

(i) for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_2(n) = |b(n)|^p$ and $a_2(n) = |a(n) + b(n)|^p$,

(ii) a_1 is summable, and

(iii) b_2 is summable.

Then a_2 is summable and $(\sum a_2)^{\frac{1}{p}} \leq (\sum a_1)^{\frac{1}{p}} + (\sum b_2)^{\frac{1}{p}}$.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

The Banach Space l^p

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Summary. We introduce the arithmetic addition and multiplication in the set of l^p real sequences and also introduce the norm. This set has the structure of the Banach space.

MML Identifier: `LP_SPACE`.

The notation and terminology used in this paper have been introduced in the following articles: [16], [5], [19], [20], [3], [4], [1], [15], [7], [18], [2], [17], [10], [9], [8], [12], [11], [6], [14], and [13].

1. The Real Norm Space of l^p Real Sequences

Let x be a sequence of real numbers and let p be a real number. The functor x^p yielding a sequence of real numbers is defined as follows:

(Def. 1) For every natural number n holds $x^p(n) = |x(n)|^p$.

Let p be a real number. Let us assume that $p \geq 1$. The functor l^p yielding a non empty subset of the carrier of the linear space of real sequences is defined as follows:

(Def. 2) For every set x holds $x \in l^p$ iff $x \in$ the set of real sequences and $(\mathrm{id}_{\mathrm{seq}}(x))^p$ is summable.

In the sequel a, b, c are real numbers.

We now state several propositions:

(1) If $a \geq 0$ and $a < b$ and $c > 0$, then $a^c < b^c$.

(2) Let p be a real number. Suppose $1 \leq p$. Let a, b be sequences of real numbers and n be a natural number. Then $(\sum_{\alpha=0}^{\kappa}((a+b)^p)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}} \leq (\sum_{\alpha=0}^{\kappa}(a^p)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}} + (\sum_{\alpha=0}^{\kappa}(b^p)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}}$.

ISSN 1426–2630

(3) Let a, b be sequences of real numbers and p be a real number. Suppose $1 \leq p$ and a^p is summable and b^p is summable. Then $(a+b)^p$ is summable and $(\sum((a+b)^p))^{\frac{1}{p}} \leq (\sum(a^p))^{\frac{1}{p}} + (\sum(b^p))^{\frac{1}{p}}$.

(4) For every real number p such that $1 \leq p$ holds l^p is linearly closed.

(5) Let p be a real number. Suppose $1 \leq p$. Then $\langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences})\rangle$ is a subspace of the linear space of real sequences.

(6) Let p be a real number. Suppose $1 \leq p$. Then $\langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences})\rangle$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

(7) Let p be a real number. Suppose $1 \leq p$. Then $\langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences})\rangle$ is a real linear space.

Let p be a real number. The functor l^p-norm yielding a function from l^p into $\mathbb{R}$ is defined by:

(Def. 3) For every set x such that $x \in l^p$ holds $l^p\text{-norm}(x) = (\sum((\mathrm{id_{seq}}(x))^p))^{\frac{1}{p}}$.

The following two propositions are true:

(8) Let p be a real number. Suppose $1 \leq p$. Then $\langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$ is a real linear space.

(9) Let p be a real number. Suppose $p \geq 1$. Then $\langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$ is a subspace of the linear space of real sequences.

2. The Banach Space of l^p Real Sequences

Next we state several propositions:

(10) Let p be a real number. Suppose $1 \leq p$. Let l_1 be a non empty normed structure. Suppose $l_1 = \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$. Then the carrier of $l_1 = l^p$ and for every set x holds x is a vector of l_1 iff x is a sequence of real numbers and $(\mathrm{id_{seq}}(x))^p$ is summable and $0_{(l_1)} = \mathrm{Zeroseq}$ and for every vector x of l_1 holds $x = \mathrm{id_{seq}}(x)$ and for all vectors x, y of l_1 holds $x + y = \mathrm{id_{seq}}(x) + \mathrm{id_{seq}}(y)$ and for every real number r and for every

vector x of l_1 holds $r \cdot x = r\ \mathrm{id}_{\mathrm{seq}}(x)$ and for every vector x of l_1 holds $-x = -\mathrm{id}_{\mathrm{seq}}(x)$ and $\mathrm{id}_{\mathrm{seq}}(-x) = -\mathrm{id}_{\mathrm{seq}}(x)$ and for all vectors x, y of l_1 holds $x-y = \mathrm{id}_{\mathrm{seq}}(x)-\mathrm{id}_{\mathrm{seq}}(y)$ and for every vector x of l_1 holds $(\mathrm{id}_{\mathrm{seq}}(x))^p$ is summable and for every vector x of l_1 holds $\|x\| = (\sum((\mathrm{id}_{\mathrm{seq}}(x))^p))^{\frac{1}{p}}$.

(11) Let p be a real number. Suppose $p \geq 1$. Let r_1 be a sequence of real numbers. Suppose that for every natural number n holds $r_1(n) = 0$. Then ${r_1}^p$ is summable and $(\sum({r_1}^p))^{\frac{1}{p}} = 0$.

(12) Let p be a real number. Suppose $1 \leq p$. Let r_1 be a sequence of real numbers. Suppose ${r_1}^p$ is summable and $(\sum({r_1}^p))^{\frac{1}{p}} = 0$. Let n be a natural number. Then $r_1(n) = 0$.

(13) Let p be a real number. Suppose $1 \leq p$. Let l_1 be a non empty normed structure. Suppose $l_1 = \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$. Let x, y be points of l_1 and a be a real number. Then $\|x\| = 0$ iff $x = 0_{(l_1)}$ and $0 \leq \|x\|$ and $\|x+y\| \leq \|x\| + \|y\|$ and $\|a \cdot x\| = |a| \cdot \|x\|$.

(14) Let p be a real number. Suppose $p \geq 1$. Let l_1 be a non empty normed structure. Suppose $l_1 = \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$. Then l_1 is real normed space-like.

(15) Let p be a real number. Suppose $p \geq 1$. Let l_1 be a non empty normed structure. Suppose $l_1 = \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$. Then l_1 is a real normed space.

(16) Let p be a real number. Suppose $1 \leq p$. Let l_1 be a real normed space. Suppose $l_1 = \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$. Let v_1 be a sequence of l_1. If v_1 is Cauchy sequence by norm, then v_1 is convergent.

Let p be a real number. Let us assume that $1 \leq p$. The functor l^p-space yielding a real Banach space is defined by the condition (Def. 4).

(Def. 4) $l^p\text{-space} = \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Lebesgue Integral of Simple Valued Function[1]

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Summary. In this article, the authors introduce Lebesgue integral of simple valued function.

MML Identifier: `MESFUNC3`.

The terminology and notation used in this paper are introduced in the following papers: [23], [12], [25], [21], [26], [10], [11], [3], [22], [24], [7], [14], [1], [2], [20], [4], [5], [6], [8], [9], [19], [13], [15], [16], [17], and [18].

1. Integral of Simple Valued Function

The following propositions are true:

(1) Let n, m be natural numbers, a be a function from $[\!: \operatorname{Seg} n, \operatorname{Seg} m :\!]$ into $\mathbb{R}$, and p, q be finite sequences of elements of $\mathbb{R}$. Suppose that

(i) $\operatorname{dom} p = \operatorname{Seg} n$,

(ii) for every natural number i such that $i \in \operatorname{dom} p$ there exists a finite sequence r of elements of $\mathbb{R}$ such that $\operatorname{dom} r = \operatorname{Seg} m$ and $p(i) = \sum r$ and for every natural number j such that $j \in \operatorname{dom} r$ holds $r(j) = a(\langle i, j \rangle)$,

(iii) $\operatorname{dom} q = \operatorname{Seg} m$, and

(iv) for every natural number j such that $j \in \operatorname{dom} q$ there exists a finite sequence s of elements of $\mathbb{R}$ such that $\operatorname{dom} s = \operatorname{Seg} n$ and $q(j) = \sum s$ and for every natural number i such that $i \in \operatorname{dom} s$ holds $s(i) = a(\langle i, j \rangle)$.

Then $\sum p = \sum q$.

[1]This work has been partially supported by the MEXT grant Grant-in-Aid for Young Scientists (B)16700156.

ISSN 1426–2630

(2) Let F be a finite sequence of elements of $\overline{\mathbb{R}}$ and f be a finite sequence of elements of $\mathbb{R}$. If $F = f$, then $\sum F = \sum f$.

(3) Let X be a non empty set, S be a σ-field of subsets of X, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S. Then there exists a finite sequence F of separated subsets of S and there exists a finite sequence a of elements of $\overline{\mathbb{R}}$ such that

(i) $\operatorname{dom} f = \bigcup \operatorname{rng} F$,

(ii) $\operatorname{dom} F = \operatorname{dom} a$,

(iii) for every natural number n such that $n \in \operatorname{dom} F$ and for every set x such that $x \in F(n)$ holds $f(x) = a(n)$, and

(iv) for every set x such that $x \in \operatorname{dom} f$ there exists a finite sequence a_1 of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} a_1 = \operatorname{dom} a$ and for every natural number n such that $n \in \operatorname{dom} a_1$ holds $a_1(n) = a(n) \cdot \chi_{F(n),X}(x)$.

(4) Let X be a set and F be a finite sequence of elements of X. Then F is disjoint valued if and only if for all natural numbers i, j such that $i \in \operatorname{dom} F$ and $j \in \operatorname{dom} F$ and $i \neq j$ holds $F(i)$ misses $F(j)$.

(5) Let X be a non empty set, A be a set, S be a σ-field of subsets of X, F be a finite sequence of separated subsets of S, and G be a finite sequence of elements of S. Suppose $\operatorname{dom} G = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} G$ holds $G(i) = A \cap F(i)$. Then G is a finite sequence of separated subsets of S.

(6) Let X be a non empty set, A be a set, and F, G be finite sequences of elements of X. Suppose $\operatorname{dom} G = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} G$ holds $G(i) = A \cap F(i)$. Then $\bigcup \operatorname{rng} G = A \cap \bigcup \operatorname{rng} F$.

(7) Let X be a set, F be a finite sequence of elements of X, and i be a natural number. If $i \in \operatorname{dom} F$, then $F(i) \subseteq \bigcup \operatorname{rng} F$ and $F(i) \cap \bigcup \operatorname{rng} F = F(i)$.

(8) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and F be a finite sequence of separated subsets of S. Then $\operatorname{dom} F = \operatorname{dom}(M \cdot F)$.

(9) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and F be a finite sequence of separated subsets of S. Then $M(\bigcup \operatorname{rng} F) = \sum (M \cdot F)$.

(10) Let F, G be finite sequences of elements of $\overline{\mathbb{R}}$ and a be an extended real number. Suppose that

(i) $a \neq +\infty$ and $a \neq -\infty$ or for every natural number i such that $i \in \operatorname{dom} F$ holds $F(i) < 0_{\overline{\mathbb{R}}}$ or for every natural number i such that $i \in \operatorname{dom} F$ holds $0_{\overline{\mathbb{R}}} < F(i)$,

(ii) $\operatorname{dom} F = \operatorname{dom} G$, and

(iii) for every natural number i such that $i \in \operatorname{dom} G$ holds $G(i) = a \cdot F(i)$.

Then $\sum G = a \cdot \sum F$.

(11) Every finite sequence of elements of $\mathbb{R}$ is a finite sequence of elements of $\overline{\mathbb{R}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, let f be a partial function from X to $\overline{\mathbb{R}}$, let F be a finite sequence of separated subsets of S, and let a be a finite sequence of elements of $\overline{\mathbb{R}}$. We say that F and a are re-presentation of f if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $\operatorname{dom} f = \bigcup \operatorname{rng} F$,

(ii) $\operatorname{dom} F = \operatorname{dom} a$, and

(iii) for every natural number n such that $n \in \operatorname{dom} F$ and for every set x such that $x \in F(n)$ holds $f(x) = a(n)$.

One can prove the following propositions:

(12) Let X be a non empty set, S be a σ-field of subsets of X, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S. Then there exists a finite sequence F of separated subsets of S and there exists a finite sequence a of elements of $\overline{\mathbb{R}}$ such that F and a are re-presentation of f.

(13) Let X be a non empty set, S be a σ-field of subsets of X, and F be a finite sequence of separated subsets of S. Then there exists a finite sequence G of separated subsets of S such that

(i) $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} G$, and

(ii) for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) \neq \emptyset$ and there exists a natural number m such that $m \in \operatorname{dom} F$ and $F(m) = G(n)$.

(14) Let X be a non empty set, S be a σ-field of subsets of X, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and for every set x such that $x \in \operatorname{dom} f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$. Then there exists a finite sequence F of separated subsets of S and there exists a finite sequence a of elements of $\overline{\mathbb{R}}$ such that

(i) F and a are re-presentation of f,

(ii) $a(1) = 0_{\overline{\mathbb{R}}}$, and

(iii) for every natural number n such that $2 \leq n$ and $n \in \operatorname{dom} a$ holds $0_{\overline{\mathbb{R}}} < a(n)$ and $a(n) < +\infty$.

(15) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, F be a finite sequence of separated subsets of S, a be a finite sequence of elements of $\overline{\mathbb{R}}$, and x be an element of X. Suppose F and a are re-presentation of f and $x \in \operatorname{dom} f$. Then there exists a finite sequence a_1 of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} a_1 = \operatorname{dom} a$ and for every natural number n such that $n \in \operatorname{dom} a_1$ holds $a_1(n) = a(n) \cdot \chi_{F(n),X}(x)$ and $f(x) = \sum a_1$.

(16) Let p be a finite sequence of elements of $\overline{\mathbb{R}}$ and q be a finite sequence of elements of $\mathbb{R}$. If $p = q$, then $\sum p = \sum q$.

(17) Let p be a finite sequence of elements of $\overline{\mathbb{R}}$. Suppose for every natural number n such that $n \in \operatorname{dom} p$ holds $0_{\overline{\mathbb{R}}} \leq p(n)$ and there exists a natural number n such that $n \in \operatorname{dom} p$ and $p(n) = +\infty$. Then $\sum p = +\infty$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\overline{\mathbb{R}}$. Let us assume that f is simple function in S and $\operatorname{dom} f \neq \emptyset$ and for every set x such that $x \in \operatorname{dom} f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$. The functor integral$(X, S, M, f)$ yielding an element of $\overline{\mathbb{R}}$ is defined by the condition (Def. 2).

(Def. 2) There exists a finite sequence F of separated subsets of S and there exist finite sequences a, x of elements of $\overline{\mathbb{R}}$ such that

(i) F and a are re-presentation of f,

(ii) $a(1) = 0_{\overline{\mathbb{R}}}$,

(iii) for every natural number n such that $2 \leq n$ and $n \in \operatorname{dom} a$ holds $0_{\overline{\mathbb{R}}} < a(n)$ and $a(n) < +\infty$,

(iv) $\operatorname{dom} x = \operatorname{dom} F$,

(v) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$, and

(vi) integral$(X, S, M, f) = \sum x$.

2. Additional Lemma

We now state the proposition

(18) Let a be a finite sequence of elements of $\overline{\mathbb{R}}$ and p, N be elements of $\overline{\mathbb{R}}$. Suppose $N = \operatorname{len} a$ and for every natural number n such that $n \in \operatorname{dom} a$ holds $a(n) = p$. Then $\sum a = N \cdot p$.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Inverse Trigonometric Functions Arcsin and Arccos[1]

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Summary. Notions of inverse sine and inverse cosine have been introduced. Their basic properties have been proved.

MML Identifier: SIN_COS6.

The papers [11], [14], [1], [10], [3], [13], [12], [9], [15], [2], [16], [6], [4], [5], [7], [8], and [17] provide the terminology and notation for this paper.

1. Preliminaries

In this paper r, s are real numbers and i is an integer number.

We now state two propositions:

(1) If $0 \leq r$ and $r < s$, then $\lfloor \frac{r}{s} \rfloor = 0$.

(2) For every function f and for all sets X, Y such that $f{\restriction}X$ is one-to-one and $Y \subseteq X$ holds $f{\restriction}Y$ is one-to-one.

2. Functions sine and cosine

We now state four propositions:

(3) $-1 \leq \sin r$.

(4) $\sin r \leq 1$.

[1]The paper was written during the first author's post-doctoral fellowship granted by the Shinshu University, Japan.

ISSN 1426–2630

(5) $-1 \leq \cos r$.

(6) $\cos r \leq 1$.

One can check that π is positive.

The following propositions are true:

(7) $\sin(-\frac{\pi}{2}) = -1$ and (the function sin)$(-\frac{\pi}{2}) = -1$.

(8) (The function sin)(r) = (the function sin)$(r + 2 \cdot \pi \cdot i)$.

(9) $\cos(-\frac{\pi}{2}) = 0$ and (the function cos)$(-\frac{\pi}{2}) = 0$.

(10) (The function cos)(r) = (the function cos)$(r + 2 \cdot \pi \cdot i)$.

(11) If $2 \cdot \pi \cdot i < r$ and $r < \pi + 2 \cdot \pi \cdot i$, then $\sin r > 0$.

(12) If $\pi + 2 \cdot \pi \cdot i < r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r < 0$.

(13) If $-\frac{\pi}{2} + 2 \cdot \pi \cdot i < r$ and $r < \frac{\pi}{2} + 2 \cdot \pi \cdot i$, then $\cos r > 0$.

(14) If $\frac{\pi}{2} + 2 \cdot \pi \cdot i < r$ and $r < \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r < 0$.

(15) If $\frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i < r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r > 0$.

(16) If $2 \cdot \pi \cdot i \leq r$ and $r \leq \pi + 2 \cdot \pi \cdot i$, then $\sin r \geq 0$.

(17) If $\pi + 2 \cdot \pi \cdot i \leq r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r \leq 0$.

(18) If $-\frac{\pi}{2} + 2 \cdot \pi \cdot i \leq r$ and $r \leq \frac{\pi}{2} + 2 \cdot \pi \cdot i$, then $\cos r \geq 0$.

(19) If $\frac{\pi}{2} + 2 \cdot \pi \cdot i \leq r$ and $r \leq \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r \leq 0$.

(20) If $\frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i \leq r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r \geq 0$.

(21) If $2 \cdot \pi \cdot i \leq r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $\sin r = 0$, then $r = 2 \cdot \pi \cdot i$ or $r = \pi + 2 \cdot \pi \cdot i$.

(22) If $2 \cdot \pi \cdot i \leq r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $\cos r = 0$, then $r = \frac{\pi}{2} + 2 \cdot \pi \cdot i$ or $r = \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i$.

(23) If $\sin r = -1$, then $r = \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$.

(24) If $\sin r = 1$, then $r = \frac{\pi}{2} + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$.

(25) If $\cos r = -1$, then $r = \pi + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$.

(26) If $\cos r = 1$, then $r = 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$.

(27) If $0 \leq r$ and $r \leq 2 \cdot \pi$ and $\sin r = -1$, then $r = \frac{3}{2} \cdot \pi$.

(28) If $0 \leq r$ and $r \leq 2 \cdot \pi$ and $\sin r = 1$, then $r = \frac{\pi}{2}$.

(29) If $0 \leq r$ and $r \leq 2 \cdot \pi$ and $\cos r = -1$, then $r = \pi$.

(30) If $0 \leq r$ and $r < \frac{\pi}{2}$, then $\sin r < 1$.

(31) If $0 \leq r$ and $r < \frac{3}{2} \cdot \pi$, then $\sin r > -1$.

(32) If $\frac{3}{2} \cdot \pi < r$ and $r \leq 2 \cdot \pi$, then $\sin r > -1$.

(33) If $\frac{\pi}{2} < r$ and $r \leq 2 \cdot \pi$, then $\sin r < 1$.

(34) If $0 < r$ and $r < 2 \cdot \pi$, then $\cos r < 1$.

(35) If $0 \leq r$ and $r < \pi$, then $\cos r > -1$.

(36) If $\pi < r$ and $r \leq 2 \cdot \pi$, then $\cos r > -1$.

(37) If $2 \cdot \pi \cdot i \leq r$ and $r < \frac{\pi}{2} + 2 \cdot \pi \cdot i$, then $\sin r < 1$.

(38) If $2 \cdot \pi \cdot i \leq r$ and $r < \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r > -1$.

(39) If $\frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i < r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r > -1$.

(40) If $\frac{\pi}{2} + 2 \cdot \pi \cdot i < r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r < 1$.

(41) If $2 \cdot \pi \cdot i < r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r < 1$.

(42) If $2 \cdot \pi \cdot i \leq r$ and $r < \pi + 2 \cdot \pi \cdot i$, then $\cos r > -1$.

(43) If $\pi + 2 \cdot \pi \cdot i < r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r > -1$.

(44) If $\cos(2 \cdot \pi \cdot r) = 1$, then $r \in \mathbb{Z}$.

(45) (The function sin) $^\circ[-\frac{\pi}{2}, \frac{\pi}{2}] = [-1, 1]$.

(46) (The function sin) $^\circ]-\frac{\pi}{2}, \frac{\pi}{2}[=]-1, 1[$.

(47) (The function sin) $^\circ[\frac{\pi}{2}, \frac{3}{2} \cdot \pi] = [-1, 1]$.

(48) (The function sin) $^\circ]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[=]-1, 1[$.

(49) (The function cos) $^\circ[0, \pi] = [-1, 1]$.

(50) (The function cos) $^\circ]0, \pi[=]-1, 1[$.

(51) (The function cos) $^\circ[\pi, 2 \cdot \pi] = [-1, 1]$.

(52) (The function cos) $^\circ]\pi, 2 \cdot \pi[=]-1, 1[$.

(53) The function sin is increasing on $[-\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{\pi}{2} + 2 \cdot \pi \cdot i]$.

(54) The function sin is decreasing on $[\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i]$.

(55) The function cos is decreasing on $[2 \cdot \pi \cdot i, \pi + 2 \cdot \pi \cdot i]$.

(56) The function cos is increasing on $[\pi + 2 \cdot \pi \cdot i, 2 \cdot \pi + 2 \cdot \pi \cdot i]$.

(57) (The function sin)$\restriction[-\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{\pi}{2} + 2 \cdot \pi \cdot i]$ is one-to-one.

(58) (The function sin)$\restriction[\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i]$ is one-to-one.

One can check that (the function sin)$\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]$ is one-to-one and (the function sin)$\restriction[\frac{\pi}{2}, \frac{3}{2} \cdot \pi]$ is one-to-one.

One can check the following observations:

* (the function sin)$\restriction[-\frac{\pi}{2}, 0]$ is one-to-one,
* (the function sin)$\restriction[0, \frac{\pi}{2}]$ is one-to-one,
* (the function sin)$\restriction[\frac{\pi}{2}, \pi]$ is one-to-one,
* (the function sin)$\restriction[\pi, \frac{3}{2} \cdot \pi]$ is one-to-one, and
* (the function sin)$\restriction[\frac{3}{2} \cdot \pi, 2 \cdot \pi]$ is one-to-one.

One can verify the following observations:

* (the function sin)$\restriction]-\frac{\pi}{2}, \frac{\pi}{2}[$ is one-to-one,
* (the function sin)$\restriction]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[$ is one-to-one,
* (the function sin)$\restriction]-\frac{\pi}{2}, 0[$ is one-to-one,
* (the function sin)$\restriction]0, \frac{\pi}{2}[$ is one-to-one,
* (the function sin)$\restriction]\frac{\pi}{2}, \pi[$ is one-to-one,
* (the function sin)$\restriction]\pi, \frac{3}{2} \cdot \pi[$ is one-to-one, and
* (the function sin)$\restriction]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$ is one-to-one.

Next we state two propositions:

(59) (The function cos)$\restriction[2 \cdot \pi \cdot i, \pi + 2 \cdot \pi \cdot i]$ is one-to-one.

(60) (The function cos)$\restriction[\pi + 2 \cdot \pi \cdot i, 2 \cdot \pi + 2 \cdot \pi \cdot i]$ is one-to-one.

Let us note that (the function cos)$\restriction[0, \pi]$ is one-to-one and (the function cos)$\restriction[\pi, 2 \cdot \pi]$ is one-to-one.

One can check the following observations:

* (the function cos)$\restriction[0, \frac{\pi}{2}]$ is one-to-one,
* (the function cos)$\restriction[\frac{\pi}{2}, \pi]$ is one-to-one,
* (the function cos)$\restriction[\pi, \frac{3}{2} \cdot \pi]$ is one-to-one, and
* (the function cos)$\restriction[\frac{3}{2} \cdot \pi, 2 \cdot \pi]$ is one-to-one.

One can check the following observations:

* (the function cos)$\restriction]0, \pi[$ is one-to-one,
* (the function cos)$\restriction]\pi, 2 \cdot \pi[$ is one-to-one,
* (the function cos)$\restriction]0, \frac{\pi}{2}[$ is one-to-one,
* (the function cos)$\restriction]\frac{\pi}{2}, \pi[$ is one-to-one,
* (the function cos)$\restriction]\pi, \frac{3}{2} \cdot \pi[$ is one-to-one, and
* (the function cos)$\restriction]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$ is one-to-one.

The following proposition is true

(61) If $2 \cdot \pi \cdot i \leq r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $2 \cdot \pi \cdot i \leq s$ and $s < 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $\sin r = \sin s$ and $\cos r = \cos s$, then $r = s$.

3. Function arcsin

The function arcsin is a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:

(Def. 1) The function arcsin $= ((\text{the function sin})\restriction[-\frac{\pi}{2}, \frac{\pi}{2}])^{-1}$.

Let r be a set. The functor $\arcsin r$ is defined by:

(Def. 2) $\arcsin r = (\text{the function arcsin})(r)$.

Let r be a set. Then $\arcsin r$ is a real number.

Next we state two propositions:

(62) (The function arcsin) $^{-1} = (\text{the function sin})\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(63) $\operatorname{rng}(\text{the function arcsin}) = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Let us note that the function arcsin is one-to-one.

The following propositions are true:

(64) $\operatorname{dom}(\text{the function arcsin}) = [-1, 1]$.

(65) ((The function sin)$\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]$ **qua** function) $\cdot$(the function arcsin) $= \mathrm{id}_{[-1,1]}$.

(66) (The function arcsin) $\cdot$((the function sin)$\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]) = \mathrm{id}_{[-1,1]}$.

(67) ((The function sin)$\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]) \cdot$ (the function arcsin) $= \mathrm{id}_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$.

(68) (The function arcsin **qua** function) $\cdot$((the function sin)$\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]) = \mathrm{id}_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$.

(69) If $-1 \leq r$ and $r \leq 1$, then $\sin \arcsin r = r$.

(70) If $-\frac{\pi}{2} \leq r$ and $r \leq \frac{\pi}{2}$, then $\arcsin \sin r = r$.

(71) $\arcsin(-1) = -\frac{\pi}{2}$.

(72) $\arcsin 0 = 0$.

(73) $\arcsin 1 = \frac{\pi}{2}$.

(74) If $-1 \leq r$ and $r \leq 1$ and $\arcsin r = -\frac{\pi}{2}$, then $r = -1$.

(75) If $-1 \leq r$ and $r \leq 1$ and $\arcsin r = 0$, then $r = 0$.

(76) If $-1 \leq r$ and $r \leq 1$ and $\arcsin r = \frac{\pi}{2}$, then $r = 1$.

(77) If $-1 \leq r$ and $r \leq 1$, then $-\frac{\pi}{2} \leq \arcsin r$ and $\arcsin r \leq \frac{\pi}{2}$.

(78) If $-1 < r$ and $r < 1$, then $-\frac{\pi}{2} < \arcsin r$ and $\arcsin r < \frac{\pi}{2}$.

(79) If $-1 \leq r$ and $r \leq 1$, then $\arcsin r = -\arcsin(-r)$.

(80) If $0 \leq s$ and $r^{\mathbf{2}} + s^{\mathbf{2}} = 1$, then $\cos \arcsin r = s$.

(81) If $s \leq 0$ and $r^{\mathbf{2}} + s^{\mathbf{2}} = 1$, then $\cos \arcsin r = -s$.

(82) If $-1 \leq r$ and $r \leq 1$, then $\cos \arcsin r = \sqrt{1 - r^{\mathbf{2}}}$.

(83) The function arcsin is increasing on $[-1, 1]$.

(84) The function arcsin is differentiable on $]-1, 1[$ and if $-1 < r$ and $r < 1$, then (the function arcsin)$'(r) = \frac{1}{\sqrt{1-r^{\mathbf{2}}}}$.

(85) The function arcsin is continuous on $[-1, 1]$.

4. Function arccos

The function arccos is a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:

(Def. 3) The function arccos $=$ ((the function cos)$\restriction[0, \pi])^{-1}$.

Let r be a set. The functor $\arccos r$ is defined by:

(Def. 4) $\arccos r =$ (the function arccos)(r).

Let r be a set. Then $\arccos r$ is a real number.

One can prove the following two propositions:

(86) (The function arccos) $^{-1} =$ (the function cos)$\restriction[0, \pi]$.

(87) rng (the function arccos) $= [0, \pi]$.

Let us note that the function arccos is one-to-one.

The following propositions are true:

(88) dom (the function arccos) $= [-1, 1]$.

(89) ((The function cos)$\restriction[0, \pi]$ **qua** function) $\cdot$(the function arccos) $= \mathrm{id}_{[-1,1]}$.

(90) (The function arccos) $\cdot$((the function cos)$\restriction[0, \pi]) = \mathrm{id}_{[-1,1]}$.

(91) ((The function cos)$\restriction[0,\pi]$) $\cdot$ (the function arccos) $= \mathrm{id}_{[0,\pi]}$.

(92) (The function arccos **qua** function) $\cdot$((the function cos)$\restriction[0,\pi]$) $= \mathrm{id}_{[0,\pi]}$.

(93) If $-1 \leq r$ and $r \leq 1$, then $\cos \arccos r = r$.

(94) If $0 \leq r$ and $r \leq \pi$, then $\arccos \cos r = r$.

(95) $\arccos(-1) = \pi$.

(96) $\arccos 0 = \frac{\pi}{2}$.

(97) $\arccos 1 = 0$.

(98) If $-1 \leq r$ and $r \leq 1$ and $\arccos r = 0$, then $r = 1$.

(99) If $-1 \leq r$ and $r \leq 1$ and $\arccos r = \frac{\pi}{2}$, then $r = 0$.

(100) If $-1 \leq r$ and $r \leq 1$ and $\arccos r = \pi$, then $r = -1$.

(101) If $-1 \leq r$ and $r \leq 1$, then $0 \leq \arccos r$ and $\arccos r \leq \pi$.

(102) If $-1 < r$ and $r < 1$, then $0 < \arccos r$ and $\arccos r < \pi$.

(103) If $-1 \leq r$ and $r \leq 1$, then $\arccos r = \pi - \arccos(-r)$.

(104) If $0 \leq s$ and $r^{\mathbf{2}} + s^{\mathbf{2}} = 1$, then $\sin \arccos r = s$.

(105) If $s \leq 0$ and $r^{\mathbf{2}} + s^{\mathbf{2}} = 1$, then $\sin \arccos r = -s$.

(106) If $-1 \leq r$ and $r \leq 1$, then $\sin \arccos r = \sqrt{1 - r^{\mathbf{2}}}$.

(107) The function arccos is decreasing on $[-1, 1]$.

(108) The function arccos is differentiable on $]-1, 1[$ and if $-1 < r$ and $r < 1$, then (the function arccos)$'(r) = -\frac{1}{\sqrt{1-r^{\mathbf{2}}}}$.

(109) The function arccos is continuous on $[-1, 1]$.

(110) If $-1 \leq r$ and $r \leq 1$, then $\arcsin r + \arccos r = \frac{\pi}{2}$.

(111) If $-1 \leq r$ and $r \leq 1$, then $\arccos(-r) - \arcsin r = \frac{\pi}{2}$.

(112) If $-1 \leq r$ and $r \leq 1$, then $\arccos r - \arcsin(-r) = \frac{\pi}{2}$.

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Received September 5, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

On Some Points of a Simple Closed Curve[1]

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MML Identifier: JORDAN21.

The notation and terminology used here are introduced in the following papers: [26], [28], [2], [13], [1], [29], [5], [18], [17], [3], [14], [24], [9], [23], [4], [25], [7], [10], [11], [12], [19], [20], [22], [21], [6], [8], [15], [16], and [27].

1. On the Subsets of $\mathcal{E}^2_{\mathrm{T}}$

For simplicity, we follow the rules: C denotes a simple closed curve, P denotes a subset of $\mathcal{E}^2_{\mathrm{T}}$, R denotes a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, p denotes a point of $\mathcal{E}^2_{\mathrm{T}}$, and i, j, k, m, n denote natural numbers.

One can prove the following propositions:

(1) For every point p of $\mathcal{E}^n_{\mathrm{T}}$ holds $\{p\}$ is Bounded.

(2) For all real numbers s_1, t and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{[s, t]; s$ ranges over real numbers: $s_1 < s\}$ holds P is convex.

(3) For all real numbers s_2, t and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{[s, t]; s$ ranges over real numbers: $s < s_2\}$ holds P is convex.

(4) For all real numbers s, t_1 and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{[s, t]; t$ ranges over real numbers: $t_1 < t\}$ holds P is convex.

(5) For all real numbers s, t_2 and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{[s, t]; t$ ranges over real numbers: $t < t_2\}$ holds P is convex.

(6) NorthHalfline $p \setminus \{p\}$ is convex.

(7) SouthHalfline $p \setminus \{p\}$ is convex.

(8) WestHalfline $p \setminus \{p\}$ is convex.

(9) EastHalfline $p \setminus \{p\}$ is convex.

[1]The paper has been completed during the author's post-doctoral fellowship granted by the Shinshu University, Japan.

ISSN 1426–2630

(10) For every subset A of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{UBD}\,A$ misses A.

(11) Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $p_1 \neq q_1$ and $p_2 \neq q_2$. Then $p_1 \notin \mathrm{Segment}(P, p_1, p_2, q_1, q_2)$ and $p_2 \notin \mathrm{Segment}(P, p_1, p_2, q_1, q_2)$.

(12) $\mathrm{proj2}^\circ(C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}))$ is not empty.

(13) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{proj2}^\circ(C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}))$ is closed, lower bounded, and upper bounded.

2. Gauges

The following propositions are true:

(14) $\langle 1,\ 1\rangle \in$ the indices of $\mathrm{Gauge}(R, n)$.

(15) $\langle 1,\ 2\rangle \in$ the indices of $\mathrm{Gauge}(R, n)$.

(16) $\langle 2,\ 1\rangle \in$ the indices of $\mathrm{Gauge}(R, n)$.

(17) Let C be a non vertical non horizontal compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $m > k$ and $\langle i,\ j\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$ and $\langle i,\ j+1\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$. Then $\rho(\mathrm{Gauge}(C, m) \circ (i, j), \mathrm{Gauge}(C, m) \circ (i, j+1)) < \rho(\mathrm{Gauge}(C, k) \circ (i, j), \mathrm{Gauge}(C, k) \circ (i, j+1))$.

(18) For every non vertical non horizontal compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $m > k$ holds $\rho(\mathrm{Gauge}(C, m) \circ (1, 1), \mathrm{Gauge}(C, m) \circ (1, 2)) < \rho(\mathrm{Gauge}(C, k) \circ (1, 1), \mathrm{Gauge}(C, k) \circ (1, 2))$.

(19) Let C be a non vertical non horizontal compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $m > k$ and $\langle i,\ j\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$ and $\langle i+1,\ j\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$. Then $\rho(\mathrm{Gauge}(C, m) \circ (i, j), \mathrm{Gauge}(C, m) \circ (i+1, j)) < \rho(\mathrm{Gauge}(C, k) \circ (i, j), \mathrm{Gauge}(C, k) \circ (i+1, j))$.

(20) For every non vertical non horizontal compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $m > k$ holds $\rho(\mathrm{Gauge}(C, m) \circ (1, 1), \mathrm{Gauge}(C, m) \circ (2, 1)) < \rho(\mathrm{Gauge}(C, k) \circ (1, 1), \mathrm{Gauge}(C, k) \circ (2, 1))$.

(21) Let r, t be real numbers. Suppose $r > 0$ and $t > 0$. Then there exists a natural number n such that $i < n$ and $\rho(\mathrm{Gauge}(C, n) \circ (1, 1), \mathrm{Gauge}(C, n) \circ (1, 2)) < r$ and $\rho(\mathrm{Gauge}(C, n) \circ (1, 1), \mathrm{Gauge}(C, n) \circ (2, 1)) < t$.

3. Middle Points

We now state four propositions:

(22) $\mathrm{UpperMiddlePoint}\,C \in C$.

(23) $\mathrm{LowerMiddlePoint}\,C \in C$.

(24) $(\mathrm{LowerMiddlePoint}\,C)_{\mathbf{2}} \neq (\mathrm{UpperMiddlePoint}\,C)_{\mathbf{2}}$.

(25) LowerMiddlePoint $C \neq$ UpperMiddlePoint C.

4. UpperArc and LowerArc

Next we state several propositions:

(26) W-bound(C) = W-bound(UpperArc(C)).

(27) E-bound(C) = E-bound(UpperArc(C)).

(28) W-bound(C) = W-bound(LowerArc(C)).

(29) E-bound(C) = E-bound(LowerArc(C)).

(30) UpperArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ is not empty and proj2°(UpperArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$) is not empty.

(31) LowerArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ is not empty and proj2°(LowerArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$) is not empty.

(32) For every compact connected subset P of $\mathcal{E}^2_{\rm T}$ such that $P \subseteq C$ and $\mathrm{W_{min}}(C) \in P$ and $\mathrm{E_{max}}(C) \in P$ holds UpperArc$(C) \subseteq P$ or LowerArc$(C) \subseteq P$.

5. UMP and LMP

Let P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. The functor UMP P yielding a point of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 1) UMP $P = [\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2}$, sup(proj2°($P \cap$ VerticalLine$(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})$))].

The functor LMP P yielding a point of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 2) LMP $P = [\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2}$, inf(proj2°($P \cap$ VerticalLine$(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})$))].

We now state a number of propositions:

(33) $(\text{UMP}\, P)_{\mathbf{1}} = \frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}$.

(34) $(\text{UMP}\, P)_{\mathbf{2}}$ = sup(proj2°($P \cap$ VerticalLine$(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})$)).

(35) $(\text{LMP}\, P)_{\mathbf{1}} = \frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}$.

(36) $(\text{LMP}\, P)_{\mathbf{2}}$ = inf(proj2°($P \cap$ VerticalLine$(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})$)).

(37) For every non vertical compact subset C of $\mathcal{E}^2_{\rm T}$ holds UMP $C \neq \mathrm{W_{min}}(C)$.

(38) For every non vertical compact subset C of $\mathcal{E}^2_{\rm T}$ holds UMP $C \neq \mathrm{E_{max}}(C)$.

(39) For every non vertical compact subset C of $\mathcal{E}^2_{\rm T}$ holds LMP $C \neq \mathrm{W_{min}}(C)$.

(40) For every non vertical compact subset C of $\mathcal{E}^2_{\rm T}$ holds LMP $C \neq \mathrm{E_{max}}(C)$.

(41) For every compact subset C of $\mathcal{E}^2_{\rm T}$ such that $p \in C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ holds $p_{\mathbf{2}} \leq (\mathrm{UMP}\, C)_{\mathbf{2}}$.

(42) For every compact subset C of $\mathcal{E}^2_{\rm T}$ such that $p \in C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ holds $(\mathrm{LMP}\, C)_{\mathbf{2}} \leq p_{\mathbf{2}}$.

(43) $\mathrm{UMP}\, C \in C$.

(44) $\mathrm{LMP}\, C \in C$.

(45) $\mathcal{L}(\mathrm{UMP}\, P, [\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}, \text{N-bound}(P)])$ is vertical.

(46) $\mathcal{L}(\mathrm{LMP}\, P, [\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}, \text{S-bound}(P)])$ is vertical.

(47) $\mathcal{L}(\mathrm{UMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{N-bound}(C)]) \cap C = \{\mathrm{UMP}\, C\}$.

(48) $\mathcal{L}(\mathrm{LMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{S-bound}(C)]) \cap C = \{\mathrm{LMP}\, C\}$.

(49) $(\mathrm{LMP}\, C)_{\mathbf{2}} < (\mathrm{UMP}\, C)_{\mathbf{2}}$.

(50) $\mathrm{UMP}\, C \neq \mathrm{LMP}\, C$.

(51) $\text{S-bound}(C) < (\mathrm{UMP}\, C)_{\mathbf{2}}$.

(52) $(\mathrm{UMP}\, C)_{\mathbf{2}} \leq \text{N-bound}(C)$.

(53) $\text{S-bound}(C) \leq (\mathrm{LMP}\, C)_{\mathbf{2}}$.

(54) $(\mathrm{LMP}\, C)_{\mathbf{2}} < \text{N-bound}(C)$.

(55) $\mathcal{L}(\mathrm{UMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{N-bound}(C)])$ misses $\mathcal{L}(\mathrm{LMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{S-bound}(C)])$.

(56) Let A, B be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $A \subseteq B$ and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^{\circ}(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2}))$ is upper bounded. Then $(\mathrm{UMP}\, A)_{\mathbf{2}} \leq (\mathrm{UMP}\, B)_{\mathbf{2}}$.

(57) Let A, B be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $A \subseteq B$ and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^{\circ}(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2}))$ is lower bounded. Then $(\mathrm{LMP}\, B)_{\mathbf{2}} \leq (\mathrm{LMP}\, A)_{\mathbf{2}}$.

(58) Let A, B be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $A \subseteq B$ and $\mathrm{UMP}\, B \in A$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^{\circ}(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(B)+\text{E-bound}(B)}{2}))$ is upper bounded and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$. Then $\mathrm{UMP}\, A = \mathrm{UMP}\, B$.

(59) Let A, B be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $A \subseteq B$ and $\mathrm{LMP}\, B \in A$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^{\circ}(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(B)+\text{E-bound}(B)}{2}))$ is lower bounded and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$. Then $\mathrm{LMP}\, A = \mathrm{LMP}\, B$.

(60) $(\mathrm{UMP}\,\mathrm{UpperArc}(C))_{\mathbf{2}} \leq \text{N-bound}(C)$.

(61) $\text{S-bound}(C) \leq (\mathrm{LMP}\,\mathrm{LowerArc}(C))_{\mathbf{2}}$.

(62) $\mathrm{LMP}\,C \notin \mathrm{LowerArc}(C)$ or $\mathrm{UMP}\,C \notin \mathrm{LowerArc}(C)$.

(63) $\mathrm{LMP}\,C \notin \mathrm{UpperArc}(C)$ or $\mathrm{UMP}\,C \notin \mathrm{UpperArc}(C)$.

(64) If $0 < n$, then $\sup(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), \mathrm{len}\,\mathrm{Gauge}(C,n))))) = \sup(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathrm{VerticalLine}(\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))+\text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2})))$.

(65) If $0 < n$, then $\inf(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), \mathrm{len}\,\mathrm{Gauge}(C,n))))) = \inf(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathrm{VerticalLine}(\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))+\text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2})))$.

(66) If $0 < n$, then $\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = [\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))+\text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2}, \sup(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), \mathrm{len}\,\mathrm{Gauge}(C,n)))))]$.

(67) If $0 < n$, then $\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = [\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))+\text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2}, \inf(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), \mathrm{len}\,\mathrm{Gauge}(C,n)))))]$.

(68) $(\mathrm{UMP}\,C)_{\mathbf{2}} < (\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))_{\mathbf{2}}$.

(69) $(\mathrm{LMP}\,C)_{\mathbf{2}} > (\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))_{\mathbf{2}}$.

(70) $\mathrm{UMP}\,\mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \in \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(71) $\mathrm{LMP}\,\mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \in \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(72) If $0 < n$, then there exists a natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), i)$.

(73) If $0 < n$, then there exists a natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), i)$.

(74) If $0 < n$, then $\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{UMP}\,\mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(75) If $0 < n$, then $\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{LMP}\,\mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(76) If $0 < n$, then $(\mathrm{UMP}\,C)_{\mathbf{2}} < (\mathrm{UMP}\,\mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))))_{\mathbf{2}}$.

(77) If $0 < n$, then $(\mathrm{LMP}\,\mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))))_{\mathbf{2}} < (\mathrm{LMP}\,C)_{\mathbf{2}}$.

(78) If $i \leq j$, then $(\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j)))_{\mathbf{2}} \leq (\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i)))_{\mathbf{2}}$.

(79) If $i \leq j$, then $(\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i)))_{\mathbf{2}} \leq (\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j)))_{\mathbf{2}}$.

(80) If $0 < i$ and $i \leq j$, then $(\mathrm{UMP}\,\mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j))))_{\mathbf{2}} \leq (\mathrm{UMP}\,\mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i))))_{\mathbf{2}}$.

(81) If $0 < i$ and $i \leq j$, then $(\mathrm{LMP}\,\mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i))))_{\mathbf{2}} \leq (\mathrm{LMP}\,\mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j))))_{\mathbf{2}}$.

Acknowledgments

The author would like to acknowledge Professor Andrzej Trybulec for his continuous encouragement to complete the proof of the Jordan Curve Theorem in Mizar as well as his precious suggestions regarding the proof itself.

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Received October 6, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

On Some Points of a Simple Closed Curve. Part II

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Summary. In the paper we formalize some lemmas needed by the proof of the Jordan Curve Theorem according to [23]. We show basic properties of the upper and the lower approximations of a simple closed curve (as its compactness and connectedness) and some facts about special points of such approximations.

MML Identifier: `JORDAN22`.

The notation and terminology used in this paper are introduced in the following papers: [25], [28], [1], [24], [29], [4], [16], [15], [2], [12], [22], [7], [27], [21], [13], [3], [5], [8], [9], [10], [18], [19], [20], [26], [6], [11], [17], and [14].

1. Properties of the Approximations

In this paper C denotes a simple closed curve and i denotes a natural number.

We now state two propositions:

(1) $(\mathrm{UpperAppr}(C))(i) \subseteq \overline{\mathrm{RightComp}(\mathrm{Cage}(C,0))}$.

(2) $(\mathrm{LowerAppr}(C))(i) \subseteq \overline{\mathrm{RightComp}(\mathrm{Cage}(C,0))}$.

Let C be a simple closed curve. One can verify that $\mathrm{UpperArc}(C)$ is connected and $\mathrm{LowerArc}(C)$ is connected.

We now state two propositions:

(3) $(\mathrm{UpperAppr}(C))(i)$ is compact and connected.

(4) $(\mathrm{LowerAppr}(C))(i)$ is compact and connected.

[1]The paper has been completed during the first author's post-doctoral fellowship granted by the Shinshu University, Japan.

[2]This work has been partially supported by the KBN grant 4 T11C 039 24 and the FP6 IST grant TYPES No. 510996.

ISSN 1426–2630

Let C be a simple closed curve. Observe that NorthArc(C) is compact and SouthArc(C) is compact.

2. On Special Points of Approximations

One can prove the following propositions:

(5) $\mathrm{W_{min}}(C) \in \mathrm{NorthArc}(C)$.

(6) $\mathrm{E_{max}}(C) \in \mathrm{NorthArc}(C)$.

(7) $\mathrm{W_{min}}(C) \in \mathrm{SouthArc}(C)$.

(8) $\mathrm{E_{max}}(C) \in \mathrm{SouthArc}(C)$.

(9) $\mathrm{UMP}\, C \in \mathrm{NorthArc}(C)$.

(10) $\mathrm{LMP}\, C \in \mathrm{SouthArc}(C)$.

(11) $\mathrm{NorthArc}(C) \subseteq C$.

(12) $\mathrm{SouthArc}(C) \subseteq C$.

(13) $\mathrm{LMP}\, C \in \mathrm{LowerArc}(C)$ and $\mathrm{UMP}\, C \in \mathrm{UpperArc}(C)$ or $\mathrm{UMP}\, C \in \mathrm{LowerArc}(C)$ and $\mathrm{LMP}\, C \in \mathrm{UpperArc}(C)$.

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Received October 6, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Uniform Continuity of Functions on Normed Complex Linear Spaces

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MML Identifier: NCFCONT2.

The papers [19], [22], [1], [17], [10], [23], [4], [24], [5], [13], [20], [21], [18], [3], [12], [11], [2], [25], [16], [6], [8], [15], [7], [14], and [9] provide the notation and terminology for this paper.

1. Uniform Continuity of Functions on Real and Complex Normed Linear Spaces

For simplicity, we follow the rules: X, X_1 denote sets, r, s denote real numbers, z denotes a complex number, R_1 denotes a real normed space, and C_1, C_2, C_3 denote complex normed spaces.

Let X be a set, let C_2, C_3 be complex normed spaces, and let f be a partial function from C_2 to C_3. We say that f is uniformly continuous on X if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of C_2 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

Let X be a set, let R_1 be a real normed space, let C_1 be a complex normed space, and let f be a partial function from C_1 to R_1. We say that f is uniformly continuous on X if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

ISSN 1426–2630

Let X be a set, let R_1 be a real normed space, let C_1 be a complex normed space, and let f be a partial function from R_1 to C_1. We say that f is uniformly continuous on X if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of R_1 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

Let X be a set, let C_1 be a complex normed space, and let f be a partial function from the carrier of C_1 to $\mathbb{C}$. We say that f is uniformly continuous on X if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $|f_{x_1} - f_{x_2}| < r$.

Let X be a set, let C_1 be a complex normed space, and let f be a partial function from the carrier of C_1 to $\mathbb{R}$. We say that f is uniformly continuous on X if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $|f_{x_1} - f_{x_2}| < r$.

Let X be a set, let R_1 be a real normed space, and let f be a partial function from the carrier of R_1 to $\mathbb{C}$. We say that f is uniformly continuous on X if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of R_1 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $|f_{x_1} - f_{x_2}| < r$.

Next we state a number of propositions:

(1) Let f be a partial function from C_2 to C_3. Suppose f is uniformly continuous on X and $X_1 \subseteq X$. Then f is uniformly continuous on X_1.

(2) Let f be a partial function from C_1 to R_1. Suppose f is uniformly continuous on X and $X_1 \subseteq X$. Then f is uniformly continuous on X_1.

(3) Let f be a partial function from R_1 to C_1. Suppose f is uniformly continuous on X and $X_1 \subseteq X$. Then f is uniformly continuous on X_1.

(4) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 + f_2$ is uniformly continuous on $X \cap X_1$.

(5) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 + f_2$ is

uniformly continuous on $X \cap X_1$.

(6) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 + f_2$ is uniformly continuous on $X \cap X_1$.

(7) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

(8) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

(9) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

(10) Let f be a partial function from C_2 to C_3. If f is uniformly continuous on X, then $z\, f$ is uniformly continuous on X.

(11) Let f be a partial function from C_1 to R_1. If f is uniformly continuous on X, then $r\, f$ is uniformly continuous on X.

(12) Let f be a partial function from R_1 to C_1. If f is uniformly continuous on X, then $z\, f$ is uniformly continuous on X.

(13) Let f be a partial function from C_2 to C_3. If f is uniformly continuous on X, then $-f$ is uniformly continuous on X.

(14) Let f be a partial function from C_1 to R_1. If f is uniformly continuous on X, then $-f$ is uniformly continuous on X.

(15) Let f be a partial function from R_1 to C_1. If f is uniformly continuous on X, then $-f$ is uniformly continuous on X.

(16) Let f be a partial function from C_2 to C_3. If f is uniformly continuous on X, then $\|f\|$ is uniformly continuous on X.

(17) Let f be a partial function from C_1 to R_1. If f is uniformly continuous on X, then $\|f\|$ is uniformly continuous on X.

(18) Let f be a partial function from R_1 to C_1. If f is uniformly continuous on X, then $\|f\|$ is uniformly continuous on X.

(19) For every partial function f from C_2 to C_3 such that f is uniformly continuous on X holds f is continuous on X.

(20) For every partial function f from C_1 to R_1 such that f is uniformly continuous on X holds f is continuous on X.

(21) For every partial function f from R_1 to C_1 such that f is uniformly continuous on X holds f is continuous on X.

(22) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$. If f is uniformly continuous on X, then f is continuous on X.

(23) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. If f is uniformly continuous on X, then f is continuous on X.

(24) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$. If f is uniformly continuous on X, then f is continuous on X.

(25) For every partial function f from C_2 to C_3 such that f is Lipschitzian on X holds f is uniformly continuous on X.

(26) For every partial function f from C_1 to R_1 such that f is Lipschitzian on X holds f is uniformly continuous on X.

(27) For every partial function f from R_1 to C_1 such that f is Lipschitzian on X holds f is uniformly continuous on X.

(28) Let f be a partial function from C_2 to C_3 and Y be a subset of C_2. Suppose Y is compact and f is continuous on Y. Then f is uniformly continuous on Y.

(29) Let f be a partial function from C_1 to R_1 and Y be a subset of C_1. Suppose Y is compact and f is continuous on Y. Then f is uniformly continuous on Y.

(30) Let f be a partial function from R_1 to C_1 and Y be a subset of R_1. Suppose Y is compact and f is continuous on Y. Then f is uniformly continuous on Y.

(31) Let f be a partial function from C_2 to C_3 and Y be a subset of C_2. Suppose $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y. Then $f^\circ Y$ is compact.

(32) Let f be a partial function from C_1 to R_1 and Y be a subset of C_1. Suppose $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y. Then $f^\circ Y$ is compact.

(33) Let f be a partial function from R_1 to C_1 and Y be a subset of R_1. Suppose $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y. Then $f^\circ Y$ is compact.

(34) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$ and Y be a subset of C_1. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y. Then there exist points x_1, x_2 of C_1 such that $x_1 \in Y$ and $x_2 \in Y$ and $f_{x_1} = \sup(f^\circ Y)$ and $f_{x_2} = \inf(f^\circ Y)$.

(35) Let f be a partial function from C_2 to C_3. If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is uniformly continuous on X.

(36) Let f be a partial function from C_1 to R_1. If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is uniformly continuous on X.

(37) Let f be a partial function from R_1 to C_1. If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is uniformly continuous on X.

2. Contraction Mapping Principle on Normed Complex Linear Spaces

Let M be a complex Banach space. A function from the carrier of M into the carrier of M is said to be a contraction of M if:

(Def. 7) There exists a real number L such that $0 < L$ and $L < 1$ and for all points x, y of M holds $\|\mathrm{it}(x) - \mathrm{it}(y)\| \leq L \cdot \|x - y\|$.

One can prove the following four propositions:

(38) For every complex normed space X and for all points x, y of X holds $\|x - y\| > 0$ iff $x \neq y$.

(39) For every complex normed space X and for all points x, y of X holds $\|x - y\| = \|y - x\|$.

(40) Let X be a complex Banach space and f be a function from X into X. Suppose f is a contraction of X. Then there exists a point x_3 of X such that $f(x_3) = x_3$ and for every point x of X such that $f(x) = x$ holds $x_3 = x$.

(41) Let X be a complex Banach space and f be a function from X into X. Given a natural number n_0 such that f^{n_0} is a contraction of X. Then there exists a point x_3 of X such that $f(x_3) = x_3$ and for every point x of X such that $f(x) = x$ holds $x_3 = x$.

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Received October 6, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Introduction to Real Linear Topological Spaces[1]

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MML Identifier: `RLTOPSP1`.

The terminology and notation used in this paper are introduced in the following articles: [20], [7], [23], [10], [15], [19], [1], [4], [24], [5], [6], [3], [13], [18], [17], [25], [9], [16], [8], [14], [2], [21], [22], [12], and [11].

1. Preliminaries

In this paper X is a non empty RLS structure and r, s, t are real numbers.

Let us note that there exists a real number which is non zero.

We now state a number of propositions:

(2)[2] Let T be a non empty topological space, X be a non empty subset of T, and F_1 be a family of subsets of T. Suppose F_1 is a cover of X. Let x be a point of T. If $x \in X$, then there exists a subset W of T such that $x \in W$ and $W \in F_1$.

(4)[3] Let X be a non empty loop structure, M, N be subsets of X, and F be a family of subsets of X. If $F = \{x + N; x$ ranges over points of X: $x \in M\}$, then $M + N = \bigcup F$.

(5) Let X be an add-associative right zeroed right complementable non empty loop structure and M be a subset of X. Then $0_X + M = M$.

(6) Let X be an add-associative non empty loop structure, x, y be points of X, and M be a subset of X. Then $(x + y) + M = x + (y + M)$.

[1]This work has been partially supported by the KBN grant 4 T11C 039 24.
[2]The proposition (1) has been removed.
[3]The proposition (3) has been removed.

ISSN 1426–2630

(7) Let X be an add-associative non empty loop structure, x be a point of X, and M, N be subsets of X. Then $(x+M)+N = x+(M+N)$.

(8) Let X be a non empty loop structure, M, N be subsets of X, and x be a point of X. If $M \subseteq N$, then $x+M \subseteq x+N$.

(9) Let X be a non empty real linear space, M be a subset of X, and x be a point of X. If $x \in M$, then $0_X \in -x+M$.

(10) For every non empty loop structure X and for all subsets M, N, V of X such that $M \subseteq N$ holds $M+V \subseteq N+V$.

(11) For every non empty loop structure X and for all subsets V_1, V_2, W_1, W_2 of X such that $V_1 \subseteq W_1$ and $V_2 \subseteq W_2$ holds $V_1+V_2 \subseteq W_1+W_2$.

(12) For every non empty real linear space X and for all subsets V_1, V_2 of X such that $0_X \in V_2$ holds $V_1 \subseteq V_1+V_2$.

(13) For every non empty real linear space X and for every real number r holds $r \cdot \{0_X\} = \{0_X\}$.

(14) Let X be a non empty real linear space, M be a subset of X, and r be a non zero real number. If $0_X \in r \cdot M$, then $0_X \in M$.

(15) Let X be a non empty real linear space, M, N be subsets of X, and r be a non zero real number. Then $(r \cdot M) \cap (r \cdot N) = r \cdot (M \cap N)$.

(16) Let X be a non empty topological space, x be a point of X, A be a neighbourhood of x, and B be a subset of X. If $A \subseteq B$, then B is a neighbourhood of x.

Let V be a non empty real linear space and let M be a subset of V. Let us observe that M is convex if and only if:

(Def. 1) For all points u, v of V and for every real number r such that $0 \leq r$ and $r \leq 1$ and $u \in M$ and $v \in M$ holds $r \cdot u + (1-r) \cdot v \in M$.

One can prove the following proposition

(17) Let X be a non empty real linear space, M be a convex subset of X, and r_1, r_2 be real numbers. If $0 \leq r_1$ and $0 \leq r_2$, then $r_1 \cdot M + r_2 \cdot M = (r_1+r_2) \cdot M$.

Let X be a non empty real linear space and let M be an empty subset of X. One can check that $\operatorname{conv} M$ is empty.

Next we state several propositions:

(18) For every non empty real linear space X and for every convex subset M of X holds $\operatorname{conv} M = M$.

(19) For every non empty real linear space X and for every subset M of X and for every real number r holds $r \cdot \operatorname{conv} M = \operatorname{conv} r \cdot M$.

(20) For every non empty real linear space X and for all subsets M_1, M_2 of X such that $M_1 \subseteq M_2$ holds Convex-Family $M_2 \subseteq$ Convex-Family M_1.

(21) For every non empty real linear space X and for all subsets M_1, M_2 of X such that $M_1 \subseteq M_2$ holds $\operatorname{conv} M_1 \subseteq \operatorname{conv} M_2$.

(22) Let X be a non empty real linear space, M be a convex subset of X, and r be a real number. If $0 \leq r$ and $r \leq 1$ and $0_X \in M$, then $r \cdot M \subseteq M$.

Let X be a non empty real linear space and let v, w be points of X. The functor $\mathcal{L}(v, w)$ yields a subset of X and is defined as follows:

(Def. 2) $\mathcal{L}(v, w) = \{(1 - r) \cdot v + r \cdot w : 0 \leq r \ \wedge \ r \leq 1\}$.

Let X be a non empty real linear space and let v, w be points of X. Note that $\mathcal{L}(v, w)$ is non empty and convex.

Next we state the proposition

(23) Let X be a non empty real linear space and M be a subset of X. Then M is convex if and only if for all points u, w of X such that $u \in M$ and $w \in M$ holds $\mathcal{L}(u, w) \subseteq M$.

Let V be a non empty RLS structure and let P be a family of subsets of V. We say that P is convex-membered if and only if:

(Def. 3) For every subset M of V such that $M \in P$ holds M is convex.

Let V be a non empty RLS structure. One can verify that there exists a family of subsets of V which is non empty and convex-membered.

We now state the proposition

(24) For every non empty RLS structure V and for every convex-membered family F of subsets of V holds $\bigcap F$ is convex.

Let X be a non empty RLS structure and let A be a subset of X. The functor $-A$ yielding a subset of X is defined by:

(Def. 4) $-A = (-1) \cdot A$.

One can prove the following proposition

(25) Let X be a non empty real linear space, M, N be subsets of X, and v be a point of X. Then $v + M$ meets N if and only if $v \in N + -M$.

Let X be a non empty RLS structure and let A be a subset of X. We say that A is symmetric if and only if:

(Def. 5) $A = -A$.

Let X be a non empty real linear space. Observe that there exists a subset of X which is non empty and symmetric.

One can prove the following proposition

(26) Let X be a non empty real linear space, A be a symmetric subset of X, and x be a point of X. If $x \in A$, then $-x \in A$.

Let X be a non empty RLS structure and let A be a subset of X. We say that A is circled if and only if:

(Def. 6) For every real number r such that $|r| \leq 1$ holds $r \cdot A \subseteq A$.

Let X be a non empty real linear space. Note that $\emptyset_X$ is circled.

We now state the proposition

(27) For every non empty real linear space X holds $\{0_X\}$ is circled.

Let X be a non empty real linear space. Observe that there exists a subset of X which is non empty and circled.

The following proposition is true

(28) For every non empty real linear space X and for every non empty circled subset B of X holds $0_X \in B$.

Let X be a non empty real linear space and let A, B be circled subsets of X. One can verify that $A + B$ is circled.

We now state the proposition

(29) Let X be a non empty real linear space, A be a circled subset of X, and r be a real number. If $|r| = 1$, then $r \cdot A = A$.

Let X be a non empty real linear space. One can check that every subset of X which is circled is also symmetric.

Let X be a non empty real linear space and let M be a circled subset of X. One can check that $\operatorname{conv} M$ is circled.

Let X be a non empty RLS structure and let F be a family of subsets of X. We say that F is circled-membered if and only if:

(Def. 7) For every subset V of X such that $V \in F$ holds V is circled.

Let V be a non empty real linear space. Note that there exists a family of subsets of V which is non empty and circled-membered.

The following two propositions are true:

(30) For every non empty real linear space X and for every circled-membered family F of subsets of X holds $\bigcup F$ is circled.

(31) For every non empty real linear space X and for every circled-membered family F of subsets of X holds $\bigcap F$ is circled.

2. Real Linear Topological Space

We introduce real linear topological structures which are extensions of RLS structure and topological structure and are systems

$\langle$ a carrier, a zero, an addition, an external multiplication, a topology $\rangle$,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, the external multiplication is a function from $[:\mathbb{R}, \text{the carrier}:]$ into the carrier, and the topology is a family of subsets of the carrier.

Let X be a non empty set, let O be an element of X, let F be a binary operation on X, let G be a function from $[:\mathbb{R}, X:]$ into X, and let T be a family of subsets of X. Observe that $\langle X, O, F, G, T\rangle$ is non empty.

Let us note that there exists a real linear topological structure which is strict and non empty.

Let X be a non empty real linear topological structure. We say that X is add-continuous if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let x_1, x_2 be points of X and V be a subset of X. Suppose V is open and $x_1 + x_2 \in V$. Then there exist subsets V_1, V_2 of X such that V_1 is open and V_2 is open and $x_1 \in V_1$ and $x_2 \in V_2$ and $V_1 + V_2 \subseteq V$.

We say that X is mult-continuous if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let a be a real number, x be a point of X, and V be a subset of X. Suppose V is open and $a \cdot x \in V$. Then there exists a positive real number r and there exists a subset W of X such that W is open and $x \in W$ and for every real number s such that $|s - a| < r$ holds $s \cdot W \subseteq V$.

Let us note that there exists a non empty real linear topological structure which is non empty, strict, add-continuous, mult-continuous, topological space-like, Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

A linear topological space is an add-continuous mult-continuous topological space-like Abelian add-associative right zeroed right complementable real linear space-like non empty real linear topological structure.

One can prove the following two propositions:

(32) Let X be a non empty linear topological space, x_1, x_2 be points of X, and V be a neighbourhood of $x_1 + x_2$. Then there exists a neighbourhood V_1 of x_1 and there exists a neighbourhood V_2 of x_2 such that $V_1 + V_2 \subseteq V$.

(33) Let X be a non empty linear topological space, a be a real number, x be a point of X, and V be a neighbourhood of $a \cdot x$. Then there exists a positive real number r and there exists a neighbourhood W of x such that for every real number s if $|s - a| < r$, then $s \cdot W \subseteq V$.

Let X be a non empty real linear topological structure and let a be a point of X. The functor $\mathrm{transl}(a, X)$ yields a map from X into X and is defined by:

(Def. 10) For every point x of X holds $(\mathrm{transl}(a, X))(x) = a + x$.

The following propositions are true:

(34) Let X be a non empty real linear topological structure, a be a point of X, and V be a subset of X. Then $(\mathrm{transl}(a, X))^\circ V = a + V$.

(35) For every non empty linear topological space X and for every point a of X holds $\mathrm{rng}\,\mathrm{transl}(a, X) = \Omega_X$.

(36) For every non empty linear topological space X and for every point a of X holds $(\mathrm{transl}(a, X))^{-1} = \mathrm{transl}(-a, X)$.

Let X be a non empty linear topological space and let a be a point of X. Note that $\mathrm{transl}(a, X)$ is homeomorphism.

Let X be a non empty linear topological space, let E be an open subset of X, and let x be a point of X. Note that $x + E$ is open.

Let X be a non empty linear topological space, let E be an open subset of X, and let x be a point of X. Observe that $x + E$ is open.

Let X be a non empty linear topological space, let E be an open subset of X, and let K be a subset of X. Observe that $K + E$ is open.

Let X be a non empty linear topological space, let D be a closed subset of X, and let x be a point of X. Note that $x + D$ is closed.

We now state several propositions:

(37) For every non empty linear topological space X and for all subsets V_1, V_2, V of X such that $V_1 + V_2 \subseteq V$ holds $\operatorname{Int} V_1 + \operatorname{Int} V_2 \subseteq \operatorname{Int} V$.

(38) For every non empty linear topological space X and for every point x of X and for every subset V of X holds $x + \operatorname{Int} V = \operatorname{Int}(x + V)$.

(39) For every non empty linear topological space X and for every point x of X and for every subset V of X holds $x + \overline{V} = \overline{x + V}$.

(40) Let X be a non empty linear topological space, x, v be points of X, and V be a neighbourhood of x. Then $v + V$ is a neighbourhood of $v + x$.

(41) Let X be a non empty linear topological space, x be a point of X, and V be a neighbourhood of x. Then $-x + V$ is a neighbourhood of 0_X.

Let X be a non empty real linear topological structure. A local base of X is a generalized basis of 0_X.

Let X be a non empty real linear topological structure. We say that X is locally-convex if and only if:

(Def. 11) There exists a local base of X which is convex-membered.

Let X be a non empty linear topological space and let E be a subset of X. We say that E is bounded if and only if:

(Def. 12) For every neighbourhood V of 0_X there exists s such that $s > 0$ and for every t such that $t > s$ holds $E \subseteq t \cdot V$.

Let X be a non empty linear topological space. Note that $\emptyset_X$ is bounded.

Let X be a non empty linear topological space. Observe that there exists a subset of X which is bounded.

The following propositions are true:

(42) For every non empty linear topological space X and for all bounded subsets V_1, V_2 of X holds $V_1 \cup V_2$ is bounded.

(43) Let X be a non empty linear topological space, P be a bounded subset of X, and Q be a subset of X. If $Q \subseteq P$, then Q is bounded.

(44) Let X be a non empty linear topological space and F be a family of subsets of X. Suppose F is finite and $F = \{P : P$ ranges over bounded subsets of $X\}$. Then $\bigcup F$ is bounded.

(45) Let X be a non empty linear topological space and P be a family of subsets of X. Suppose $P = \{U : U$ ranges over neighbourhoods of $0_X\}$. Then P is a local base of X.

(46) Let X be a non empty linear topological space, O be a local base of X, and P be a family of subsets of X. Suppose $P = \{a + U; a$ ranges over points of X, U ranges over subsets of X: $U \in O\}$. Then P is a generalized basis of X.

Let X be a non empty real linear topological structure and let r be a real number. The functor $r \bullet X$ yielding a map from X into X is defined as follows:

(Def. 13) For every point x of X holds $(r \bullet X)(x) = r \cdot x$.

The following propositions are true:

(47) Let X be a non empty real linear topological structure, V be a subset of X, and r be a non zero real number. Then $(r \bullet X)^\circ V = r \cdot V$.

(48) For every non empty linear topological space X and for every non zero real number r holds $\mathrm{rng}(r \bullet X) = \Omega_X$.

(49) For every non empty linear topological space X and for every non zero real number r holds $(r \bullet X)^{-1} = r^{-1} \bullet X$.

Let X be a non empty linear topological space and let r be a non zero real number. One can check that $r \bullet X$ is homeomorphism.

Next we state several propositions:

(50) Let X be a non empty linear topological space, V be an open subset of X, and r be a non zero real number. Then $r \cdot V$ is open.

(51) Let X be a non empty linear topological space, V be a closed subset of X, and r be a non zero real number. Then $r \cdot V$ is closed.

(52) Let X be a non empty linear topological space, V be a subset of X, and r be a non zero real number. Then $r \cdot \mathrm{Int}\, V = \mathrm{Int}(r \cdot V)$.

(53) Let X be a non empty linear topological space, A be a subset of X, and r be a non zero real number. Then $r \cdot \overline{A} = \overline{r \cdot A}$.

(54) Let X be a non empty linear topological space and A be a subset of X. If X is a T_1 space, then $0 \cdot \overline{A} = \overline{0 \cdot A}$.

(55) Let X be a non empty linear topological space, x be a point of X, V be a neighbourhood of x, and r be a non zero real number. Then $r \cdot V$ is a neighbourhood of $r \cdot x$.

(56) Let X be a non empty linear topological space, V be a neighbourhood of 0_X, and r be a non zero real number. Then $r \cdot V$ is a neighbourhood of 0_X.

Let X be a non empty linear topological space, let V be a bounded subset of X, and let r be a real number. Observe that $r \cdot V$ is bounded.

We now state four propositions:

(57) Let X be a non empty linear topological space and W be a neighbourhood of 0_X. Then there exists an open neighbourhood U of 0_X such that U is symmetric and $U + U \subseteq W$.

(58) Let X be a non empty linear topological space, K be a compact subset of X, and C be a closed subset of X. Suppose K misses C. Then there exists a neighbourhood V of 0_X such that $K + V$ misses $C + V$.

(59) Let X be a non empty linear topological space, B be a local base of X, and V be a neighbourhood of 0_X. Then there exists a neighbourhood W of 0_X such that $W \in B$ and $\overline{W} \subseteq V$.

(60) Let X be a non empty linear topological space and V be a neighbourhood of 0_X. Then there exists a neighbourhood W of 0_X such that $\overline{W} \subseteq V$.

Let us observe that every non empty linear topological space which is T_1 is also Hausdorff.

We now state three propositions:

(61) Let X be a non empty linear topological space and A be a subset of X. Then $\overline{A} = \bigcap\{A + V : V$ ranges over neighbourhoods of $0_X\}$.

(62) For every non empty linear topological space X and for all subsets A, B of X holds $\operatorname{Int} A + \operatorname{Int} B \subseteq \operatorname{Int}(A + B)$.

(63) For every non empty linear topological space X and for all subsets A, B of X holds $\overline{A} + \overline{B} \subseteq \overline{A + B}$.

Let X be a non empty linear topological space and let C be a convex subset of X. Note that $\overline{C}$ is convex.

Let X be a non empty linear topological space and let C be a convex subset of X. Note that $\operatorname{Int} C$ is convex.

Let X be a non empty linear topological space and let B be a circled subset of X. One can check that $\overline{B}$ is circled.

One can prove the following proposition

(64) Let X be a non empty linear topological space and B be a circled subset of X. If $0_X \in \operatorname{Int} B$, then $\operatorname{Int} B$ is circled.

Let X be a non empty linear topological space and let E be a bounded subset of X. Note that $\overline{E}$ is bounded.

The following propositions are true:

(65) Let X be a non empty linear topological space and U be a neighbourhood of 0_X. Then there exists a neighbourhood W of 0_X such that W is circled and $W \subseteq U$.

(66) Let X be a non empty linear topological space and U be a neighbourhood of 0_X. Suppose U is convex. Then there exists a neighbourhood W of 0_X such that W is circled and convex and $W \subseteq U$.

(67) For every non empty linear topological space X holds there exists a local base of X which is circled-membered.

(68) For every non empty linear topological space X such that X is locally-convex holds there exists a local base of X which is convex-membered.

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Received October 6, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Some Properties of Rectangles on the Plane[1]

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MML Identifier: `TOPREALA`.

The terminology and notation used in this paper have been introduced in the following articles: [25], [9], [28], [2], [29], [5], [30], [8], [6], [16], [3], [23], [24], [27], [1], [4], [7], [22], [17], [21], [20], [26], [13], [10], [19], [31], [14], [12], [11], [18], and [15].

1. Real Numbers

We adopt the following rules: i is an integer and a, b, r, s are real numbers.

The following propositions are true:

(1) $\operatorname{frac}(r+i) = \operatorname{frac} r$.

(2) If $r \leq a$ and $a < \lfloor r \rfloor + 1$, then $\lfloor a \rfloor = \lfloor r \rfloor$.

(3) If $r \leq a$ and $a < \lfloor r \rfloor + 1$, then $\operatorname{frac} r \leq \operatorname{frac} a$.

(4) If $r < a$ and $a < \lfloor r \rfloor + 1$, then $\operatorname{frac} r < \operatorname{frac} a$.

(5) If $a \geq \lfloor r \rfloor + 1$ and $a \leq r + 1$, then $\lfloor a \rfloor = \lfloor r \rfloor + 1$.

(6) If $a \geq \lfloor r \rfloor + 1$ and $a < r + 1$, then $\operatorname{frac} a < \operatorname{frac} r$.

(7) If $r \leq a$ and $a < r + 1$ and $r \leq b$ and $b < r + 1$ and $\operatorname{frac} a = \operatorname{frac} b$, then $a = b$.

[1]The paper was written during the first author's post-doctoral fellowship granted by the Shinshu University, Japan.

ISSN 1426–2630

2. Subsets of $\mathbb{R}$

Let r be a real number and let s be a positive real number. One can verify the following observations:

* $]r, r+s[$ is non empty,
* $[r, r+s[$ is non empty,
* $]r, r+s]$ is non empty,
* $[r, r+s]$ is non empty,
* $]r-s, r[$ is non empty,
* $[r-s, r[$ is non empty,
* $]r-s, r]$ is non empty, and
* $[r-s, r]$ is non empty.

Let r be a non positive real number and let s be a positive real number. One can verify the following observations:

* $]r, s[$ is non empty,
* $[r, s[$ is non empty,
* $]r, s]$ is non empty, and
* $[r, s]$ is non empty.

Let r be a negative real number and let s be a non negative real number. One can check the following observations:

* $]r, s[$ is non empty,
* $[r, s[$ is non empty,
* $]r, s]$ is non empty, and
* $[r, s]$ is non empty.

We now state a number of propositions:

(8) If $r \leq a$ and $b < s$, then $[a, b] \subseteq [r, s[$.

(9) If $r < a$ and $b \leq s$, then $[a, b] \subseteq]r, s]$.

(10) If $r < a$ and $b < s$, then $[a, b] \subseteq]r, s[$.

(11) If $r \leq a$ and $b \leq s$, then $[a, b[\subseteq [r, s]$.

(12) If $r \leq a$ and $b \leq s$, then $[a, b[\subseteq [r, s[$.

(13) If $r < a$ and $b \leq s$, then $[a, b[\subseteq]r, s]$.

(14) If $r < a$ and $b \leq s$, then $[a, b[\subseteq]r, s[$.

(15) If $r \leq a$ and $b \leq s$, then $]a, b] \subseteq [r, s]$.

(16) If $r \leq a$ and $b < s$, then $]a, b] \subseteq [r, s[$.

(17) If $r \leq a$ and $b \leq s$, then $]a, b] \subseteq]r, s]$.

(18) If $r \leq a$ and $b < s$, then $]a, b] \subseteq]r, s[$.

(19) If $r \leq a$ and $b \leq s$, then $]a, b[\subseteq [r, s]$.

(20) If $r \leq a$ and $b \leq s$, then $]a, b[\subseteq [r, s[$.

(21) If $r \leq a$ and $b \leq s$, then $]a, b[\subseteq]r, s]$.

3. Functions

The following propositions are true:

(22) For every function f and for all sets x, X such that $x \in \operatorname{dom} f$ and $f(x) \in f^\circ X$ and f is one-to-one holds $x \in X$.

(23) For every finite sequence f and for every natural number i such that $i + 1 \in \operatorname{dom} f$ holds $i \in \operatorname{dom} f$ or $i = 0$.

(24) For all sets x, y, X, Y and for every function f such that $x \neq y$ and $f \in \prod[x \longmapsto X, y \longmapsto Y]$ holds $f(x) \in X$ and $f(y) \in Y$.

(25) For all sets a, b holds $\langle a, b \rangle = [1 \longmapsto a, 2 \longmapsto b]$.

4. General Topology

Let us note that there exists a topological space which is constituted finite sequences, non empty, and strict.

Let T be a constituted finite sequences topological space. Note that every subspace of T is constituted finite sequences.

One can prove the following proposition

(26) Let T be a non empty topological space, Z be a non empty subspace of T, t be a point of T, z be a point of Z, N be an open neighbourhood of t, and M be a subset of Z. If $t = z$ and $M = N \cap \Omega_Z$, then M is an open neighbourhood of z.

Let us note that every topological space which is empty is also discrete and anti-discrete.

Let X be a discrete topological space and let Y be a topological space. Note that every map from X into Y is continuous.

The following proposition is true

(27) Let X be a topological space, Y be a topological structure, and f be a map from X into Y. If f is empty, then f is continuous.

Let X be a topological space and let Y be a topological structure. Observe that every map from X into Y which is empty is also continuous.

One can prove the following propositions:

(28) Let X be a topological structure, Y be a non empty topological structure, and Z be a non empty subspace of Y. Then every map from X into Z is a map from X into Y.

(29) Let S, T be non empty topological spaces, X be a subset of S, Y be a subset of T, f be a continuous map from S into T, and g be a map from $S \restriction X$ into $T \restriction Y$. If $g = f \restriction X$, then g is continuous.

(30) Let S, T be non empty topological spaces, Z be a non empty subspace of T, f be a map from S into T, and g be a map from S into Z. If $f = g$ and f is open, then g is open.

(31) Let S, T be non empty topological spaces, S_1 be a subset of S, T_1 be a subset of T, f be a map from S into T, and g be a map from $S \restriction S_1$ into $T \restriction T_1$. If $g = f \restriction S_1$ and g is onto and f is open and one-to-one, then g is open.

(32) Let X, Y, Z be non empty topological spaces, f be a map from X into Y, and g be a map from Y into Z. If f is open and g is open, then $g \cdot f$ is open.

(33) Let X, Y be topological spaces, Z be an open subspace of Y, f be a map from X into Y, and g be a map from X into Z. If $f = g$ and g is open, then f is open.

(34) Let S, T be non empty topological spaces and f be a map from S into T. Suppose f is one-to-one and onto. Then f is continuous if and only if f^{-1} is open.

(35) Let S, T be non empty topological spaces and f be a map from S into T. Suppose f is one-to-one and onto. Then f is open if and only if f^{-1} is continuous.

(36) Let S be a topological space and T be a non empty topological space. Then S and T are homeomorphic if and only if the topological structure of S and the topological structure of T are homeomorphic.

(37) Let S, T be non empty topological spaces and f be a map from S into T. Suppose f is one-to-one, onto, continuous, and open. Then f is a homeomorphism.

5. $\mathbb{R}^{\mathbf{1}}$

One can prove the following propositions:

(38) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that $f = \mathbb{R} \longmapsto r$ holds f is continuous on $\mathbb{R}$.

(39) Let f, f_1, f_2 be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $\operatorname{dom} f = \operatorname{dom} f_1 \cup \operatorname{dom} f_2$ and $\operatorname{dom} f_1$ is open and $\operatorname{dom} f_2$ is open and f_1 is continuous on $\operatorname{dom} f_1$ and f_2 is continuous on $\operatorname{dom} f_2$ and for every set z such that $z \in \operatorname{dom} f_1$ holds $f(z) = f_1(z)$ and for every set z such that $z \in \operatorname{dom} f_2$ holds $f(z) = f_2(z)$. Then f is continuous on $\operatorname{dom} f$.

(40) Let x be a point of $\mathbb{R}^{\mathbf{1}}$, N be a subset of $\mathbb{R}$, and M be a subset of $\mathbb{R}^{\mathbf{1}}$. Suppose $M = N$. If N is a neighbourhood of x, then M is a neighbourhood of x.

(41) For every point x of $\mathbb{R}^{\mathbf{1}}$ and for every neighbourhood M of x there exists a neighbourhood N of x such that $N \subseteq M$.

(42) Let f be a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$, g be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and x be a point of $\mathbb{R}^{\mathbf{1}}$. If $f = g$ and g is continuous in x, then f is continuous at x.

(43) Let f be a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$ and g be a function from $\mathbb{R}$ into $\mathbb{R}$. If $f = g$ and g is continuous on $\mathbb{R}$, then f is continuous.

(44) If $a \leq r$ and $s \leq b$, then $[r, s]$ is a closed subset of $[a,\ b]_{\mathrm{T}}$.

(45) If $a \leq r$ and $s \leq b$, then $]r, s[$ is an open subset of $[a,\ b]_{\mathrm{T}}$.

(46) If $a \leq b$ and $a \leq r$, then $]r, b]$ is an open subset of $[a,\ b]_{\mathrm{T}}$.

(47) If $a \leq b$ and $r \leq b$, then $[a, r[$ is an open subset of $[a,\ b]_{\mathrm{T}}$.

(48) If $a \leq b$ and $r \leq s$, then the carrier of $[\![[a,\ b]_{\mathrm{T}},\ [r,\ s]_{\mathrm{T}}]\!] = [\![[a, b],\ [r, s]]\!]$.

6. $\mathcal{E}^2_{\mathrm{T}}$

Next we state four propositions:

(49) $[a, b] = [1 \longmapsto a, 2 \longmapsto b]$.

(50) $[a, b](1) = a$ and $[a, b](2) = b$.

(51) $\mathrm{ClosedInsideOfRectangle}(a, b, r, s) = \prod[1 \longmapsto [a, b], 2 \longmapsto [r, s]]$.

(52) If $a \leq b$ and $r \leq s$, then $[a, r] \in \mathrm{ClosedInsideOfRectangle}(a, b, r, s)$.

Let a, b, c, d be real numbers. The functor $\mathrm{Trectangle}(a, b, c, d)$ yielding a subspace of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 1) $\mathrm{Trectangle}(a, b, c, d) = (\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$.

The following propositions are true:

(53) The carrier of $\mathrm{Trectangle}(a, b, r, s) = \mathrm{ClosedInsideOfRectangle}(a, b, r, s)$.

(54) If $a \leq b$ and $r \leq s$, then $\mathrm{Trectangle}(a, b, r, s)$ is non empty.

Let a, c be non positive real numbers and let b, d be non negative real numbers. Observe that $\mathrm{Trectangle}(a, b, c, d)$ is non empty.

The map R2Homeo from $[\![\mathbb{R}^{\mathbf{1}},\ \mathbb{R}^{\mathbf{1}}]\!]$ into $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 2) For all real numbers x, y holds $\mathrm{R2Homeo}(\langle\!\langle x,\ y \rangle\!\rangle) = \langle x, y \rangle$.

Next we state several propositions:

(55) For all subsets A, B of $\mathbb{R}$ holds $\mathrm{R2Homeo}^\circ[\![A,\ B]\!] = \prod[1 \longmapsto A, 2 \longmapsto B]$.

(56) R2Homeo is a homeomorphism.

(57) If $a \leq b$ and $r \leq s$, then R2Homeo $\restriction$the carrier of $[\![[a, b]_{\rm T}, [r, s]_{\rm T}]\!]$ is a map from $[\![[a, b]_{\rm T}, [r, s]_{\rm T}]\!]$ into Trectangle(a, b, r, s).

(58) Suppose $a \leq b$ and $r \leq s$. Let h be a map from $[\![[a, b]_{\rm T}, [r, s]_{\rm T}]\!]$ into Trectangle(a, b, r, s). If $h =$ R2Homeo $\restriction$the carrier of $[\![[a, b]_{\rm T}, [r, s]_{\rm T}]\!]$, then h is a homeomorphism.

(59) If $a \leq b$ and $r \leq s$, then $[\![[a, b]_{\rm T}, [r, s]_{\rm T}]\!]$ and Trectangle(a, b, r, s) are homeomorphic.

(60) If $a \leq b$ and $r \leq s$, then for every subset A of $[a, b]_{\rm T}$ and for every subset B of $[r, s]_{\rm T}$ holds $\prod[1 \longmapsto A, 2 \longmapsto B]$ is a subset of Trectangle(a, b, r, s).

(61) Suppose $a \leq b$ and $r \leq s$. Let A be an open subset of $[a, b]_{\rm T}$ and B be an open subset of $[r, s]_{\rm T}$. Then $\prod[1 \longmapsto A, 2 \longmapsto B]$ is an open subset of Trectangle(a, b, r, s).

(62) Suppose $a \leq b$ and $r \leq s$. Let A be a closed subset of $[a, b]_{\rm T}$ and B be a closed subset of $[r, s]_{\rm T}$. Then $\prod[1 \longmapsto A, 2 \longmapsto B]$ is a closed subset of Trectangle(a, b, r, s).

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Received October 18, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Some Properties of Circles on the Plane[1]

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MML Identifier: `TOPREALB`.

The articles [30], [34], [1], [5], [35], [7], [6], [23], [29], [17], [4], [33], [2], [27], [24], [26], [31], [9], [25], [37], [12], [18], [11], [10], [28], [3], [14], [36], [15], [32], [13], [16], [20], [19], [21], [8], and [22] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we follow the rules: n is a natural number, i is an integer, a, b, r are real numbers, and x is a point of $\mathcal{E}^n_{\rm T}$.

One can check the following observations:

* $]0,1[$ is non empty,
* $[-1,1]$ is non empty, and
* $]\frac{1}{2},\frac{3}{2}[$ is non empty.

One can verify the following observations:

* the function sin is continuous,
* the function cos is continuous,
* the function arcsin is continuous, and
* the function arccos is continuous.

Next we state two propositions:

(1) $\sin(a \cdot r + b) = (({\rm the\ function\ sin}) \cdot {\rm AffineMap}(a, b))(r)$.

(2) $\cos(a \cdot r + b) = (({\rm the\ function\ cos}) \cdot {\rm AffineMap}(a, b))(r)$.

[1]The paper was written during the first author's post-doctoral fellowship granted by the Shinshu University, Japan.

ISSN 1426–2630

Let a be a non zero real number and let b be a real number. Note that AffineMap(a, b) is onto and one-to-one.

Let a, b be real numbers. The functor IntIntervals(a, b) is defined as follows:

(Def. 1) IntIntervals$(a, b) = \{]a + n, b + n[: n$ ranges over elements of $\mathbb{Z}\}$.

One can prove the following proposition

(3) For every set x holds $x \in$ IntIntervals(a, b) iff there exists an element n of $\mathbb{Z}$ such that $x =]a + n, b + n[$.

Let a, b be real numbers. Observe that IntIntervals(a, b) is non empty.

Next we state the proposition

(4) If $b - a \leq 1$, then IntIntervals(a, b) is mutually-disjoint.

Let a, b be real numbers. Then IntIntervals(a, b) is a family of subsets of $\mathbb{R}^{\mathbf{1}}$.

Let a, b be real numbers. Then IntIntervals(a, b) is an open family of subsets of $\mathbb{R}^{\mathbf{1}}$.

2. Correspondence between $\mathbb{R}$ and $\mathbb{R}^{\mathbf{1}}$

Let r be a real number. The functor $R^1 r$ yielding a point of $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def. 2) $R^1 r = r$.

Let A be a subset of $\mathbb{R}$. The functor $R^1 A$ yielding a subset of $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def. 3) $R^1 A = A$.

Let A be a non empty subset of $\mathbb{R}$. Observe that $R^1 A$ is non empty.

Let A be an open subset of $\mathbb{R}$. Note that $R^1 A$ is open.

Let A be a closed subset of $\mathbb{R}$. Observe that $R^1 A$ is closed.

Let A be an open subset of $\mathbb{R}$. Observe that $\mathbb{R}^{\mathbf{1}} \restriction R^1 A$ is open.

Let A be a closed subset of $\mathbb{R}$. One can verify that $\mathbb{R}^{\mathbf{1}} \restriction R^1 A$ is closed.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $R^1 f$ yielding a map from $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{dom} f$ into $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{rng} f$ is defined as follows:

(Def. 4) $R^1 f = f$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. One can check that $R^1 f$ is onto.

Let f be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. Observe that $R^1 f$ is one-to-one.

One can prove the following four propositions:

(5) $\mathbb{R}^{\mathbf{1}} \restriction R^1(\Omega_{\mathbb{R}}) = \mathbb{R}^{\mathbf{1}}$.

(6) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} f = \mathbb{R}$ holds $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{dom} f = \mathbb{R}^{\mathbf{1}}$.

(7) Every function f from $\mathbb{R}$ into $\mathbb{R}$ is a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{rng} f$.

(8) Every function from $\mathbb{R}$ into $\mathbb{R}$ is a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$.

Let f be a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$. Note that $R^1 f$ is continuous.

Let a be a non zero real number and let b be a real number. One can verify that $R^1\,\mathrm{AffineMap}(a,b)$ is open.

3. Circles

Let S be a subspace of $\mathcal{E}^2_{\mathrm{T}}$. We say that S satisfies conditions of simple closed curve if and only if:

(Def. 5) The carrier of S is a simple closed curve.

Let us note that every subspace of $\mathcal{E}^2_{\mathrm{T}}$ which satisfies conditions of simple closed curve is also non empty, arcwise connected, and compact.

Let r be a positive real number and let x be a point of $\mathcal{E}^2_{\mathrm{T}}$. Observe that $\mathrm{Sphere}(x,r)$ satisfies conditions of simple closed curve.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a real number. The functor $\mathrm{Tcircle}(x,r)$ yielding a subspace of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 6) $\mathrm{Tcircle}(x,r) = (\mathcal{E}^n_{\mathrm{T}}) \restriction \mathrm{Sphere}(x,r)$.

Let n be a non empty natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a non negative real number. Note that $\mathrm{Tcircle}(x,r)$ is strict and non empty.

One can prove the following proposition

(9) The carrier of $\mathrm{Tcircle}(x,r) = \mathrm{Sphere}(x,r)$.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be an empty real number. Note that $\mathrm{Tcircle}(x,r)$ is trivial.

Next we state the proposition

(10) $\mathrm{Tcircle}(0_{\mathcal{E}^2_{\mathrm{T}}},r)$ is a subspace of $\mathrm{Trectangle}(-r,r,-r,r)$.

Let x be a point of $\mathcal{E}^2_{\mathrm{T}}$ and let r be a positive real number. One can verify that $\mathrm{Tcircle}(x,r)$ satisfies conditions of simple closed curve.

Let us mention that there exists a subspace of $\mathcal{E}^2_{\mathrm{T}}$ which is strict and satisfies conditions of simple closed curve.

Next we state the proposition

(11) For all subspaces S, T of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve holds S and T are homeomorphic.

Let n be a natural number. The functor $\mathrm{TopUnitCircle}\,n$ yields a subspace of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 7) $\mathrm{TopUnitCircle}\,n = \mathrm{Tcircle}(0_{\mathcal{E}^n_{\mathrm{T}}},1)$.

Let n be a non empty natural number. Note that $\mathrm{TopUnitCircle}\,n$ is non empty.

We now state several propositions:

(12) For every non empty natural number n and for every point x of $\mathcal{E}^n_{\rm T}$ such that x is a point of TopUnitCircle n holds $|x| = 1$.

(13) For every point x of $\mathcal{E}^2_{\rm T}$ such that x is a point of TopUnitCircle 2 holds $-1 \leq x_{\mathbf{1}}$ and $x_{\mathbf{1}} \leq 1$ and $-1 \leq x_{\mathbf{2}}$ and $x_{\mathbf{2}} \leq 1$.

(14) For every point x of $\mathcal{E}^2_{\rm T}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{1}} = 1$ holds $x_{\mathbf{2}} = 0$.

(15) For every point x of $\mathcal{E}^2_{\rm T}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{1}} = -1$ holds $x_{\mathbf{2}} = 0$.

(16) For every point x of $\mathcal{E}^2_{\rm T}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{2}} = 1$ holds $x_{\mathbf{1}} = 0$.

(17) For every point x of $\mathcal{E}^2_{\rm T}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{2}} = -1$ holds $x_{\mathbf{1}} = 0$.

The following propositions are true:

(18) TopUnitCircle 2 is a subspace of Trectangle$(-1, 1, -1, 1)$.

(19) Let n be a non empty natural number, r be a positive real number, x be a point of $\mathcal{E}^n_{\rm T}$, and f be a map from TopUnitCircle n into Tcircle(x, r). Suppose that for every point a of TopUnitCircle n and for every point b of $\mathcal{E}^n_{\rm T}$ such that $a = b$ holds $f(a) = r \cdot b + x$. Then f is a homeomorphism.

Let us observe that TopUnitCircle 2 satisfies conditions of simple closed curve.

One can prove the following proposition

(20) Let n be a non empty natural number, r, s be positive real numbers, and x, y be points of $\mathcal{E}^n_{\rm T}$. Then Tcircle(x, r) and Tcircle(y, s) are homeomorphic.

Let x be a point of $\mathcal{E}^2_{\rm T}$ and let r be a non negative real number. Observe that Tcircle(x, r) is arcwise connected.

The point $c[10]$ of TopUnitCircle 2 is defined as follows:

(Def. 8) $c[10] = [1, 0]$.

The point $c[-10]$ of TopUnitCircle 2 is defined as follows:

(Def. 9) $c[-10] = [-1, 0]$.

Next we state the proposition

(21) $c[10] \neq c[-10]$.

Let p be a point of TopUnitCircle 2. The functor TopOpenUnitCircle p yielding a strict subspace of TopUnitCircle 2 is defined by:

(Def. 10) The carrier of TopOpenUnitCircle p = (the carrier of TopUnitCircle 2) $\setminus$ $\{p\}$.

Let p be a point of TopUnitCircle 2. Note that TopOpenUnitCircle p is non empty.

We now state several propositions:

(22) For every point p of TopUnitCircle 2 holds p is not a point of TopOpenUnitCircle p.

(23) For every point p of TopUnitCircle 2 holds TopOpenUnitCircle p = TopUnitCircle $2\restriction(\Omega_{\mathrm{TopUnitCircle}\,2} \setminus \{p\})$.

(24) For all points p, q of TopUnitCircle 2 such that $p \neq q$ holds q is a point of TopOpenUnitCircle p.

(25) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that p is a point of TopOpenUnitCircle $c[10]$ and $p_{\mathbf{2}} = 0$ holds $p = c[-10]$.

(26) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that p is a point of TopOpenUnitCircle $c[-10]$ and $p_{\mathbf{2}} = 0$ holds $p = c[10]$.

Next we state three propositions:

(27) Let p be a point of TopUnitCircle 2 and x be a point of $\mathcal{E}^2_{\mathrm{T}}$. If x is a point of TopOpenUnitCircle p, then $-1 \leq x_{\mathbf{1}}$ and $x_{\mathbf{1}} \leq 1$ and $-1 \leq x_{\mathbf{2}}$ and $x_{\mathbf{2}} \leq 1$.

(28) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopOpenUnitCircle $c[10]$ holds $-1 \leq x_{\mathbf{1}}$ and $x_{\mathbf{1}} < 1$.

(29) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopOpenUnitCircle $c[-10]$ holds $-1 < x_{\mathbf{1}}$ and $x_{\mathbf{1}} \leq 1$.

Let p be a point of TopUnitCircle 2. Note that TopOpenUnitCircle p is open.

We now state two propositions:

(30) For every point p of TopUnitCircle 2 holds TopOpenUnitCircle p and $I(01)$ are homeomorphic.

(31) For all points p, q of TopUnitCircle 2 holds TopOpenUnitCircle p and TopOpenUnitCircle q are homeomorphic.

4. Correspondence between the Real Line and Circles

The map CircleMap from $\mathbb{R}^{\mathbf{1}}$ into TopUnitCircle 2 is defined by:

(Def. 11) For every real number x holds $\mathrm{CircleMap}(x) = [\cos(2 \cdot \pi \cdot x), \sin(2 \cdot \pi \cdot x)]$.

Next we state several propositions:

(32) $\mathrm{CircleMap}(r) = \mathrm{CircleMap}(r + i)$.

(33) $\mathrm{CircleMap}(i) = c[10]$.

(34) $\mathrm{CircleMap}^{-1}(\{c[10]\}) = \mathbb{Z}$.

(35) If $\mathrm{frac}\, r = \frac{1}{2}$, then $\mathrm{CircleMap}(r) = [-1, 0]$.

(36) If $\mathrm{frac}\, r = \frac{1}{4}$, then $\mathrm{CircleMap}(r) = [0, 1]$.

(37) If $\mathrm{frac}\, r = \frac{3}{4}$, then $\mathrm{CircleMap}(r) = [0, -1]$.

(38) For all integers i, j holds $\mathrm{CircleMap}(r) = [((\text{the function } \cos) \cdot \mathrm{AffineMap}(2 \cdot \pi, 2 \cdot \pi \cdot i))(r), ((\text{the function } \sin) \cdot \mathrm{AffineMap}(2 \cdot \pi, 2 \cdot \pi \cdot j))(r)]$.

Let us note that CircleMap is continuous.

The following proposition is true

(39) For every subset B of $\mathbb{R}^{\mathbf{1}}$ and for every map f from $\mathbb{R}^{\mathbf{1}} \restriction B$ into TopUnitCircle 2 such that $[0,1[\subseteq B$ and $f = \mathrm{CircleMap} \restriction B$ holds f is onto.

Let us observe that CircleMap is onto.

Let r be a real number. One can verify that CircleMap $\restriction [r, r+1[$ is one-to-one.

Let r be a real number. One can verify that CircleMap $\restriction]r, r+1[$ is one-to-one.

One can prove the following two propositions:

(40) If $b - a \leq 1$, then for every set d such that $d \in \mathrm{IntIntervals}(a, b)$ holds CircleMap $\restriction d$ is one-to-one.

(41) For every set d such that $d \in \mathrm{IntIntervals}(a, b)$ holds $\mathrm{CircleMap}^{\circ} d = \mathrm{CircleMap}^{\circ} \bigcup \mathrm{IntIntervals}(a, b)$.

Let r be a point of $\mathbb{R}^{\mathbf{1}}$. The functor CircleMap r yielding a map from $\mathbb{R}^{\mathbf{1}} \restriction R^1]r, r+1[$ into TopOpenUnitCircle CircleMap(r) is defined by:

(Def. 12) $\mathrm{CircleMap}\, r = \mathrm{CircleMap} \restriction]r, r+1[$.

One can prove the following proposition

(42) $\mathrm{CircleMap}\, R^1(a+i) = \mathrm{CircleMap}\, R^1 a \cdot (\mathrm{AffineMap}(1, -i) \restriction]a+i, a+i+1[)$.

Let r be a point of $\mathbb{R}^{\mathbf{1}}$. One can check that CircleMap r is one-to-one, onto, and continuous.

The map Circle2IntervalR from TopOpenUnitCircle $c[10]$ into $\mathbb{R}^{\mathbf{1}} \restriction R^1]0, 1[$ is defined by the condition (Def. 13).

(Def. 13) Let p be a point of TopOpenUnitCircle $c[10]$. Then there exist real numbers x, y such that $p = [x, y]$ and if $y \geq 0$, then $\mathrm{Circle2IntervalR}(p) = \frac{\arccos x}{2 \cdot \pi}$ and if $y \leq 0$, then $\mathrm{Circle2IntervalR}(p) = 1 - \frac{\arccos x}{2 \cdot \pi}$.

The map Circle2IntervalL from TopOpenUnitCircle $c[-10]$ into $\mathbb{R}^{\mathbf{1}} \restriction R^1]\frac{1}{2}, \frac{3}{2}[$ is defined by the condition (Def. 14).

(Def. 14) Let p be a point of TopOpenUnitCircle $c[-10]$. Then there exist real numbers x, y such that $p = [x, y]$ and if $y \geq 0$, then $\mathrm{Circle2IntervalL}(p) = 1 + \frac{\arccos x}{2 \cdot \pi}$ and if $y \leq 0$, then $\mathrm{Circle2IntervalL}(p) = 1 - \frac{\arccos x}{2 \cdot \pi}$.

We now state two propositions:

(43) $(\mathrm{CircleMap}\, R^1 0)^{-1} = \mathrm{Circle2IntervalR}$.

(44) $(\mathrm{CircleMap}\, R^1(\frac{1}{2}))^{-1} = \mathrm{Circle2IntervalL}$.

Let us observe that Circle2IntervalR is one-to-one, onto, and continuous and Circle2IntervalL is one-to-one, onto, and continuous.

Let i be an integer. Observe that CircleMap $R^1 i$ is open and CircleMap $R^1(\frac{1}{2} + i)$ is open.

Let us observe that Circle2IntervalR is open and Circle2IntervalL is open.

Next we state several propositions:

(45) CircleMap $R^1 0$ is a homeomorphism.

(46) CircleMap $R^1(\frac{1}{2})$ is a homeomorphism.

(47) Circle2IntervalR is a homeomorphism.

(48) Circle2IntervalL is a homeomorphism.

(49) There exists a family F of subsets of TopUnitCircle 2 such that

(i) $F = \{\text{CircleMap}^\circ]0, 1[, \text{CircleMap}^\circ]\frac{1}{2}, \frac{3}{2}[\}$,

(ii) F is a cover of TopUnitCircle 2 and open, and

(iii) for every subset U of TopUnitCircle 2 holds if $U = \text{CircleMap}^\circ]0, 1[$, then $\bigcup \text{IntIntervals}(0, 1) = \text{CircleMap}^{-1}(U)$ and for every subset d of $\mathbb{R}^{\mathbf{1}}$ such that $d \in \text{IntIntervals}(0, 1)$ and for every map f from $\mathbb{R}^{\mathbf{1}} \restriction d$ into TopUnitCircle $2 \restriction U$ such that $f = \text{CircleMap} \restriction d$ holds f is a homeomorphism and if $U = \text{CircleMap}^\circ]\frac{1}{2}, \frac{3}{2}[$, then $\bigcup \text{IntIntervals}(\frac{1}{2}, \frac{3}{2}) = \text{CircleMap}^{-1}(U)$ and for every subset d of $\mathbb{R}^{\mathbf{1}}$ such that $d \in \text{IntIntervals}(\frac{1}{2}, \frac{3}{2})$ and for every map f from $\mathbb{R}^{\mathbf{1}} \restriction d$ into TopUnitCircle $2 \restriction U$ such that $f = \text{CircleMap} \restriction d$ holds f is a homeomorphism.

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Received October 18, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

On the Characterization of Collineations of the Segre Product of Strongly Connected Partial Linear Spaces[1]

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Summary. In this paper we characterize the automorphisms (collineations) of the Segre product of partial linear spaces. In particular, we show that if all components of the product are strongly connected, then every collineation is determined by a set of isomorphisms between its components. The formalization follows the ideas presented in the *Journal of Geometry* paper [16] by Naumowicz and Prażmowski.

MML Identifier: `PENCIL_3`.

The articles [20], [10], [2], [23], [22], [6], [8], [9], [19], [24], [7], [1], [11], [5], [3], [17], [21], [12], [13], [18], [4], [15], and [14] provide the terminology and notation for this paper.

1. Preliminaries

The following propositions are true:

(1) Let S be a non empty non void topological structure, f be a collineation of S, and p, q be points of S. If p, q are collinear, then $f(p)$, $f(q)$ are collinear.

(2) Let I be a non empty set, i be an element of I, and A be a non-Trivial-yielding 1-sorted yielding many sorted set indexed by I. Then $A(i)$ is non trivial.

[1]This work has been partially supported by the KBN grant 4 T11C 039 24 and the FP6 IST grant TYPES No. 510096.

ISSN 1426–2630

(3) Let S be a non void identifying close blocks topological structure such that S is strongly connected. Then S has no isolated points.

(4) Let S be a non empty non void identifying close blocks topological structure. If S is strongly connected, then S is connected.

(5) Let S be a non void non degenerated topological structure and L be a block of S. Then there exists a point x of S such that $x \notin L$.

(6) Let I be a non empty set and A be a nonempty TopStruct-yielding many sorted set indexed by I. Then every point of SegreProduct A is an element of the support of A.

(7) Let I be a non empty set, A be a 1-sorted yielding many sorted set indexed by I, and x be an element of I. Then (the support of A)$(x) = \Omega_{A(x)}$.

(8) Let I be a non empty set, i be an element of I, and A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I. Then there exists a Segre-like non trivial-yielding many sorted subset L indexed by the support of A such that $\mathrm{index}(L) = i$ and $\prod L$ is a Segre coset of A.

(9) Let I be a non empty set, i be an element of I, A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I, and p be a point of SegreProduct A. Then there exists a Segre-like non trivial-yielding many sorted subset L indexed by the support of A such that $\mathrm{index}(L) = i$ and $\prod L$ is a Segre coset of A and $p \in \prod L$.

(10) Let I be a non empty set, A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I, and b be a Segre-like non trivial-yielding many sorted subset indexed by the support of A. If $\prod b$ is a Segre coset of A, then $b(\mathrm{index}(b)) = \Omega_{A(\mathrm{index}(b))}$.

(11) Let I be a non empty set, A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I, and L_1, L_2 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. Suppose $\prod L_1$ is a Segre coset of A and $\prod L_2$ is a Segre coset of A and $\mathrm{index}(L_1) = \mathrm{index}(L_2)$ and $\prod L_1$ meets $\prod L_2$. Then $L_1 = L_2$.

(12) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, L be a Segre-like non trivial-yielding many sorted subset indexed by the support of A, and B be a block of $A(\mathrm{index}(L))$. Then $\prod(L +\cdot (\mathrm{index}(L), B))$ is a block of SegreProduct A.

(13) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, i be an element of I, p be a point of $A(i)$, and L be a Segre-like non trivial-yielding many sorted subset indexed by the support of A. Suppose $i \neq \mathrm{index}(L)$. Then $L +\cdot (i, \{p\})$ is a Segre-like non trivial-yielding many sorted subset indexed by the support of A.

(14) Let I be a non empty set, A be a PLS-yielding many sorted set indexed

by I, i be an element of I, S be a subset of $A(i)$, and L be a Segre-like non trivial-yielding many sorted subset indexed by the support of A. Then $\prod(L +\cdot (i, S))$ is a subset of SegreProduct A.

(15) Let I be a non empty set, P be a many sorted set indexed by I, and i be an element of I. Then $\{P\}(i)$ is non empty and trivial.

(16) Let I be a non empty set, i be an element of I, A be a PLS-yielding many sorted set indexed by I, B be a block of $A(i)$, and P be an element of the support of A. Then $\prod(\{P\} +\cdot (i, B))$ is a block of SegreProduct A.

(17) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, and p, q be points of SegreProduct A. Suppose $p \neq q$. Then p, q are collinear if and only if for all many sorted sets p_1, q_1 indexed by I such that $p_1 = p$ and $q_1 = q$ there exists an element i of I such that for all points a, b of $A(i)$ such that $a = p_1(i)$ and $b = q_1(i)$ holds $a \neq b$ and a, b are collinear and for every element j of I such that $j \neq i$ holds $p_1(j) = q_1(j)$.

(18) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, b be a Segre-like non trivial-yielding many sorted subset indexed by the support of A, and x be a point of $A(\mathrm{index}(b))$. Then there exists a many sorted set p indexed by I such that $p \in \prod b$ and $\{(p +\cdot (\mathrm{index}(b), x)$ **qua** set$)\ \} = \prod(b +\cdot (\mathrm{index}(b), \{x\}))$.

Let I be a finite non empty set and let b_1, b_2 be many sorted sets indexed by I. The functor ${b_1}'(b_2)$ yields a natural number and is defined by:

(Def. 1) ${b_1}'(b_2) = \overline{\overline{\{i; i \text{ ranges over elements of } I\text{: } b_1(i) \neq b_2(i)\}}}$.

One can prove the following proposition

(19) Let I be a finite non empty set, b_1, b_2 be many sorted sets indexed by I, and i be an element of I. If $b_1(i) \neq b_2(i)$, then ${b_1}'(b_2) = {b_1}'(b_2 +\cdot (i, b_1(i))) + 1$.

2. The Adherence of Segre Cosets

Let I be a non empty set, let A be a PLS-yielding many sorted set indexed by I, and let B_1, B_2 be Segre cosets of A. The predicate $B_1||B_2$ is defined as follows:

(Def. 2) For every point x of SegreProduct A such that $x \in B_1$ there exists a point y of SegreProduct A such that $y \in B_2$ and x, y are collinear.

Next we state several propositions:

(20) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, and B_1, B_2 be Segre cosets of A. Suppose $B_1||B_2$. Let f be a collineation of SegreProduct A and C_1, C_2 be Segre cosets of A. If $C_1 = f^\circ B_1$ and $C_2 = f^\circ B_2$, then $C_1||C_2$.

(21) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, and B_1, B_2 be Segre cosets of A. Suppose B_1 misses B_2. Then $B_1 || B_2$ if and only if for all Segre-like non trivial-yielding many sorted subsets b_1, b_2 indexed by the support of A such that $B_1 = \prod b_1$ and $B_2 = \prod b_2$ holds $\mathrm{index}(b_1) = \mathrm{index}(b_2)$ and there exists an element r of I such that $r \neq \mathrm{index}(b_1)$ and for every element i of I such that $i \neq r$ holds $b_1(i) = b_2(i)$ and for all points c_1, c_2 of $A(r)$ such that $b_1(r) = \{c_1\}$ and $b_2(r) = \{c_2\}$ holds c_1, c_2 are collinear.

(22) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is connected. Let i be an element of I, p be a point of $A(i)$, and b_1, b_2 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. Suppose $\prod b_1$ is a Segre coset of A and $\prod b_2$ is a Segre coset of A and $b_1 = b_2 +\cdot (i, \{p\})$ and $p \notin b_2(i)$. Then there exists a finite sequence D of elements of $2^{\text{the carrier of SegreProduct}\,A}$ such that

(i) $D(1) = \prod b_1$,

(ii) $D(\mathrm{len}\,D) = \prod b_2$,

(iii) for every natural number i such that $i \in \mathrm{dom}\,D$ holds $D(i)$ is a Segre coset of A, and

(iv) for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\,D$ and for all Segre cosets D_1, D_2 of A such that $D_1 = D(i)$ and $D_2 = D(i+1)$ holds D_1 misses D_2 and $D_1 || D_2$.

(23) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is connected. Let B_1, B_2 be Segre cosets of A. Suppose B_1 misses B_2. Let b_1, b_2 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. Suppose $B_1 = \prod b_1$ and $B_2 = \prod b_2$. Then $\mathrm{index}(b_1) = \mathrm{index}(b_2)$ if and only if there exists a finite sequence D of elements of $2^{\text{the carrier of SegreProduct}\,A}$ such that $D(1) = B_1$ and $D(\mathrm{len}\,D) = B_2$ and for every natural number i such that $i \in \mathrm{dom}\,D$ holds $D(i)$ is a Segre coset of A and for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\,D$ and for all Segre cosets D_1, D_2 of A such that $D_1 = D(i)$ and $D_2 = D(i+1)$ holds D_1 misses D_2 and $D_1 || D_2$.

(24) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A, B_1, B_2 be Segre cosets of A, and b_1, b_2, b_3, b_4 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. If $B_1 = \prod b_1$ and $B_2 = \prod b_2$ and $f^\circ B_1 = \prod b_3$ and $f^\circ B_2 = \prod b_4$, then if $\mathrm{index}(b_1) = \mathrm{index}(b_2)$, then $\mathrm{index}(b_3) = \mathrm{index}(b_4)$.

(25) Let I be a finite non empty set and A be a PLS-yielding many sorted set

indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A. Then there exists a permutation s of I such that for all elements i, j of I holds $s(i) = j$ if and only if for every Segre coset B_1 of A and for all Segre-like non trivial-yielding many sorted subsets b_1, b_2 indexed by the support of A such that $B_1 = \prod b_1$ and $f^\circ B_1 = \prod b_2$ holds if $\mathrm{index}(b_1) = i$, then $\mathrm{index}(b_2) = j$.

Let I be a finite non empty set and let A be a PLS-yielding many sorted set indexed by I. Let us assume that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A. The functor IndPerm(f) yields a permutation of I and is defined by the condition (Def. 3).

(Def. 3) Let i, j be elements of I. Then $(\mathrm{IndPerm}(f))(i) = j$ if and only if for every Segre coset B_1 of A and for all Segre-like non trivial-yielding many sorted subsets b_1, b_2 indexed by the support of A such that $B_1 = \prod b_1$ and $f^\circ B_1 = \prod b_2$ holds if $\mathrm{index}(b_1) = i$, then $\mathrm{index}(b_2) = j$.

3. Canonical Embeddings and Automorphisms of Segre Product

Let I be a finite non empty set and let A be a PLS-yielding many sorted set indexed by I. Let us assume that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A and let b_1 be a Segre-like non trivial-yielding many sorted subset indexed by the support of A. Let us assume that $\prod b_1$ is a Segre coset of A. The functor $\mathrm{CanEmb}(f, b_1)$ yields a map from $A(\mathrm{index}(b_1))$ into $A((\mathrm{IndPerm}(f))(\mathrm{index}(b_1)))$ and is defined by the conditions (Def. 4).

(Def. 4)(i) $\mathrm{CanEmb}(f, b_1)$ is isomorphic, and

(ii) for every many sorted set a indexed by I such that a is a point of SegreProduct A and $a \in \prod b_1$ and for every many sorted set b indexed by I such that $b = f(a)$ holds $b((\mathrm{IndPerm}(f))(\mathrm{index}(b_1))) = (\mathrm{CanEmb}(f, b_1))(a(\mathrm{index}(b_1)))$.

Next we state two propositions:

(26) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A and B_1, B_2 be Segre cosets of A. Suppose B_1 misses B_2 and $B_1 || B_2$. Let b_1, b_2 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. If $\prod b_1 = B_1$ and $\prod b_2 = B_2$, then $\mathrm{CanEmb}(f, b_1) = \mathrm{CanEmb}(f, b_2)$.

(27) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A and b_1, b_2 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A.

Suppose $\prod b_1$ is a Segre coset of A and $\prod b_2$ is a Segre coset of A and $\mathrm{index}(b_1) = \mathrm{index}(b_2)$. Then $\mathrm{CanEmb}(f, b_1) = \mathrm{CanEmb}(f, b_2)$.

Let I be a finite non empty set and let A be a PLS-yielding many sorted set indexed by I. Let us assume that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A and let i be an element of I. The functor $\mathrm{CanEmb}(f, i)$ yields a map from $A(i)$ into $A((\mathrm{IndPerm}(f))(i))$ and is defined by the condition (Def. 5).

(Def. 5) Let b be a Segre-like non trivial-yielding many sorted subset indexed by the support of A. If $\prod b$ is a Segre coset of A and $\mathrm{index}(b) = i$, then $\mathrm{CanEmb}(f, i) = \mathrm{CanEmb}(f, b)$.

Next we state the proposition

(28) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A. Then there exists a permutation s of I and there exists a function yielding many sorted set B indexed by I such that for every element i of I holds

(i) $B(i)$ is a map from $A(i)$ into $A(s(i))$,

(ii) for every map m from $A(i)$ into $A(s(i))$ such that $m = B(i)$ holds m is isomorphic, and

(iii) for every point p of SegreProduct A and for every many sorted set a indexed by I such that $a = p$ and for every many sorted set b indexed by I such that $b = f(p)$ and for every map m from $A(i)$ into $A(s(i))$ such that $m = B(i)$ holds $b(s(i)) = m(a(i))$.

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Received October 18, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Spaces of Pencils, Grassmann Spaces, and Generalized Veronese Spaces[1]

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Summary. In this paper we construct several examples of partial linear spaces. First, we define two algebraic structures, namely the spaces of k-pencils and Grassmann spaces for vector spaces over an arbitrary field. Then we introduce the notion of generalized Veronese spaces following the definition presented in the paper [8] by Naumowicz and Prażmowski. For all spaces defined, we state the conditions under which they are not degenerated to a single line.

MML Identifier: `PENCIL_4`.

The terminology and notation used here are introduced in the following articles: [11], [16], [4], [2], [9], [3], [1], [5], [10], [7], [15], [6], [14], [13], [12], and [17].

1. Spaces of k-Pencils

One can prove the following propositions:

(1) For all natural numbers k, n such that $1 \leq k$ and $k < n$ and $3 \leq n$ holds $k+1 < n$ or $2 \leq k$.

(2) For every finite set X and for every natural number n such that $n \leq \operatorname{card} X$ there exists a subset Y of X such that $\operatorname{card} Y = n$.

(3) For every field F and for every vector space V over F holds every subspace of V is a subspace of Ω_V.

(4) For every field F and for every vector space V over F holds every subspace of Ω_V is a subspace of V.

[1]This work has been partially supported by the KBN grant 4 T11C 039 24OB and the FP6 IST grant TYPES No. 510096.

ISSN 1426–2630

(5) For every field F and for every vector space V over F and for every subspace W of V holds Ω_W is a subspace of V.

(6) Let F be a field and V, W be vector spaces over F. If Ω_W is a subspace of V, then W is a subspace of V.

Let F be a field, let V be a vector space over F, and let W_1, W_2 be subspaces of V. The functor $\mathrm{segment}(W_1, W_2)$ yielding a subset of $\mathrm{Subspaces}\,V$ is defined by:

(Def. 1)(i) For every strict subspace W of V holds $W \in \mathrm{segment}(W_1, W_2)$ iff W_1 is a subspace of W and W is a subspace of W_2 if W_1 is a subspace of W_2,

(ii) $\mathrm{segment}(W_1, W_2) = \emptyset$, otherwise.

We now state the proposition

(7) Let F be a field, V be a vector space over F, and W_1, W_2, W_3, W_4 be subspaces of V. Suppose W_1 is a subspace of W_2 and W_3 is a subspace of W_4 and $\Omega_{(W_1)} = \Omega_{(W_3)}$ and $\Omega_{(W_2)} = \Omega_{(W_4)}$. Then $\mathrm{segment}(W_1, W_2) = \mathrm{segment}(W_3, W_4)$.

Let F be a field, let V be a vector space over F, and let W_1, W_2 be subspaces of V. The functor $\mathrm{pencil}(W_1, W_2)$ yielding a subset of $\mathrm{Subspaces}\,V$ is defined by:

(Def. 2) $\mathrm{pencil}(W_1, W_2) = \mathrm{segment}(W_1, W_2) \setminus \{\Omega_{(W_1)}, \Omega_{(W_2)}\}$.

Next we state the proposition

(8) Let F be a field, V be a vector space over F, and W_1, W_2, W_3, W_4 be subspaces of V. Suppose W_1 is a subspace of W_2 and W_3 is a subspace of W_4 and $\Omega_{(W_1)} = \Omega_{(W_3)}$ and $\Omega_{(W_2)} = \Omega_{(W_4)}$. Then $\mathrm{pencil}(W_1, W_2) = \mathrm{pencil}(W_3, W_4)$.

Let F be a field, let V be a finite dimensional vector space over F, let W_1, W_2 be subspaces of V, and let k be a natural number. The functor $\mathrm{pencil}(W_1, W_2, k)$ yielding a subset of $\mathrm{Sub}_k(V)$ is defined by:

(Def. 3) $\mathrm{pencil}(W_1, W_2, k) = \mathrm{pencil}(W_1, W_2) \cap \mathrm{Sub}_k(V)$.

We now state two propositions:

(9) Let F be a field, V be a finite dimensional vector space over F, k be a natural number, and W_1, W_2, W be subspaces of V. If $W \in \mathrm{pencil}(W_1, W_2, k)$, then W_1 is a subspace of W and W is a subspace of W_2.

(10) Let F be a field, V be a finite dimensional vector space over F, k be a natural number, and W_1, W_2, W_3, W_4 be subspaces of V. Suppose W_1 is a subspace of W_2 and W_3 is a subspace of W_4 and $\Omega_{(W_1)} = \Omega_{(W_3)}$ and $\Omega_{(W_2)} = \Omega_{(W_4)}$. Then $\mathrm{pencil}(W_1, W_2, k) = \mathrm{pencil}(W_3, W_4, k)$.

Let F be a field, let V be a finite dimensional vector space over F, and let k be a natural number. k pencils of V yields a family of subsets of $\mathrm{Sub}_k(V)$ and is defined by the condition (Def. 4).

(Def. 4) Let X be a set. Then $X \in k$ pencils of V if and only if there exist subspaces W_1, W_2 of V such that W_1 is a subspace of W_2 and $\dim(W_1) + 1 = k$ and $\dim(W_2) = k + 1$ and $X = \mathrm{pencil}(W_1, W_2, k)$.

We now state several propositions:

(11) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$, then k pencils of V is non empty.

(12) Let F be a field, V be a finite dimensional vector space over F, W_1, W_2, P, Q be subspaces of V, and k be a natural number. Suppose $1 \leq k$ and $k < \dim(V)$ and $\dim(W_1) + 1 = k$ and $\dim(W_2) = k + 1$ and $P \in \mathrm{pencil}(W_1, W_2, k)$ and $Q \in \mathrm{pencil}(W_1, W_2, k)$ and $P \neq Q$. Then $P \cap Q = \Omega_{(W_1)}$ and $P + Q = \Omega_{(W_2)}$.

(13) Let F be a field, V be a finite dimensional vector space over F, and v be a vector of V. If $v \neq 0_V$, then $\dim(\mathrm{Lin}(\{v\})) = 1$.

(14) Let F be a field, V be a finite dimensional vector space over F, W be a subspace of V, and v be a vector of V. If $v \notin W$, then $\dim(W + \mathrm{Lin}(\{v\})) = \dim(W) + 1$.

(15) Let F be a field, V be a finite dimensional vector space over F, W be a subspace of V, and v, u be vectors of V. Suppose $v \notin W$ and $u \notin W$ and $v \neq u$ and $\{v, u\}$ is linearly independent and $W \cap \mathrm{Lin}(\{v, u\}) = \mathbf{0}_V$. Then $\dim(W + \mathrm{Lin}(\{v, u\})) = \dim(W) + 2$.

(16) Let F be a field, V be a finite dimensional vector space over F, and W_1, W_2 be subspaces of V. Suppose W_1 is a subspace of W_2. Let k be a natural number. Suppose $1 \leq k$ and $k < \dim(V)$ and $\dim(W_1) + 1 = k$ and $\dim(W_2) = k + 1$. Let v be a vector of V. If $v \in W_2$ and $v \notin W_1$, then $W_1 + \mathrm{Lin}(\{v\}) \in \mathrm{pencil}(W_1, W_2, k)$.

(17) Let F be a field, V be a finite dimensional vector space over F, and W_1, W_2 be subspaces of V. Suppose W_1 is a subspace of W_2. Let k be a natural number. If $1 \leq k$ and $k < \dim(V)$ and $\dim(W_1) + 1 = k$ and $\dim(W_2) = k + 1$, then $\mathrm{pencil}(W_1, W_2, k)$ is non trivial.

Let F be a field, let V be a finite dimensional vector space over F, and let k be a natural number. The functor $\mathrm{PencilSpace}(V, k)$ yielding a strict topological structure is defined by:

(Def. 5) $\mathrm{PencilSpace}(V, k) = \langle \mathrm{Sub}_k(V), k \text{ pencils of } V \rangle$.

Next we state several propositions:

(18) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $k \leq \dim(V)$, then $\mathrm{PencilSpace}(V, k)$ is non empty.

(19) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$, then $\mathrm{PencilSpace}(V, k)$ is non void.

(20) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$ and $3 \leq \dim(V)$, then PencilSpace(V,k) is non degenerated.

(21) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$, then PencilSpace(V,k) has non trivial blocks.

(22) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$, then PencilSpace(V,k) is identifying close blocks.

(23) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$ and $3 \leq \dim(V)$, then PencilSpace(V,k) is a PLS.

2. Grassmann Spaces

Let F be a field, let V be a finite dimensional vector space over F, and let m, n be natural numbers. The functor SubspaceSet(V,m,n) yields a family of subsets of $\mathrm{Sub}_m(V)$ and is defined by:

(Def. 6) For every set X holds $X \in$ SubspaceSet(V,m,n) iff there exists a subspace W of V such that $\dim(W) = n$ and $X = \mathrm{Sub}_m(W)$.

One can prove the following propositions:

(24) Let F be a field, V be a finite dimensional vector space over F, and m, n be natural numbers. If $n \leq \dim(V)$, then SubspaceSet(V,m,n) is non empty.

(25) Let F be a field and W_1, W_2 be finite dimensional vector spaces over F. If $\Omega_{(W_1)} = \Omega_{(W_2)}$, then for every natural number m holds $\mathrm{Sub}_m(W_1) = \mathrm{Sub}_m(W_2)$.

(26) Let F be a field, V be a finite dimensional vector space over F, W be a subspace of V, and m be a natural number. If $1 \leq m$ and $m \leq \dim(V)$ and $\mathrm{Sub}_m(V) \subseteq \mathrm{Sub}_m(W)$, then $\Omega_V = \Omega_W$.

Let F be a field, let V be a finite dimensional vector space over F, and let m, n be natural numbers. The functor GrassmannSpace(V,m,n) yields a strict topological structure and is defined as follows:

(Def. 7) $\mathrm{GrassmannSpace}(V,m,n) = \langle \mathrm{Sub}_m(V), \mathrm{SubspaceSet}(V,m,n) \rangle$.

We now state several propositions:

(27) Let F be a field, V be a finite dimensional vector space over F, and m, n be natural numbers. If $m \leq \dim(V)$, then GrassmannSpace(V,m,n) is non empty.

(28) Let F be a field, V be a finite dimensional vector space over F, and m, n be natural numbers. If $n \leq \dim(V)$, then GrassmannSpace(V, m, n) is non void.

(29) Let F be a field, V be a finite dimensional vector space over F, and m, n be natural numbers. If $1 \leq m$ and $m < n$ and $n < \dim(V)$, then GrassmannSpace(V, m, n) is non degenerated.

(30) Let F be a field, V be a finite dimensional vector space over F, and m, n be natural numbers. If $1 \leq m$ and $m < n$ and $n \leq \dim(V)$, then GrassmannSpace(V, m, n) has non trivial blocks.

(31) Let F be a field, V be a finite dimensional vector space over F, and m be a natural number. If $1 \leq m$ and $m + 1 \leq \dim(V)$, then GrassmannSpace$(V, m, m + 1)$ is identifying close blocks.

(32) Let F be a field, V be a finite dimensional vector space over F, and m be a natural number. If $1 \leq m$ and $m + 1 < \dim(V)$, then GrassmannSpace$(V, m, m + 1)$ is a PLS.

3. Veronese Spaces

Let X be a set. The functor PairSet X is defined as follows:

(Def. 8) For every set z holds $z \in$ PairSet X iff there exist sets x, y such that $x \in X$ and $y \in X$ and $z = \{x, y\}$.

Let X be a non empty set. One can verify that PairSet X is non empty.

Let t, X be sets. The functor PairSet(t, X) is defined as follows:

(Def. 9) For every set z holds $z \in$ PairSet(t, X) iff there exists a set y such that $y \in X$ and $z = \{t, y\}$.

Let t be a set and let X be a non empty set. One can verify that PairSet(t, X) is non empty.

Let t be a set and let X be a non trivial set. One can verify that PairSet(t, X) is non trivial.

Let X be a set and let L be a family of subsets of X. The functor PairSetFamily L yields a family of subsets of PairSet X and is defined as follows:

(Def. 10) For every set S holds $S \in$ PairSetFamily L iff there exists a set t and there exists a subset l of X such that $t \in X$ and $l \in L$ and $S =$ PairSet(t, l).

Let X be a non empty set and let L be a non empty family of subsets of X. Note that PairSetFamily L is non empty.

Let S be a topological structure. The functor VeroneseSpace S yielding a strict topological structure is defined by:

(Def. 11) VeroneseSpace $S = \langle$PairSet (the carrier of S),PairSetFamily (the topology of S)$\rangle$.

Let S be a non empty topological structure. One can verify that VeroneseSpace S is non empty.

Let S be a non empty non void topological structure. One can check that VeroneseSpace S is non void.

Let S be a non empty non void non degenerated topological structure. Note that VeroneseSpace S is non degenerated.

Let S be a non empty non void topological structure with non trivial blocks. One can check that VeroneseSpace S has non trivial blocks.

Let S be an identifying close blocks topological structure. Note that VeroneseSpace S is identifying close blocks.

Let S be a PLS. Then VeroneseSpace S is a strict PLS.

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Received November 8, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

On the Boundary and Derivative of a Set[1]

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Summary. This is the first Mizar article in a series aiming at a complete formalization of the textbook "General Topology" by Engelking [7]. We cover the first part of Section 1.3, by defining such notions as a derivative of a subset A of a topological space (usually denoted by A^{d}, but Der A in our notation), the derivative and the boundary of families of subsets, points of accumulation and isolated points. We also introduce dense-in-itself, perfect and scattered topological spaces and formulate the notion of the density of a space. Some basic properties are given as well as selected exercises from [7].

MML Identifier: `TOPGEN_1`.

The terminology and notation used in this paper are introduced in the following papers: [13], [15], [1], [2], [12], [3], [5], [10], [16], [9], [14], [4], [6], [8], and [11].

1. Preliminaries

Let T be a set, let A be a subset of T, and let B be a set. Then $A \setminus B$ is a subset of T.

The following three propositions are true:

(1) For every 1-sorted structure T and for all subsets A, B of T holds A meets B^{c} iff $A \setminus B \neq \emptyset$.

(2) For every 1-sorted structure T holds T is countable iff Ω_T is countable.

(3) For every 1-sorted structure T holds T is countable iff $\overline{\overline{\Omega_T}} \leq \aleph_0$.

Let T be a finite 1-sorted structure. Note that Ω_T is finite.

Let us note that every 1-sorted structure which is finite is also countable.

[1]This work has been partially supported by the KBN grant 4 T11C 039 24 and the FP6 IST grant TYPES No. 510996.

ISSN 1426–2630

Let us observe that there exists a 1-sorted structure which is countable and non empty and there exists a topological space which is countable and non empty.

Let T be a countable 1-sorted structure. Observe that Ω_T is countable.

Let us observe that there exists a topological space which is T_1 and non empty.

2. Boundary of a Subset

Next we state two propositions:

(4) For every topological structure T and for every subset A of T holds $A \cup \Omega_T = \Omega_T$.

(5) For every topological space T and for every subset A of T holds $\mathrm{Int}\, A = \overline{A^{\mathrm{c}}}^{\mathrm{c}}$.

Let T be a topological space and let F be a family of subsets of T. The functor $\mathrm{Fr}\, F$ yielding a family of subsets of T is defined by:

(Def. 1) For every subset A of T holds $A \in \mathrm{Fr}\, F$ iff there exists a subset B of T such that $A = \mathrm{Fr}\, B$ and $B \in F$.

The following propositions are true:

(6) For every topological space T and for every family F of subsets of T such that $F = \emptyset$ holds $\mathrm{Fr}\, F = \emptyset$.

(7) Let T be a topological space, F be a family of subsets of T, and A be a subset of T. If $F = \{A\}$, then $\mathrm{Fr}\, F = \{\mathrm{Fr}\, A\}$.

(8) For every topological space T and for all families F, G of subsets of T such that $F \subseteq G$ holds $\mathrm{Fr}\, F \subseteq \mathrm{Fr}\, G$.

(9) For every topological space T and for all families F, G of subsets of T holds $\mathrm{Fr}(F \cup G) = \mathrm{Fr}\, F \cup \mathrm{Fr}\, G$.

(10) For every topological structure T and for every subset A of T holds $\mathrm{Fr}\, A = \overline{A} \setminus \mathrm{Int}\, A$.

(11) Let T be a non empty topological space, A be a subset of T, and p be a point of T. Then $p \in \mathrm{Fr}\, A$ if and only if for every subset U of T such that U is open and $p \in U$ holds A meets U and $U \setminus A \neq \emptyset$.

(12) Let T be a non empty topological space, A be a subset of T, and p be a point of T. Then $p \in \mathrm{Fr}\, A$ if and only if for every basis B of p and for every subset U of T such that $U \in B$ holds A meets U and $U \setminus A \neq \emptyset$.

(13) Let T be a non empty topological space, A be a subset of T, and p be a point of T. Then $p \in \mathrm{Fr}\, A$ if and only if there exists a basis B of p such that for every subset U of T such that $U \in B$ holds A meets U and $U \setminus A \neq \emptyset$.

(14) For every topological space T and for all subsets A, B of T holds $\operatorname{Fr}(A \cap B) \subseteq \overline{A} \cap \operatorname{Fr} B \cup \operatorname{Fr} A \cap \overline{B}$.

(15) For every topological space T and for every subset A of T holds the carrier of $T = \operatorname{Int} A \cup \operatorname{Fr} A \cup \operatorname{Int}(A^{\mathrm{c}})$.

(16) For every topological space T and for every subset A of T holds A is open and closed iff $\operatorname{Fr} A = \emptyset$.

3. Accumulation Points and Derivative of a Set

Let T be a topological structure, let A be a subset of T, and let x be a set. We say that x is an accumulation point of A if and only if:

(Def. 2) $x \in \overline{A \setminus \{x\}}$.

We now state the proposition

(17) Let T be a topological space, A be a subset of T, and x be a set. If x is an accumulation point of A, then x is a point of T.

Let T be a topological structure and let A be a subset of T. The functor $\operatorname{Der} A$ yielding a subset of T is defined by:

(Def. 3) For every set x such that $x \in$ the carrier of T holds $x \in \operatorname{Der} A$ iff x is an accumulation point of A.

Next we state four propositions:

(18) Let T be a non empty topological space, A be a subset of T, and x be a set. Then $x \in \operatorname{Der} A$ if and only if x is an accumulation point of A.

(19) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \operatorname{Der} A$ if and only if for every open subset U of T such that $x \in U$ there exists a point y of T such that $y \in A \cap U$ and $x \neq y$.

(20) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \operatorname{Der} A$ if and only if for every basis B of x and for every subset U of T such that $U \in B$ there exists a point y of T such that $y \in A \cap U$ and $x \neq y$.

(21) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \operatorname{Der} A$ if and only if there exists a basis B of x such that for every subset U of T such that $U \in B$ there exists a point y of T such that $y \in A \cap U$ and $x \neq y$.

4. Isolated Points

Let T be a topological space, let A be a subset of T, and let x be a set. We say that x is isolated in A if and only if:

(Def. 4) $x \in A$ and x is not an accumulation point of A.

The following three propositions are true:

(22) Let T be a non empty topological space, A be a subset of T, and p be a set. Then $p \in A \setminus \operatorname{Der} A$ if and only if p is isolated in A.

(23) Let T be a non empty topological space, A be a subset of T, and p be a point of T. Then p is an accumulation point of A if and only if for every open subset U of T such that $p \in U$ there exists a point q of T such that $q \neq p$ and $q \in A$ and $q \in U$.

(24) Let T be a non empty topological space, A be a subset of T, and p be a point of T. Then p is isolated in A if and only if there exists an open subset G of T such that $G \cap A = \{p\}$.

Let T be a topological space and let p be a point of T. We say that p is isolated if and only if:

(Def. 5) p is isolated in Ω_T.

Next we state the proposition

(25) For every non empty topological space T and for every point p of T holds p is isolated iff $\{p\}$ is open.

5. Derivative of a Subset-Family

Let T be a topological space and let F be a family of subsets of T. The functor $\operatorname{Der} F$ yielding a family of subsets of T is defined by:

(Def. 6) For every subset A of T holds $A \in \operatorname{Der} F$ iff there exists a subset B of T such that $A = \operatorname{Der} B$ and $B \in F$.

For simplicity, we follow the rules: T is a non empty topological space, A, B are subsets of T, F, G are families of subsets of T, and x is a set.

One can prove the following propositions:

(26) If $F = \emptyset$, then $\operatorname{Der} F = \emptyset$.

(27) If $F = \{A\}$, then $\operatorname{Der} F = \{\operatorname{Der} A\}$.

(28) If $F \subseteq G$, then $\operatorname{Der} F \subseteq \operatorname{Der} G$.

(29) $\operatorname{Der}(F \cup G) = \operatorname{Der} F \cup \operatorname{Der} G$.

(30) For every non empty topological space T and for every subset A of T holds $\operatorname{Der} A \subseteq \overline{A}$.

(31) For every topological space T and for every subset A of T holds $\overline{A} = A \cup \operatorname{Der} A$.

(32) For every non empty topological space T and for all subsets A, B of T such that $A \subseteq B$ holds $\operatorname{Der} A \subseteq \operatorname{Der} B$.

(33) For every non empty topological space T and for all subsets A, B of T holds $\operatorname{Der}(A \cup B) = \operatorname{Der} A \cup \operatorname{Der} B$.

(34) For every non empty topological space T and for every subset A of T such that T is T_1 holds $\operatorname{Der}\operatorname{Der} A \subseteq \operatorname{Der} A$.

(35) For every topological space T and for every subset A of T such that T is T_1 holds $\overline{\operatorname{Der} A} = \operatorname{Der} A$.

Let T be a T_1 non empty topological space and let A be a subset of T. Observe that $\operatorname{Der} A$ is closed.

One can prove the following two propositions:

(36) For every non empty topological space T and for every family F of subsets of T holds $\bigcup \operatorname{Der} F \subseteq \operatorname{Der} \bigcup F$.

(37) If $A \subseteq B$ and x is an accumulation point of A, then x is an accumulation point of B.

6. Density-in-itself

Let T be a topological space and let A be a subset of T. We say that A is dense-in-itself if and only if:

(Def. 7) $A \subseteq \operatorname{Der} A$.

Let T be a non empty topological space. We say that T is dense-in-itself if and only if:

(Def. 8) Ω_T is dense-in-itself.

Next we state the proposition

(38) If T is T_1 and A is dense-in-itself, then $\overline{A}$ is dense-in-itself.

Let T be a topological space and let F be a family of subsets of T. We say that F is dense-in-itself if and only if:

(Def. 9) For every subset A of T such that $A \in F$ holds A is dense-in-itself.

The following propositions are true:

(39) For every family F of subsets of T such that F is dense-in-itself holds $\bigcup F \subseteq \bigcup \operatorname{Der} F$.

(40) If F is dense-in-itself, then $\bigcup F$ is dense-in-itself.

(41) $\operatorname{Fr}(\emptyset_T) = \emptyset$.

Let T be a topological space and let A be an open closed subset of T. Note that $\operatorname{Fr} A$ is empty.

Let T be a non empty non discrete topological space. Note that there exists a subset of T which is non open and there exists a subset of T which is non closed.

Let T be a non empty non discrete topological space and let A be a non open subset of T. Observe that $\operatorname{Fr} A$ is non empty.

Let T be a non empty non discrete topological space and let A be a non closed subset of T. One can check that $\operatorname{Fr} A$ is non empty.

7. Perfect Sets

Let T be a topological space and let A be a subset of T. We say that A is perfect if and only if:

(Def. 10) A is closed and dense-in-itself.

Let T be a topological space. One can check that every subset of T which is perfect is also closed and dense-in-itself and every subset of T which is closed and dense-in-itself is also perfect.

We now state three propositions:

(42) For every topological space T holds $\mathrm{Der}(\emptyset_T) = \emptyset_T$.

(43) For every topological space T and for every subset A of T holds A is perfect iff $\mathrm{Der}\, A = A$.

(44) For every topological space T holds $\emptyset_T$ is perfect.

Let T be a topological space. Note that every subset of T which is empty is also perfect.

Let T be a topological space. Observe that there exists a subset of T which is perfect.

8. Scattered Subsets

Let T be a topological space and let A be a subset of T. We say that A is scattered if and only if:

(Def. 11) It is not true that there exists a subset B of T such that B is non empty and $B \subseteq A$ and B is dense-in-itself.

Let T be a non empty topological space. Observe that every subset of T which is non empty and scattered is also non dense-in-itself and every subset of T which is dense-in-itself and non empty is also non scattered.

The following proposition is true

(45) For every topological space T holds $\emptyset_T$ is scattered.

Let T be a topological space. Note that every subset of T which is empty is also scattered.

One can prove the following proposition

(46) Let T be a non empty topological space. Suppose T is T_1. Then there exist subsets A, B of T such that $A \cup B = \Omega_T$ and A misses B and A is perfect and B is scattered.

Let T be a discrete topological space and let A be a subset of T. Observe that $\mathrm{Fr}\, A$ is empty.

Let T be a discrete topological space. Observe that every subset of T is closed and open.

The following proposition is true

(47) For every discrete topological space T and for every subset A of T holds $\operatorname{Der} A = \emptyset$.

Let T be a discrete non empty topological space and let A be a subset of T. Note that $\operatorname{Der} A$ is empty.

One can prove the following proposition

(48) For every discrete non empty topological space T and for every subset A of T such that A is dense holds $A = \Omega_T$.

9. Density of a Topological Space and Separable Spaces

Let T be a topological space. The functor density T yielding a cardinal number is defined by:

(Def. 12) There exists a subset A of T such that A is dense and density $T = \overline{\overline{A}}$ and for every subset B of T such that B is dense holds density $T \leq \overline{\overline{B}}$.

Let T be a topological space. We say that T is separable if and only if:

(Def. 13) density $T \leq \aleph_0$.

One can prove the following proposition

(49) Every countable topological space is separable.

Let us observe that every topological space which is countable is also separable.

10. Exercises

The following propositions are true:

(50) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{Q}$ holds $A^{\mathrm{c}} = \mathbb{IQ}$.

(51) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{IQ}$ holds $A^{\mathrm{c}} = \mathbb{Q}$.

(52) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{Q}$ holds $\operatorname{Int} A = \emptyset$.

(53) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{IQ}$ holds $\operatorname{Int} A = \emptyset$.

(54) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{Q}$ holds A is dense.

(55) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{IQ}$ holds A is dense.

(56) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{Q}$ holds A is boundary.

(57) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{IQ}$ holds A is boundary.

(58) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{R}$ holds A is non boundary.

(59) There exist subsets A, B of $\mathbb{R}^{\mathbf{1}}$ such that A is boundary and B is boundary and $A \cup B$ is non boundary.

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Received November 8, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Construction of Gröbner Bases: Avoiding S-Polynomials – Buchberger's First Criterium[1]

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Summary. We continue the formalization of Groebner bases following the book "Groebner Bases – A Computational Approach to Commutative Algebra" by Becker and Weispfenning. Here we prove Buchberger's first criterium on avoiding S-polynomials: S-polynomials for polynomials with disjoint head terms need not be considered when constructing Groebner bases. In the course of formalizing this theorem we also introduced the splitting of a polynomial in an upper and a lower polynomial containing the greater resp. smaller terms of the original polynomial with respect to a given term order.

MML Identifier: GROEB_3.

The terminology and notation used in this paper have been introduced in the following articles: [24], [28], [29], [31], [1], [3], [12], [2], [8], [30], [9], [10], [17], [25], [16], [26], [11], [7], [5], [15], [13], [19], [27], [6], [4], [14], [23], [20], [22], [21], and [18].

1. Preliminaries

One can prove the following propositions:

(1) For every set X and for all bags b_1, b_2 of X holds $\frac{b_1+b_2}{b_2} = b_1$.

(2) Let n be an ordinal number, T be an admissible term order of n, and b_1, b_2, b_3 be bags of n. If $b_1 \leq_T b_2$, then $b_1 + b_3 \leq_T b_2 + b_3$.

(3) Let n be an ordinal number, T be a term order of n, and b_1, b_2, b_3 be bags of n. If $b_1 \leq_T b_2$ and $b_2 <_T b_3$, then $b_1 <_T b_3$.

[1]This work has been partially supported by grant BW 5100-5-0147-4.

ISSN 1426–2630

(4) Let n be an ordinal number, T be an admissible term order of n, and b_1, b_2, b_3 be bags of n. If $b_1 <_T b_2$, then $b_1 + b_3 <_T b_2 + b_3$.

(5) Let n be an ordinal number, T be an admissible term order of n, and b_1, b_2, b_3, b_4 be bags of n. If $b_1 <_T b_2$ and $b_3 \leq_T b_4$, then $b_1 + b_3 <_T b_2 + b_4$.

(6) Let n be an ordinal number, T be an admissible term order of n, and b_1, b_2, b_3, b_4 be bags of n. If $b_1 \leq_T b_2$ and $b_3 <_T b_4$, then $b_1 + b_3 <_T b_2 + b_4$.

2. More on Polynomials

One can prove the following propositions:

(7) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, and m_1, m_2 be non-zero monomials of n, L. Then $\operatorname{term} m_1 * m_2 = \operatorname{term} m_1 + \operatorname{term} m_2$.

(8) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, m be a non-zero monomial of n, L, and b be a bag of n. Then $b \in \operatorname{Support} p$ if and only if $\operatorname{term} m + b \in \operatorname{Support}(m * p)$.

(9) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, and m be a non-zero monomial of n, L. Then $\operatorname{Support}(m * p) = \{\operatorname{term} m + b; b$ ranges over elements of $\operatorname{Bags} n : b \in \operatorname{Support} p\}$.

(10) Let n be an ordinal number, L be an add-associative right complementable left zeroed right zeroed unital distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, and m be a non-zero monomial of n, L. Then $\operatorname{card} \operatorname{Support} p = \operatorname{card} \operatorname{Support}(m * p)$.

(11) Let n be an ordinal number, T be a connected term order of n, and L be an add-associative right complementable right zeroed non trivial loop structure. Then $\operatorname{Red}(0_n L, T) = 0_n L$.

(12) Let n be an ordinal number, L be an Abelian add-associative right zeroed right complementable commutative unital distributive non trivial double loop structure, and p, q be polynomials of n, L. If $p - q = 0_n L$, then $p = q$.

(13) Let X be a set and L be an add-associative right zeroed right complementable non empty loop structure. Then $-0_X L = 0_X L$.

(14) Let X be a set, L be an add-associative right zeroed right complementable non empty loop structure, and f be a series of X, L. Then $0_X L - f = -f$.

(15) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial double loop structure, and p be a polynomial of n, L. Then $p - \mathrm{Red}(p, T) = \mathrm{HM}(p, T)$.

Let n be an ordinal number, let L be an add-associative right complementable right zeroed non empty loop structure, and let p be a polynomial of n, L. Observe that $\mathrm{Support}\, p$ is finite.

Let n be an ordinal number, let L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, and let p, q be polynomials of n, L. Then $\{p, q\}$ is a non empty finite subset of Polynom-Ring(n, L).

3. Restriction and Splitting of Polynomials

Let X be a set, let L be a non empty zero structure, let s be a series of X, L, and let Y be a subset of $\mathrm{Bags}\, X$. The functor $s \restriction Y$ yields a series of X, L and is defined as follows:

(Def. 1) $s \restriction Y = s {+\cdot} (\mathrm{Support}\, s \setminus Y \longmapsto 0_L)$.

Let n be an ordinal number, let L be a non empty zero structure, let p be a polynomial of n, L, and let Y be a subset of $\mathrm{Bags}\, n$. Note that $p \restriction Y$ is finite-Support.

Next we state three propositions:

(16) Let X be a set, L be a non empty zero structure, s be a series of X, L, and Y be a subset of $\mathrm{Bags}\, X$. Then $\mathrm{Support}(s \restriction Y) = \mathrm{Support}\, s \cap Y$ and for every bag b of X such that $b \in \mathrm{Support}(s \restriction Y)$ holds $(s \restriction Y)(b) = s(b)$.

(17) Let X be a set, L be a non empty zero structure, s be a series of X, L, and Y be a subset of $\mathrm{Bags}\, X$. Then $\mathrm{Support}(s \restriction Y) \subseteq \mathrm{Support}\, s$.

(18) For every set X and for every non empty zero structure L and for every series s of X, L holds $s \restriction \mathrm{Support}\, s = s$ and $s \restriction \emptyset_{\mathrm{Bags}\, X} = 0_X L$.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right zeroed right complementable non empty loop structure, let p be a polynomial of n, L, and let i be a natural number. Let us assume that $i \leq \mathrm{card}\, \mathrm{Support}\, p$. The functor $\mathrm{UpperSupport}(p, T, i)$ yielding a finite subset of $\mathrm{Bags}\, n$ is defined by the conditions (Def. 2).

(Def. 2)(i) $\mathrm{UpperSupport}(p, T, i) \subseteq \mathrm{Support}\, p$,

(ii) $\mathrm{card}\, \mathrm{UpperSupport}(p, T, i) = i$, and

(iii) for all bags b, b' of n such that $b \in \mathrm{UpperSupport}(p, T, i)$ and $b' \in \mathrm{Support}\, p$ and $b \leq_T b'$ holds $b' \in \mathrm{UpperSupport}(p, T, i)$.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right zeroed right complementable non empty loop structure, let p be a polynomial of n, L, and let i be a natural number. The functor $\mathrm{LowerSupport}(p, T, i)$ yielding a finite subset of $\mathrm{Bags}\, n$ is defined by:

(Def. 3) $\mathrm{LowerSupport}(p,T,i) = \mathrm{Support}\,p \setminus \mathrm{UpperSupport}(p,T,i)$.

We now state several propositions:

(19) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i \leq \mathrm{card}\,\mathrm{Support}\,p$, then $\mathrm{UpperSupport}(p,T,i) \cup \mathrm{LowerSupport}(p,T,i) = \mathrm{Support}\,p$ and $\mathrm{UpperSupport}(p,T,i) \cap \mathrm{LowerSupport}(p,T,i) = \emptyset$.

(20) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \mathrm{card}\,\mathrm{Support}\,p$. Let b, b' be bags of n. If $b \in \mathrm{UpperSupport}(p,T,i)$ and $b' \in \mathrm{LowerSupport}(p,T,i)$, then $b' <_T b$.

(21) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, and p be a polynomial of n, L. Then $\mathrm{UpperSupport}(p,T,0) = \emptyset$ and $\mathrm{LowerSupport}(p,T,0) = \mathrm{Support}\,p$.

(22) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, and p be a polynomial of n, L. Then $\mathrm{UpperSupport}(p,T,\mathrm{card}\,\mathrm{Support}\,p) = \mathrm{Support}\,p$ and $\mathrm{LowerSupport}(p,T,\mathrm{card}\,\mathrm{Support}\,p) = \emptyset$.

(23) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non trivial loop structure, p be a non-zero polynomial of n, L, and i be a natural number. If $1 \leq i$ and $i \leq \mathrm{card}\,\mathrm{Support}\,p$, then $\mathrm{HT}(p,T) \in \mathrm{UpperSupport}(p,T,i)$.

(24) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \mathrm{card}\,\mathrm{Support}\,p$. Then $\mathrm{LowerSupport}(p,T,i) \subseteq \mathrm{Support}\,p$ and $\mathrm{card}\,\mathrm{LowerSupport}(p,T,i) = \mathrm{card}\,\mathrm{Support}\,p - i$ and for all bags b, b' of n such that $b \in \mathrm{LowerSupport}(p,T,i)$ and $b' \in \mathrm{Support}\,p$ and $b' \leq_T b$ holds $b' \in \mathrm{LowerSupport}(p,T,i)$.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right zeroed right complementable non empty loop structure, let p be a polynomial of n, L, and let i be a natural number. The functor $\mathrm{Up}(p,T,i)$ yields a polynomial of n, L and is defined by:

(Def. 4) $\mathrm{Up}(p,T,i) = p{\restriction}\,\mathrm{UpperSupport}(p,T,i)$.

The functor $\mathrm{Low}(p,T,i)$ yielding a polynomial of n, L is defined by:

(Def. 5) $\mathrm{Low}(p,T,i) = p{\restriction}\,\mathrm{LowerSupport}(p,T,i)$.

One can prove the following propositions:

(25) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i \leq \mathrm{card}\,\mathrm{Support}\,p$, then $\mathrm{Support}\,\mathrm{Up}(p, T, i) = \mathrm{UpperSupport}(p, T, i)$ and $\mathrm{Support}\,\mathrm{Low}(p, T, i) = \mathrm{LowerSupport}(p, T, i)$.

(26) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i \leq \mathrm{card}\,\mathrm{Support}\,p$, then $\mathrm{Support}\,\mathrm{Up}(p, T, i) \subseteq \mathrm{Support}\,p$ and $\mathrm{Support}\,\mathrm{Low}(p, T, i) \subseteq \mathrm{Support}\,p$.

(27) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, p be a polynomial of n, L, and i be a natural number. If $1 \leq i$ and $i \leq \mathrm{card}\,\mathrm{Support}\,p$, then $\mathrm{Support}\,\mathrm{Low}(p, T, i) \subseteq \mathrm{Support}\,\mathrm{Red}(p, T)$.

(28) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \mathrm{card}\,\mathrm{Support}\,p$. Let b be a bag of n. If $b \in \mathrm{Support}\,p$, then $b \in \mathrm{Support}\,\mathrm{Up}(p, T, i)$ or $b \in \mathrm{Support}\,\mathrm{Low}(p, T, i)$ but $b \notin \mathrm{Support}\,\mathrm{Up}(p, T, i) \cap \mathrm{Support}\,\mathrm{Low}(p, T, i)$.

(29) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \mathrm{card}\,\mathrm{Support}\,p$. Let b, b' be bags of n. If $b \in \mathrm{Support}\,\mathrm{Low}(p, T, i)$ and $b' \in \mathrm{Support}\,\mathrm{Up}(p, T, i)$, then $b <_T b'$.

(30) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $1 \leq i$ and $i \leq \mathrm{card}\,\mathrm{Support}\,p$, then $\mathrm{HT}(p, T) \in \mathrm{Support}\,\mathrm{Up}(p, T, i)$.

(31) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \mathrm{card}\,\mathrm{Support}\,p$. Let b be a bag of n. If $b \in \mathrm{Support}\,\mathrm{Low}(p, T, i)$, then $(\mathrm{Low}(p, T, i))(b) = p(b)$ and $(\mathrm{Up}(p, T, i))(b) = 0_L$.

(32) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \mathrm{card}\,\mathrm{Support}\,p$. Let b be a bag of n. If $b \in \mathrm{Support}\,\mathrm{Up}(p, T, i)$, then $(\mathrm{Up}(p, T, i))(b) = p(b)$ and $(\mathrm{Low}(p, T, i))(b) = 0_L$.

(33) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop

structure, p be a polynomial of n, L, and i be a natural number. If $i \leq \operatorname{card} \operatorname{Support} p$, then $\operatorname{Up}(p, T, i) + \operatorname{Low}(p, T, i) = p$.

(34) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, and p be a polynomial of n, L. Then $\operatorname{Up}(p, T, 0) = 0_n L$ and $\operatorname{Low}(p, T, 0) = p$.

(35) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable Abelian non empty double loop structure, and p be a polynomial of n, L. Then $\operatorname{Up}(p, T, \operatorname{card} \operatorname{Support} p) = p$ and $\operatorname{Low}(p, T, \operatorname{card} \operatorname{Support} p) = 0_n L$.

(36) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable Abelian non trivial double loop structure, and p be a non-zero polynomial of n, L. Then $\operatorname{Up}(p, T, 1) = \operatorname{HM}(p, T)$ and $\operatorname{Low}(p, T, 1) = \operatorname{Red}(p, T)$.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right zeroed right complementable non trivial loop structure, and let p be a non-zero polynomial of n, L. Observe that $\operatorname{Up}(p, T, 0)$ is monomial-like.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right zeroed right complementable Abelian non trivial double loop structure, and let p be a non-zero polynomial of n, L. Note that $\operatorname{Up}(p, T, 1)$ is non-zero and monomial-like and $\operatorname{Low}(p, T, \operatorname{card} \operatorname{Support} p)$ is monomial-like.

The following propositions are true:

(37) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non trivial loop structure, p be a polynomial of n, L, and j be a natural number. If $j = \operatorname{card} \operatorname{Support} p - 1$, then $\operatorname{Low}(p, T, j)$ is a non-zero monomial of n, L.

(38) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i < \operatorname{card} \operatorname{Support} p$, then $\operatorname{HT}(\operatorname{Low}(p, T, i+1), T) \leq_T \operatorname{HT}(\operatorname{Low}(p, T, i), T)$.

(39) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $0 < i$ and $i < \operatorname{card} \operatorname{Support} p$, then $\operatorname{HT}(\operatorname{Low}(p, T, i), T) <_T \operatorname{HT}(p, T)$.

(40) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, m be a non-zero monomial of n, L, and i be a natural number. Suppose $i \leq \operatorname{card} \operatorname{Support} p$. Let b be a bag of n. Then $\operatorname{term} m + b \in \operatorname{Support} \operatorname{Low}(m * p, T, i)$ if and only if $b \in \operatorname{Support} \operatorname{Low}(p, T, i)$.

(41) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i < \operatorname{card} \operatorname{Support} p$, then $\operatorname{Support} \operatorname{Low}(p, T, i+1) \subseteq \operatorname{Support} \operatorname{Low}(p, T, i)$.

(42) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i < \operatorname{card} \operatorname{Support} p$, then $\operatorname{Support} \operatorname{Low}(p, T, i) \setminus \operatorname{Support} \operatorname{Low}(p, T, i+1) = \{\operatorname{HT}(\operatorname{Low}(p, T, i), T)\}$.

(43) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right zeroed right complementable non trivial loop structure, p be a polynomial of n, L, and i be a natural number. If $i < \operatorname{card} \operatorname{Support} p$, then $\operatorname{Low}(p, T, i+1) = \operatorname{Red}(\operatorname{Low}(p, T, i), T)$.

(44) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, m be a non-zero monomial of n, L, and i be a natural number. If $i \leq \operatorname{card} \operatorname{Support} p$, then $\operatorname{Low}(m * p, T, i) = m * \operatorname{Low}(p, T, i)$.

4. More on Polynomial Reduction

Next we state several propositions:

(45) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and f, g, p be polynomials of n, L. If f reduces to g, p, T, then $-f$ reduces to $-g$, p, T.

(46) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and f, f_1, g, p be polynomials of n, L. Suppose f reduces to f_1, $\{p\}$, T and for every bag b_1 of n such that $b_1 \in \operatorname{Support} g$ holds $\operatorname{HT}(p, T) \nmid b_1$. Then $f + g$ reduces to $f_1 + g$, $\{p\}$, T.

(47) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and f, g be non-zero polynomials of n, L. Then $f * g$ reduces to $\operatorname{Red}(f, T) * g$, $\{g\}$, T.

(48) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative

associative left unital right unital distributive Abelian field-like non trivial double loop structure, f, g be non-zero polynomials of n, L, and p be a polynomial of n, L. If $p(\mathrm{HT}(f*g,T)) = 0_L$, then $f*g+p$ reduces to $\mathrm{Red}(f,T)*g+p$, $\{g\}$, T.

(49) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, P be a subset of Polynom-Ring(n,L), and f, g be polynomials of n, L. If PolyRedRel(P,T) reduces f to g, then PolyRedRel(P,T) reduces $-f$ to $-g$.

(50) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and f, f_1, g, p be polynomials of n, L. Suppose PolyRedRel$(\{p\},T)$ reduces f to f_1 and for every bag b_1 of n such that $b_1 \in \mathrm{Support}\,g$ holds $\mathrm{HT}(p,T) \nmid b_1$. Then PolyRedRel$(\{p\},T)$ reduces $f+g$ to f_1+g.

(51) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and f, g be non-zero polynomials of n, L. Then PolyRedRel$(\{g\},T)$ reduces $f*g$ to 0_nL.

5. The Criterium

We now state several propositions:

(52) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive field-like non trivial double loop structure, and p_1, p_2 be polynomials of n, L. Suppose $\mathrm{HT}(p_1,T)$, $\mathrm{HT}(p_2,T)$ are disjoint. Let b_1, b_2 be bags of n. If $b_1 \in \mathrm{Support}\,\mathrm{Red}(p_1,T)$ and $b_2 \in \mathrm{Support}\,\mathrm{Red}(p_2,T)$, then $\mathrm{HT}(p_1,T)+b_2 \neq \mathrm{HT}(p_2,T)+b_1$.

(53) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and p_1, p_2 be polynomials of n, L. If $\mathrm{HT}(p_1,T)$, $\mathrm{HT}(p_2,T)$ are disjoint, then $\text{S-Poly}(p_1,p_2,T) = \mathrm{HM}(p_2,T)*\mathrm{Red}(p_1,T)-\mathrm{HM}(p_1,T)*\mathrm{Red}(p_2,T)$.

(54) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double

loop structure, and p_1, p_2 be polynomials of n, L. If $\mathrm{HT}(p_1,T)$, $\mathrm{HT}(p_2,T)$ are disjoint, then $\text{S-Poly}(p_1,p_2,T) = \mathrm{Red}(p_1,T) * p_2 - \mathrm{Red}(p_2,T) * p_1$.

(55) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and p_1, p_2 be non-zero polynomials of n, L. Suppose $\mathrm{HT}(p_1,T)$, $\mathrm{HT}(p_2,T)$ are disjoint and $\mathrm{Red}(p_1,T)$ is non-zero and $\mathrm{Red}(p_2,T)$ is non-zero. Then $\mathrm{PolyRedRel}(\{p_1\},T)$ reduces $\mathrm{HM}(p_2,T) * \mathrm{Red}(p_1,T) - \mathrm{HM}(p_1,T) * \mathrm{Red}(p_2,T)$ to $p_2 * \mathrm{Red}(p_1,T)$.

(56) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and p_1, p_2 be polynomials of n, L. If $\mathrm{HT}(p_1,T)$, $\mathrm{HT}(p_2,T)$ are disjoint, then $\mathrm{PolyRedRel}(\{p_1,p_2\},T)$ reduces $\text{S-Poly}(p_1,p_2,T)$ to 0_nL.

(57) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non degenerated non empty double loop structure, and G be a subset of $\text{Polynom-Ring}(n,L)$. Suppose G is a Groebner basis wrt T. Let g_1, g_2 be polynomials of n, L. Suppose $g_1 \in G$ and $g_2 \in G$ and $\mathrm{HT}(g_1,T)$, $\mathrm{HT}(g_2,T)$ are not disjoint. Then $\mathrm{PolyRedRel}(G,T)$ reduces $\text{S-Poly}(g_1,g_2,T)$ to 0_nL.

(58) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non degenerated non trivial double loop structure, and G be a subset of $\text{Polynom-Ring}(n,L)$. Suppose $0_nL \notin G$. Suppose that for all polynomials g_1, g_2 of n, L such that $g_1 \in G$ and $g_2 \in G$ and $\mathrm{HT}(g_1,T)$, $\mathrm{HT}(g_2,T)$ are not disjoint holds $\mathrm{PolyRedRel}(G,T)$ reduces $\text{S-Poly}(g_1,g_2,T)$ to 0_nL. Let g_1, g_2, h be polynomials of n, L. Suppose $g_1 \in G$ and $g_2 \in G$ and $\mathrm{HT}(g_1,T)$, $\mathrm{HT}(g_2,T)$ are not disjoint and h is a normal form of $\text{S-Poly}(g_1,g_2,T)$ w.r.t. $\mathrm{PolyRedRel}(G,T)$. Then $h = 0_nL$.

(59) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non degenerated non empty double loop structure, and G be a subset of $\text{Polynom-Ring}(n,L)$. Suppose $0_nL \notin G$. Suppose that for all polynomials g_1, g_2, h of n, L such that $g_1 \in G$ and $g_2 \in G$ and $\mathrm{HT}(g_1,T)$, $\mathrm{HT}(g_2,T)$ are not disjoint and h is a normal form of $\text{S-Poly}(g_1,g_2,T)$ w.r.t. $\mathrm{PolyRedRel}(G,T)$ holds $h = 0_nL$. Then G is a Groebner basis wrt T.

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Received December 10, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

A Theory of Matrices of Complex Elements

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Summary. A concept of "Matrix of Complex" is defined here. Addition, subtraction, scalar multiplication and product are introduced using correspondent definitions of "Matrix of Field". Many equations for such operations consist of a case of "Matrix of Field". A calculation method of product of matrices is shown using a finite sequence of Complex in the last theorem.

MML Identifier: MATRIX_5.

The articles [11], [14], [1], [4], [2], [15], [6], [10], [9], [3], [8], [7], [13], [12], and [5] provide the terminology and notation for this paper.

The following two propositions are true:

(1) $1 = 1_{\mathbb{C}_{\mathrm{F}}}$.

(2) $0_{\mathbb{C}_{\mathrm{F}}} = 0$.

Let A be a matrix over $\mathbb{C}$. The functor $A_{\mathbb{C}_{\mathrm{F}}}$ yields a matrix over $\mathbb{C}_{\mathrm{F}}$ and is defined by:

(Def. 1) $A_{\mathbb{C}_{\mathrm{F}}} = A$.

Let A be a matrix over $\mathbb{C}_{\mathrm{F}}$. The functor $A_{\mathbb{C}}$ yielding a matrix over $\mathbb{C}$ is defined by:

(Def. 2) $A_{\mathbb{C}} = A$.

We now state four propositions:

(3) For all matrices A, B over $\mathbb{C}$ such that $A_{\mathbb{C}_{\mathrm{F}}} = B_{\mathbb{C}_{\mathrm{F}}}$ holds $A = B$.

(4) For all matrices A, B over $\mathbb{C}_{\mathrm{F}}$ such that $A_{\mathbb{C}} = B_{\mathbb{C}}$ holds $A = B$.

(5) For every matrix A over $\mathbb{C}$ holds $A = (A_{\mathbb{C}_{\mathrm{F}}})_{\mathbb{C}}$.

(6) For every matrix A over $\mathbb{C}_{\mathrm{F}}$ holds $A = (A_{\mathbb{C}})_{\mathbb{C}_{\mathrm{F}}}$.

Let A, B be matrices over $\mathbb{C}$. The functor $A + B$ yielding a matrix over $\mathbb{C}$ is defined as follows:

ISSN 1426–2630

(Def. 3) $A + B = (A_{\mathbb{C}_{\mathrm{F}}} + B_{\mathbb{C}_{\mathrm{F}}})_{\mathbb{C}}$.

Let A be a matrix over $\mathbb{C}$. The functor $-A$ yielding a matrix over $\mathbb{C}$ is defined as follows:

(Def. 4) $-A = (-A_{\mathbb{C}_{\mathrm{F}}})_{\mathbb{C}}$.

Let A, B be matrices over $\mathbb{C}$. The functor $A - B$ yields a matrix over $\mathbb{C}$ and is defined as follows:

(Def. 5) $A - B = (A_{\mathbb{C}_{\mathrm{F}}} - B_{\mathbb{C}_{\mathrm{F}}})_{\mathbb{C}}$.

Let A, B be matrices over $\mathbb{C}$. The functor $A \cdot B$ yielding a matrix over $\mathbb{C}$ is defined as follows:

(Def. 6) $A \cdot B = (A_{\mathbb{C}_{\mathrm{F}}} \cdot B_{\mathbb{C}_{\mathrm{F}}})_{\mathbb{C}}$.

Let x be a complex number and let A be a matrix over $\mathbb{C}$. The functor $x \cdot A$ yielding a matrix over $\mathbb{C}$ is defined as follows:

(Def. 7) For every element e_1 of $\mathbb{C}_{\mathrm{F}}$ such that $e_1 = x$ holds $x \cdot A = (e_1 \cdot A_{\mathbb{C}_{\mathrm{F}}})_{\mathbb{C}}$.

One can prove the following propositions:

(7) For every matrix A over $\mathbb{C}$ holds $\operatorname{len} A = \operatorname{len}(A_{\mathbb{C}_{\mathrm{F}}})$ and $\operatorname{width} A = \operatorname{width}(A_{\mathbb{C}_{\mathrm{F}}})$.

(8) For every matrix A over $\mathbb{C}_{\mathrm{F}}$ holds $\operatorname{len} A = \operatorname{len}(A_{\mathbb{C}})$ and $\operatorname{width} A = \operatorname{width}(A_{\mathbb{C}})$.

(9) For every matrix M over $\mathbb{C}$ such that $\operatorname{len} M > 0$ holds $--M = M$.

(10) For every field K and for every matrix M over K holds $1_K \cdot M = M$.

(11) For every matrix M over $\mathbb{C}$ holds $1 \cdot M = M$.

(12) For every field K and for all elements a, b of K and for every matrix M over K holds $a \cdot (b \cdot M) = (a \cdot b) \cdot M$.

(13) For every field K and for all elements a, b of K and for every matrix M over K holds $(a + b) \cdot M = a \cdot M + b \cdot M$.

(14) For every matrix M over $\mathbb{C}$ holds $M + M = 2 \cdot M$.

(15) For every matrix M over $\mathbb{C}$ holds $M + M + M = 3 \cdot M$.

Let n, m be natural numbers. The functor $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}^{n \times m}_{\mathbb{C}}$ yields a matrix over $\mathbb{C}$ and is defined by:

(Def. 8) $$\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}^{n \times m}_{\mathbb{C}} = \left(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}^{n \times m}_{\mathbb{C}_{\mathrm{F}}}\right)_{\mathbb{C}}.$$

One can prove the following propositions:

(16) For all natural numbers n, m holds $\left(\begin{matrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{matrix}\right)^{n \times m}_{\mathbb{C}}{}_{\mathbb{C}_{\rm F}} = \left(\begin{matrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{matrix}\right)^{n \times m}_{\mathbb{C}_{\rm F}}$.

(17) For every matrix M over $\mathbb{C}$ such that $\operatorname{len} M > 0$ holds $M + -M = \left(\begin{matrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{matrix}\right)^{(\operatorname{len} M) \times (\operatorname{width} M)}_{\mathbb{C}}$.

(18) For every matrix M over $\mathbb{C}$ such that $\operatorname{len} M > 0$ holds $M - M = \left(\begin{matrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{matrix}\right)^{(\operatorname{len} M) \times (\operatorname{width} M)}_{\mathbb{C}}$.

(19) For all matrices M_1, M_2, M_3 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $M_1 + M_3 = M_2 + M_3$ holds $M_1 = M_2$.

(20) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\operatorname{len} M_2 > 0$ holds $M_1 - -M_2 = M_1 + M_2$.

(21) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_1 = M_1 + M_2$ holds $M_2 = \left(\begin{matrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{matrix}\right)^{(\operatorname{len} M_1) \times (\operatorname{width} M_1)}_{\mathbb{C}}$.

(22) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_1 - M_2 = \left(\begin{matrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{matrix}\right)^{(\operatorname{len} M_1) \times (\operatorname{width} M_1)}_{\mathbb{C}}$ holds $M_1 = M_2$.

(23) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_1 + M_2 = \left(\begin{matrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{matrix}\right)^{(\operatorname{len} M_1) \times (\operatorname{width} M_1)}_{\mathbb{C}}$ holds $M_2 = -M_1$.

(24) For all natural numbers n, m such that $n > 0$ holds

$$-\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{\mathbb{C}}^{n\times m} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{\mathbb{C}}^{n\times m}.$$

(25) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_2 - M_1 = M_2$ holds $M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{\mathbb{C}}^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}$.

(26) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $M_1 = M_1 - (M_2 - M_2)$.

(27) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $-(M_1 + M_2) = -M_1 + -M_2$.

(28) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $M_1 - (M_1 - M_2) = M_2$.

(29) For all matrices M_1, M_2, M_3 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $M_1 - M_3 = M_2 - M_3$ holds $M_1 = M_2$.

(30) For all matrices M_1, M_2, M_3 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $M_3 - M_1 = M_3 - M_2$ holds $M_1 = M_2$.

(31) For all matrices M_1, M_2, M_3 over $\mathbb{C}$ such that $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{width} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_1 > 0$ and $\operatorname{len} M_2 > 0$ holds $M_1 \cdot (M_2 + M_3) = M_1 \cdot M_2 + M_1 \cdot M_3$.

(32) For all matrices M_1, M_2, M_3 over $\mathbb{C}$ such that $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_2 > 0$ and $\operatorname{len} M_1 > 0$ holds $(M_2 + M_3) \cdot M_1 = M_2 \cdot M_1 + M_3 \cdot M_1$.

(33) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ holds $M_1 + M_2 = M_2 + M_1$.

(34) For all matrices M_1, M_2, M_3 over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_1 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_3$ holds $(M_1 + M_2) + M_3 = M_1 + (M_2 + M_3)$.

(35) For every matrix M over $\mathbb{C}$ such that $\operatorname{len} M > 0$ holds $M + \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{\mathbb{C}}^{(\operatorname{len} M)\times(\operatorname{width} M)} = M$.

(36) Let K be a field, b be an element of K, and M_1, M_2 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$, then $b \cdot (M_1 + M_2) = b \cdot M_1 + b \cdot M_2$.

(37) Let M_1, M_2 be matrices over $\mathbb{C}$ and a be a complex number. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$, then $a \cdot (M_1 + M_2) = a \cdot M_1 + a \cdot M_2$.

(38) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{width} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_1 > 0$ and $\operatorname{len} M_2 > 0$ holds $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)} \cdot M_2 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M_1)\times(\operatorname{width} M_2)}$.

(39) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\operatorname{width} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_1 > 0$ and $\operatorname{len} M_2 > 0$ holds $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{C}}^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)} \cdot M_2 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{C}}^{(\operatorname{len} M_1)\times(\operatorname{width} M_2)}$.

(40) For every field K and for every matrix M_1 over K such that $\operatorname{len} M_1 > 0$ holds $0_K \cdot M_1 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}$.

(41) For every matrix M_1 over $\mathbb{C}$ such that $\operatorname{len} M_1 > 0$ holds $0 \cdot M_1 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{C}}^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}$.

Let s be a finite sequence of elements of $\mathbb{C}$ and let k be a natural number. Then $s(k)$ is an element of $\mathbb{C}$.

We now state the proposition

(42) Let i, j be natural numbers and M_1, M_2 be matrices over $\mathbb{C}$. Suppose $\operatorname{len} M_1 > 0$ and $\operatorname{len} M_2 > 0$ and $\operatorname{width} M_1 = \operatorname{len} M_2$ and $1 \leq i$ and $i \leq \operatorname{len} M_1$ and $1 \leq j$ and $j \leq \operatorname{width} M_2$. Then there exists a finite sequence s of elements of $\mathbb{C}$ such that $\operatorname{len} s = \operatorname{len} M_2$ and $s(1) = (M_1 \circ (i, 1)) \cdot (M_2 \circ (1, j))$ and for every natural number k such that $1 \leq k$ and $k < \operatorname{len} M_2$ holds $s(k+1) = s(k) + (M_1 \circ (i, k+1)) \cdot (M_2 \circ (k+1, j))$.

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Received December 10, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

On the Characteristic and Weight of a Topological Space[1]

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Summary. We continue Mizar formalization of General Topology according to the book [13] by Engelking. In the article the formalization of Section 1.1 is completed. Namely, the paper includes the formalization of theorems on correspondence of the basis and basis in a point, definitions of the character of a point and a topological space, a neighborhood system, and the weight of a topological space. The formalization is tested with almost discrete topological spaces with infinity.

MML Identifier: `TOPGEN_2`.

The notation and terminology used here are introduced in the following articles: [22], [26], [21], [16], [27], [9], [28], [10], [7], [3], [18], [5], [4], [12], [24], [1], [2], [25], [17], [29], [11], [14], [8], [19], [20], [23], [6], and [15].

1. Characteristic of Topological Spaces

One can prove the following propositions:

(1) Let T be a non empty topological space, B be a basis of T, and x be an element of T. Then $\{U; U$ ranges over subsets of T: $x \in U \ \wedge \ U \in B\}$ is a basis of x.

(2) Let T be a non empty topological space and B be a many sorted set indexed by T. Suppose that for every element x of T holds $B(x)$ is a basis of x. Then $\bigcup B$ is a basis of T.

Let T be a non empty topological structure and let x be an element of T. The functor $\mathrm{Chi}(x, T)$ yielding a cardinal number is defined as follows:

[1]This work has been partially supported by the KBN grant 4 T11C 039 24.

ISSN 1426–2630

(Def. 1) There exists a basis B of x such that $\mathrm{Chi}(x, T) = \overline{\overline{B}}$ and for every basis B of x holds $\mathrm{Chi}(x, T) \leq \overline{\overline{B}}$.

One can prove the following proposition

(3) Let X be a set. Suppose that for every set a such that $a \in X$ holds a is a cardinal number. Then $\bigcup X$ is a cardinal number.

Let T be a non empty topological structure. The functor $\mathrm{Chi}\, T$ yields a cardinal number and is defined by the conditions (Def. 2).

(Def. 2)(i) For every point x of T holds $\mathrm{Chi}(x, T) \leq \mathrm{Chi}\, T$, and

(ii) for every cardinal number M such that for every point x of T holds $\mathrm{Chi}(x, T) \leq M$ holds $\mathrm{Chi}\, T \leq M$.

The following three propositions are true:

(4) For every non empty topological structure T holds $\mathrm{Chi}\, T = \bigcup\{\mathrm{Chi}(x, T) : x$ ranges over points of $T\}$.

(5) For every non empty topological structure T and for every point x of T holds $\mathrm{Chi}(x, T) \leq \mathrm{Chi}\, T$.

(6) For every non empty topological space T holds T is first-countable iff $\mathrm{Chi}\, T \leq \aleph_0$.

2. Neighborhood Systems

Let T be a non empty topological space. A many sorted set indexed by T is said to be a neighborhood system of T if:

(Def. 3) For every element x of T holds $\mathrm{it}(x)$ is a basis of x.

Let T be a non empty topological space and let N be a neighborhood system of T. Then $\bigcup N$ is a basis of T. Let x be a point of T. Then $N(x)$ is a basis of x.

We now state several propositions:

(7) Let T be a non empty topological space, N be a neighborhood system of T, and x be an element of T. Then $N(x)$ is non empty and for every set U such that $U \in N(x)$ holds $x \in U$.

(8) Let T be a non empty topological space, x, y be points of T, B_1 be a basis of x, B_2 be a basis of y, and U be a set. If $x \in U$ and $U \in B_2$, then there exists an open subset V of T such that $V \in B_1$ and $V \subseteq U$.

(9) Let T be a non empty topological space, x be a point of T, B be a basis of x, and U_1, U_2 be sets. If $U_1 \in B$ and $U_2 \in B$, then there exists an open subset V of T such that $V \in B$ and $V \subseteq U_1 \cap U_2$.

(10) Let T be a non empty topological space, A be a subset of T, and x be an element of T. Then $x \in \overline{A}$ if and only if for every basis B of x and for every set U such that $U \in B$ holds U meets A.

(11) Let T be a non empty topological space, A be a subset of T, and x be an element of T. Then $x \in \overline{A}$ if and only if there exists a basis B of x such that for every set U such that $U \in B$ holds U meets A.

Let T be a topological space. Note that there exists a family of subsets of T which is open and non empty.

3. Weights of Topological Spaces

Next we state the proposition

(12) Let T be a non empty topological space and S be an open family of subsets of T. Then there exists an open family G of subsets of T such that $G \subseteq S$ and $\bigcup G = \bigcup S$ and $\overline{\overline{G}} \leq \operatorname{weight} T$.

Let T be a topological structure. We say that T is finite-weight if and only if:

(Def. 4) weight T is finite.

Let T be a topological structure. We introduce T is infinite-weight as an antonym of T is finite-weight.

Let us mention that every topological structure which is finite is also finite-weight and every topological structure which is infinite-weight is also infinite.

Let us note that there exists a topological space which is finite and non empty.

The following propositions are true:

(13) For every set X holds $\overline{\overline{\mathrm{SmallestPartition}(X)}} = \overline{\overline{X}}$.

(14) Let T be a discrete non empty topological structure. Then SmallestPartition(the carrier of T) is a basis of T and for every basis B of T holds SmallestPartition(the carrier of T) $\subseteq B$.

(15) For every discrete non empty topological structure T holds weight $T = \overline{\overline{\text{the carrier of } T}}$.

One can verify that there exists a topological space which is infinite-weight.

Next we state several propositions:

(16) Let T be an infinite-weight topological space and B be a basis of T. Then there exists a basis B_1 of T such that $B_1 \subseteq B$ and $\overline{\overline{B_1}} = \operatorname{weight} T$.

(17) Let T be a finite-weight non empty topological space. Then there exists a basis B_0 of T and there exists a function f from the carrier of T into the topology of T such that $B_0 = \operatorname{rng} f$ and for every point x of T holds $x \in f(x)$ and for every open subset U of T such that $x \in U$ holds $f(x) \subseteq U$.

(18) Let T be a topological space, B_0, B be bases of T, and f be a function from the carrier of T into the topology of T. Suppose $B_0 = \operatorname{rng} f$ and for every point x of T holds $x \in f(x)$ and for every open subset U of T such that $x \in U$ holds $f(x) \subseteq U$. Then $B_0 \subseteq B$.

(19) Let T be a topological space, B_0 be a basis of T, and f be a function from the carrier of T into the topology of T. Suppose $B_0 = \operatorname{rng} f$ and for every point x of T holds $x \in f(x)$ and for every open subset U of T such that $x \in U$ holds $f(x) \subseteq U$. Then $\operatorname{weight} T = \overline{\overline{B_0}}$.

(20) For every non empty topological space T and for every basis B of T there exists a basis B_1 of T such that $B_1 \subseteq B$ and $\overline{\overline{B_1}} = \operatorname{weight} T$.

4. Example of Almost Discrete Topological Space with Infinity

Let X, x_0 be sets. The functor $\operatorname{DiscrWithInfin}(X, x_0)$ yielding a strict topological structure is defined by the conditions (Def. 5).

(Def. 5)(i) The carrier of $\operatorname{DiscrWithInfin}(X, x_0) = X$, and

(ii) the topology of $\operatorname{DiscrWithInfin}(X, x_0) = \{U; U$ ranges over subsets of X: $x_0 \notin U\} \cup \{F^{\mathrm{c}}; F$ ranges over subsets of X: F is finite$\}$.

Let X, x_0 be sets. Observe that $\operatorname{DiscrWithInfin}(X, x_0)$ is topological space-like.

Let X be a non empty set and let x_0 be a set. One can verify that $\operatorname{DiscrWithInfin}(X, x_0)$ is non empty.

Next we state a number of propositions:

(21) For all sets X, x_0 and for every subset A of $\operatorname{DiscrWithInfin}(X, x_0)$ holds A is open iff $x_0 \notin A$ or A^{c} is finite.

(22) For all sets X, x_0 and for every subset A of $\operatorname{DiscrWithInfin}(X, x_0)$ holds A is closed iff if $x_0 \in X$, then $x_0 \in A$ or A is finite.

(23) For all sets X, x_0, x such that $x \in X$ holds $\{x\}$ is a closed subset of $\operatorname{DiscrWithInfin}(X, x_0)$.

(24) For all sets X, x_0, x such that $x \in X$ and $x \neq x_0$ holds $\{x\}$ is an open subset of $\operatorname{DiscrWithInfin}(X, x_0)$.

(25) For all sets X, x_0 such that X is infinite and for every subset U of $\operatorname{DiscrWithInfin}(X, x_0)$ such that $U = \{x_0\}$ holds U is not open.

(26) For all sets X, x_0 and for every subset A of $\operatorname{DiscrWithInfin}(X, x_0)$ such that A is finite holds $\overline{A} = A$.

(27) Let T be a non empty topological space and A be a subset of T. Suppose A is not closed. Let a be a point of T. If $A \cup \{a\}$ is closed, then $\overline{A} = A \cup \{a\}$.

(28) For all sets X, x_0 such that $x_0 \in X$ and for every subset A of $\operatorname{DiscrWithInfin}(X, x_0)$ such that A is infinite holds $\overline{A} = A \cup \{x_0\}$.

(29) For all sets X, x_0 and for every subset A of $\operatorname{DiscrWithInfin}(X, x_0)$ such that A^{c} is finite holds $\operatorname{Int} A = A$.

(30) For all sets X, x_0 such that $x_0 \in X$ and for every subset A of $\operatorname{DiscrWithInfin}(X, x_0)$ such that A^{c} is infinite holds $\operatorname{Int} A = A \setminus \{x_0\}$.

(31) For all sets X, x_0 there exists a basis B_0 of $\mathrm{DiscrWithInfin}(X, x_0)$ such that $B_0 = (\mathrm{SmallestPartition}(X) \setminus \{\{x_0\}\}) \cup \{F^{\mathrm{c}}; F$ ranges over subsets of X: F is finite$\}$.

In the sequel Z denotes an infinite set.

The following proposition is true

(32) $\overline{\overline{\mathrm{Fin}\, Z}} = \overline{\overline{Z}}$.

In the sequel F is a subset of Z.

One can prove the following propositions:

(33) $\overline{\overline{\{F^{\mathrm{c}} : F \text{ is finite}\}}} = \overline{\overline{Z}}$.

(34) Let X be an infinite set, x_0 be a set, and B_0 be a basis of $\mathrm{DiscrWithInfin}(X, x_0)$. If $B_0 = (\mathrm{SmallestPartition}(X) \setminus \{\{x_0\}\}) \cup \{F^{\mathrm{c}}; F$ ranges over subsets of X: F is finite$\}$, then $\overline{\overline{B_0}} = \overline{\overline{X}}$.

(35) For every infinite set X and for every set x_0 and for every basis B of $\mathrm{DiscrWithInfin}(X, x_0)$ holds $\overline{\overline{X}} \leq \overline{\overline{B}}$.

(36) For every infinite set X and for every set x_0 holds $\mathrm{weight}\, \mathrm{DiscrWithInfin}(X, x_0) = \overline{\overline{X}}$.

(37) Let X be a non empty set and x_0 be a set. Then there exists a prebasis B_0 of $\mathrm{DiscrWithInfin}(X, x_0)$ such that $B_0 = (\mathrm{SmallestPartition}(X) \setminus \{\{x_0\}\}) \cup \{\{x\}^{\mathrm{c}} : x$ ranges over elements of $X\}$.

5. Exercises

Next we state four propositions:

(38) Let T be a topological space, F be a family of subsets of T, and I be a non empty family of subsets of F. Suppose that for every set G such that $G \in I$ holds $F \setminus G$ is finite. Then $\overline{\bigcup F} = \bigcup \mathrm{clf}\, F \cup \bigcap \{\overline{\bigcup G}; G$ ranges over families of subsets of T: $G \in I\}$.

(39) Let T be a topological space and F be a family of subsets of T. Then $\overline{\bigcup F} = \bigcup \mathrm{clf}\, F \cup \bigcap \{\overline{\bigcup G}; G$ ranges over families of subsets of T: $G \subseteq F \wedge F \setminus G$ is finite$\}$.

(40) Let X be a set and O be a family of subsets of 2^X. Suppose that for every family o of subsets of X such that $o \in O$ holds $\langle X, o \rangle$ is a topological space. Then there exists a family B of subsets of X such that

(i) $B = \mathrm{Intersect}(O)$,

(ii) $\langle X, B \rangle$ is a topological space,

(iii) for every family o of subsets of X such that $o \in O$ holds $\langle X, o \rangle$ is a topological extension of $\langle X, B \rangle$, and

(iv) for every topological space T such that the carrier of $T = X$ and for every family o of subsets of X such that $o \in O$ holds $\langle X, o \rangle$ is a topological extension of T holds $\langle X, B \rangle$ is a topological extension of T.

(41) Let X be a set and O be a family of subsets of 2^X. Then there exists a family B of subsets of X such that

(i) $B = \mathrm{UniCl}(\mathrm{FinMeetCl}(\bigcup O))$,

(ii) $\langle X, B\rangle$ is a topological space,

(iii) for every family o of subsets of X such that $o \in O$ holds $\langle X, B\rangle$ is a topological extension of $\langle X, o\rangle$, and

(iv) for every topological space T such that the carrier of $T = X$ and for every family o of subsets of X such that $o \in O$ holds T is a topological extension of $\langle X, o\rangle$ holds T is a topological extension of $\langle X, B\rangle$.

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Received December 10, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

On Constructing Topological Spaces and Sorgenfrey Line[1]

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Summary. We continue Mizar formalization of General Topology according to the book [19] by Engelking. In the article the formalization of Section 1.2 is almost completed. Namely, we formalize theorems on introduction of topologies by bases, neighborhood systems, closed sets, closure operator, and interior operator. The Sorgenfrey line is defined by a basis. It is proved that the weight of it is continuum. Other techniques are used to demonstrate introduction of discrete and anti-discrete topologies.

MML Identifier: `TOPGEN_3`.

The terminology and notation used in this paper have been introduced in the following articles: [39], [17], [45], [30], [18], [38], [43], [46], [47], [15], [16], [10], [6], [7], [3], [5], [13], [20], [2], [8], [1], [14], [4], [42], [27], [44], [23], [37], [35], [11], [25], [24], [32], [33], [34], [29], [40], [26], [31], [48], [21], [22], [36], [12], [41], [28], and [9].

1. Introducing Topology

In this paper a is a set.

Let X be a set and let B be a family of subsets of X. We say that B is point-filtered if and only if:

(Def. 1) For all sets x, U_1, U_2 such that $U_1 \in B$ and $U_2 \in B$ and $x \in U_1 \cap U_2$ there exists a subset U of X such that $U \in B$ and $x \in U$ and $U \subseteq U_1 \cap U_2$.

We now state four propositions:

[1]This work has been partially supported by the KBN grant 4 T11C 039 24.

ISSN 1426–2630

(1) Let X be a set and B be a family of subsets of X. Then B is covering if and only if for every set x such that $x \in X$ there exists a subset U of X such that $U \in B$ and $x \in U$.

(2) Let X be a set and B be a family of subsets of X. Suppose B is point-filtered and covering. Let T be a topological structure. Suppose the carrier of $T = X$ and the topology of $T = \mathrm{UniCl}(B)$. Then T is a topological space and B is a basis of T.

(3) Let X be a set and B be a non-empty many sorted set indexed by X. Suppose that

(i) $\mathrm{rng}\, B \subseteq 2^{2^X}$,

(ii) for all sets x, U such that $x \in X$ and $U \in B(x)$ holds $x \in U$,

(iii) for all sets x, y, U such that $x \in U$ and $U \in B(y)$ and $y \in X$ there exists a set V such that $V \in B(x)$ and $V \subseteq U$, and

(iv) for all sets x, U_1, U_2 such that $x \in X$ and $U_1 \in B(x)$ and $U_2 \in B(x)$ there exists a set U such that $U \in B(x)$ and $U \subseteq U_1 \cap U_2$.

Then there exists a family P of subsets of X such that

(v) $P = \bigcup B$, and

(vi) for every topological structure T such that the carrier of $T = X$ and the topology of $T = \mathrm{UniCl}(P)$ holds T is a topological space and for every non empty topological space T' such that $T' = T$ holds B is a neighborhood system of T'.

(4) Let X be a set and F be a family of subsets of X. Suppose that

(i) $\emptyset \in F$,

(ii) $X \in F$,

(iii) for all sets A, B such that $A \in F$ and $B \in F$ holds $A \cup B \in F$, and

(iv) for every family G of subsets of X such that $G \subseteq F$ holds $\mathrm{Intersect}(G) \in F$.

Let T be a topological structure. Suppose the carrier of $T = X$ and the topology of $T = F^{\mathrm{c}}$. Then T is a topological space and for every subset A of T holds A is closed iff $A \in F$.

The scheme *TopDefByClosedPred* deals with a set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a strict topological space T such that the carrier of $T = \mathcal{A}$ and for every subset A of T holds A is closed iff $\mathcal{P}[A]$

provided the following conditions are satisfied:

- $\mathcal{P}[\emptyset]$ and $\mathcal{P}[\mathcal{A}]$,
- For all sets A, B such that $\mathcal{P}[A]$ and $\mathcal{P}[B]$ holds $\mathcal{P}[A \cup B]$, and
- For every family G of subsets of $\mathcal{A}$ such that for every set A such that $A \in G$ holds $\mathcal{P}[A]$ holds $\mathcal{P}[\mathrm{Intersect}(G)]$.

We now state two propositions:

(5) Let T_1, T_2 be topological spaces. Suppose that for every set A holds A is an open subset of T_1 iff A is an open subset of T_2. Then the topological structure of T_1 = the topological structure of T_2.

(6) Let T_1, T_2 be topological spaces. Suppose that for every set A holds A is a closed subset of T_1 iff A is a closed subset of T_2. Then the topological structure of T_1 = the topological structure of T_2.

Let X, Y be sets and let r be a subset of $[\!: X, 2^Y :\!]$. Then $\operatorname{rng} r$ is a family of subsets of Y.

We now state the proposition

(7) Let X be a set and c be a function from 2^X into 2^X. Suppose that
(i) $c(\emptyset) = \emptyset$,
(ii) for every subset A of X holds $A \subseteq c(A)$,
(iii) for all subsets A, B of X holds $c(A \cup B) = c(A) \cup c(B)$, and
(iv) for every subset A of X holds $c(c(A)) = c(A)$.

Let T be a topological structure. Suppose the carrier of $T = X$ and the topology of $T = (\operatorname{rng} c)^{\mathrm{c}}$. Then T is a topological space and for every subset A of T holds $\overline{A} = c(A)$.

The scheme *TopDefByClosureOp* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a strict topological space T such that the carrier of $T = \mathcal{A}$ and for every subset A of T holds $\overline{A} = \mathcal{F}(A)$

provided the parameters satisfy the following conditions:

- $\mathcal{F}(\emptyset) = \emptyset$,
- For every subset A of $\mathcal{A}$ holds $A \subseteq \mathcal{F}(A)$ and $\mathcal{F}(A) \subseteq \mathcal{A}$,
- For all subsets A, B of $\mathcal{A}$ holds $\mathcal{F}(A \cup B) = \mathcal{F}(A) \cup \mathcal{F}(B)$, and
- For every subset A of $\mathcal{A}$ holds $\mathcal{F}(\mathcal{F}(A)) = \mathcal{F}(A)$.

We now state two propositions:

(8) Let T_1, T_2 be topological spaces. Suppose that
(i) the carrier of T_1 = the carrier of T_2, and
(ii) for every subset A_1 of T_1 and for every subset A_2 of T_2 such that $A_1 = A_2$ holds $\overline{A_1} = \overline{A_2}$.

Then the topology of T_1 = the topology of T_2.

(9) Let X be a set and i be a function from 2^X into 2^X. Suppose that
(i) $i(X) = X$,
(ii) for every subset A of X holds $i(A) \subseteq A$,
(iii) for all subsets A, B of X holds $i(A \cap B) = i(A) \cap i(B)$, and
(iv) for every subset A of X holds $i(i(A)) = i(A)$.

Let T be a topological structure. Suppose the carrier of $T = X$ and the topology of $T = \operatorname{rng} i$. Then T is a topological space and for every subset A of T holds $\operatorname{Int} A = i(A)$.

The scheme *TopDefByInteriorOp* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a strict topological space T such that the carrier of $T = \mathcal{A}$ and for every subset A of T holds $\operatorname{Int} A = \mathcal{F}(A)$

provided the following conditions are met:

- $\mathcal{F}(\mathcal{A}) = \mathcal{A}$,
- For every subset A of $\mathcal{A}$ holds $\mathcal{F}(A) \subseteq A$,
- For all subsets A, B of $\mathcal{A}$ holds $\mathcal{F}(A \cap B) = \mathcal{F}(A) \cap \mathcal{F}(B)$, and
- For every subset A of $\mathcal{A}$ holds $\mathcal{F}(\mathcal{F}(A)) = \mathcal{F}(A)$.

Next we state the proposition

(10) Let T_1, T_2 be topological spaces. Suppose that
(i) the carrier of $T_1 =$ the carrier of T_2, and
(ii) for every subset A_1 of T_1 and for every subset A_2 of T_2 such that $A_1 = A_2$ holds $\operatorname{Int} A_1 = \operatorname{Int} A_2$.
Then the topology of $T_1 =$ the topology of T_2.

2. Sorgenfrey Line

In the sequel x, q denote elements of $\mathbb{R}$.

The strict non empty topological space Sorgenfrey line is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of Sorgenfrey line $= \mathbb{R}$, and
(ii) there exists a family B of subsets of $\mathbb{R}$ such that the topology of Sorgenfrey line $= \operatorname{UniCl}(B)$ and $B = \{[x, q[\colon x < q \ \wedge \ q \text{ is rational}\}$.

We now state several propositions:

(11) For all real numbers x, y holds $[x, y[$ is an open subset of Sorgenfrey line.

(12) For all real numbers x, y holds $]x, y[$ is an open subset of Sorgenfrey line.

(13) For every real number x holds $]-\infty, x[$ is an open subset of Sorgenfrey line.

(14) For every real number x holds $]x, +\infty[$ is an open subset of Sorgenfrey line.

(15) For every real number x holds $[x, +\infty[$ is an open subset of Sorgenfrey line.

(16) $\overline{\overline{\mathbb{Z}}} = \aleph_0$.

(17) $\overline{\overline{\mathbb{Q}}} = \aleph_0$.

(18) Let A be a set. Suppose A is mutually-disjoint and for every a such that $a \in A$ there exist real numbers x, y such that $x < y$ and $]x, y[\subseteq a$. Then A is countable.

Let X be a set and let x be a real number. We say that x is local minimum of X if and only if:

(Def. 3) $x \in X$ and there exists a real number y such that $y < x$ and $]y, x[$ misses X.

In the sequel x is an element of $\mathbb{R}$.

One can prove the following proposition

(19) For every subset U of $\mathbb{R}$ holds $\{x : x$ is local minimum of $U\}$ is countable.

One can check the following observations:

* $\mathbb{Z}$ is infinite,
* $\mathbb{Q}$ is infinite, and
* $\mathbb{R}$ is infinite.

Let X be an infinite set. Note that 2^X is infinite.

Let M be an aleph. Observe that 2^M is infinite.

The infinite cardinal number $\mathfrak{c}$ is defined by:

(Def. 4) $\mathfrak{c} = \overline{\overline{\mathbb{R}}}$.

In the sequel x, q are elements of $\mathbb{R}$.

One can prove the following proposition

(20) $\overline{\overline{\{[x, q[: x < q \ \wedge \ q \text{ is rational}\}}} = \mathfrak{c}$.

Let X be an infinite set. Observe that there exists a subset of X which is infinite.

Let X be a set and let r be a real number. The functor X-powers(r) yields a function from $\mathbb{N}$ into $\mathbb{R}$ and is defined by:

(Def. 5) For every natural number i holds $i \in X$ and $(X\text{-powers}(r))(i) = r^i$ or $i \notin X$ and $(X\text{-powers}(r))(i) = 0$.

Next we state the proposition

(21) For every set X and for every real number r such that $0 < r$ and $r < 1$ holds X-powers(r) is summable.

In the sequel r denotes a real number, X denotes a set, and n denotes an element of $\mathbb{N}$.

The following propositions are true:

(22) If $0 < r$ and $r < 1$, then $\sum((r^\kappa)_{\kappa \in \mathbb{N}} \uparrow n) = \frac{r^n}{1-r}$.

(23) $\sum(((\frac{1}{2})^\kappa)_{\kappa \in \mathbb{N}} \uparrow (n+1)) = (\frac{1}{2})^n$.

(24) If $0 < r$ and $r < 1$, then $\sum(X\text{-powers}(r)) \leq \sum((r^\kappa)_{\kappa \in \mathbb{N}})$.

(25) $\sum((X\text{-powers}(\frac{1}{2})) \uparrow (n+1)) \leq (\frac{1}{2})^n$.

(26) For every infinite subset X of $\mathbb{N}$ and for every natural number i holds $(\sum_{\alpha=0}^{\kappa}(X\text{-powers}(\frac{1}{2}))(\alpha))_{\kappa \in \mathbb{N}}(i) < \sum(X\text{-powers}(\frac{1}{2}))$.

(27) For all infinite subsets X, Y of $\mathbb{N}$ such that $\sum(X\text{-powers}(\frac{1}{2})) = \sum(Y\text{-powers}(\frac{1}{2}))$ holds $X = Y$.

(28) If X is countable, then $\operatorname{Fin} X$ is countable.

(29) $\mathfrak{c} = 2^{\aleph_0}$.

(30) $\aleph_0 < \mathfrak{c}$.

(31) For every family A of subsets of $\mathbb{R}$ such that $\overline{\overline{A}} < \mathfrak{c}$ holds $\overline{\overline{\{x : \bigvee_{U : \mathrm{set}} (U \in \mathrm{UniCl}(A) \ \wedge \ x \text{ is local minimum of } U)\}}} < \mathfrak{c}$.

(32) Let X be a family of subsets of $\mathbb{R}$. Suppose $\overline{\overline{X}} < \mathfrak{c}$. Then there exists a real number x and there exists a rational number q such that $x < q$ and $[x, q[\notin \mathrm{UniCl}(X)$.

(33) weight Sorgenfrey line $= \mathfrak{c}$.

3. Example: closed = finite

Let X be a set. The functor $\mathrm{ClFinTop}(X)$ yielding a strict topological space is defined by:

(Def. 6) The carrier of $\mathrm{ClFinTop}(X) = X$ and for every subset F of $\mathrm{ClFinTop}(X)$ holds F is closed iff F is finite or $F = X$.

The following two propositions are true:

(34) For every set X and for every subset A of $\mathrm{ClFinTop}(X)$ holds A is open iff $A = \emptyset$ or A^{c} is finite.

(35) For every infinite set X and for every subset A of X such that A is finite holds A^{c} is infinite.

Let X be a non empty set. Note that $\mathrm{ClFinTop}(X)$ is non empty.

The following proposition is true

(36) For every infinite set X and for all non empty open subsets U, V of $\mathrm{ClFinTop}(X)$ holds U meets V.

4. Example: one point closure

Let X, x_0 be sets. The functor x_0-PointClTop(X) yielding a strict topological space is defined as follows:

(Def. 7) The carrier of x_0-PointClTop(X) $=$ X and for every subset A of x_0-PointClTop(X) holds $\overline{A} = (A = \emptyset \rightarrow A, A \cup \{x_0\} \cap X)$.

Let X be a non empty set and let x_0 be a set. One can check that x_0-PointClTop(X) is non empty.

We now state two propositions:

(37) For every non empty set X and for every element x_0 of X and for every non empty subset A of x_0-PointClTop(X) holds $\overline{A} = A \cup \{x_0\}$.

(38) Let X be a non empty set, x_0 be an element of X, and A be a non empty subset of x_0-PointClTop(X). Then A is closed if and only if $x_0 \in A$.

Let X be a non empty set and let A be a proper subset of X. Observe that A^{c} is non empty.

The following propositions are true:

(39) Let X be a non empty set, x_0 be an element of X, and A be a proper subset of x_0-PointClTop(X). Then A is open if and only if $x_0 \notin A$.

(40) For all sets X, x_0, x such that $x_0 \in X$ holds $\{x\}$ is a closed subset of x_0-PointClTop(X) iff $x = x_0$.

(41) For all sets X, x_0, x such that $\{x_0\} \subset X$ holds $\{x\}$ is an open subset of x_0-PointClTop(X) iff $x \in X$ and $x \neq x_0$.

5. Example: discrete on subset

Let X, X_0 be sets. The functor X_0-DiscreteTop(X) yielding a strict topological space is defined as follows:

(Def. 8) The carrier of X_0-DiscreteTop(X) $= X$ and for every subset A of X_0-DiscreteTop(X) holds $\mathrm{Int}\, A = (A = X \to A, A \cap X_0)$.

Let X be a non empty set and let X_0 be a set. One can check that X_0-DiscreteTop(X) is non empty.

We now state several propositions:

(42) For every non empty set X and for every set X_0 and for every proper subset A of X_0-DiscreteTop(X) holds $\mathrm{Int}\, A = A \cap X_0$.

(43) For every non empty set X and for every set X_0 and for every proper subset A of X_0-DiscreteTop(X) holds A is open iff $A \subseteq X_0$.

(44) For every set X and for every subset X_0 of X holds the topology of X_0-DiscreteTop$(X) = \{X\} \cup 2^{X_0}$.

(45) For every set X holds ADTS$(X) = \emptyset$-DiscreteTop(X).

(46) For every set X holds $\{X\}_{\mathrm{top}} = X$-DiscreteTop(X).

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Received December 10, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

On the Real Valued Functions[1]

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MML Identifier: PARTFUN3.

The terminology and notation used here have been introduced in the following articles: [9], [12], [1], [10], [11], [13], [14], [2], [3], [4], [6], [5], [8], and [7].

Let r be a real number. Observe that $\frac{r}{r}$ is non negative.

Let r be a real number. Observe that $r \cdot r$ is non negative and $r \cdot r^{-1}$ is non negative.

Let r be a non negative real number. One can check that $\sqrt{r}$ is non negative.

Let r be a positive real number. Observe that $\sqrt{r}$ is positive.

We now state the proposition

(1) For every function f and for every set A such that f is one-to-one and $A \subseteq \mathrm{dom}(f^{-1})$ holds $f^\circ (f^{-1})^\circ A = A$.

Let f be a non-empty function. One can verify that $f^{-1}(\{0\})$ is empty.

Let R be a binary relation. We say that R is positive yielding if and only if:

(Def. 1) For every real number r such that $r \in \mathrm{rng}\, R$ holds $0 < r$.

We say that R is negative yielding if and only if:

(Def. 2) For every real number r such that $r \in \mathrm{rng}\, R$ holds $0 > r$.

We say that R is non-positive yielding if and only if:

(Def. 3) For every real number r such that $r \in \mathrm{rng}\, R$ holds $0 \geq r$.

We say that R is non-negative yielding if and only if:

(Def. 4) For every real number r such that $r \in \mathrm{rng}\, R$ holds $0 \leq r$.

Let X be a set and let r be a positive real number. Observe that $X \longmapsto r$ is positive yielding.

Let X be a set and let r be a negative real number. Note that $X \longmapsto r$ is negative yielding.

[1]The paper was written during the author's post-doctoral fellowship granted by the Shinshu University, Japan.

ISSN 1426–2630

Let X be a set and let r be a non positive real number. Note that $X \longmapsto r$ is non-positive yielding.

Let X be a set and let r be a non negative real number. Observe that $X \longmapsto r$ is non-negative yielding.

Let X be a non empty set. Note that $X \longmapsto 0$ is non non-empty.

Let us observe that every binary relation which is positive yielding is also non-negative yielding and non-empty and every binary relation which is negative yielding is also non-positive yielding and non-empty.

Let X be a set. One can check that there exists a function from X into $\mathbb{R}$ which is negative yielding and there exists a function from X into $\mathbb{R}$ which is positive yielding.

One can check that there exists a function which is non-empty and real-yielding.

We now state two propositions:

(2) For every non-empty real-yielding function f holds $\mathrm{dom}(\frac{1}{f}) = \mathrm{dom}\, f$.

(3) Let X be a non empty set, f be a partial function from X to $\mathbb{R}$, and g be a non-empty partial function from X to $\mathbb{R}$. Then $\mathrm{dom}(\frac{f}{g}) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$.

Let X be a set and let f, g be non-positive yielding partial functions from X to $\mathbb{R}$. Observe that $f + g$ is non-positive yielding.

Let X be a set and let f, g be non-negative yielding partial functions from X to $\mathbb{R}$. Note that $f + g$ is non-negative yielding.

Let X be a set, let f be a positive yielding partial function from X to $\mathbb{R}$, and let g be a non-negative yielding partial function from X to $\mathbb{R}$. Observe that $f + g$ is positive yielding.

Let X be a set, let f be a non-negative yielding partial function from X to $\mathbb{R}$, and let g be a positive yielding partial function from X to $\mathbb{R}$. One can verify that $f + g$ is positive yielding.

Let X be a set, let f be a non-positive yielding partial function from X to $\mathbb{R}$, and let g be a negative yielding partial function from X to $\mathbb{R}$. Note that $f + g$ is negative yielding.

Let X be a set, let f be a negative yielding partial function from X to $\mathbb{R}$, and let g be a non-positive yielding partial function from X to $\mathbb{R}$. Note that $f + g$ is negative yielding.

Let X be a set, let f be a non-negative yielding partial function from X to $\mathbb{R}$, and let g be a non-positive yielding partial function from X to $\mathbb{R}$. Note that $f - g$ is non-negative yielding.

Let X be a set, let f be a non-positive yielding partial function from X to $\mathbb{R}$, and let g be a non-negative yielding partial function from X to $\mathbb{R}$. Observe that $f - g$ is non-positive yielding.

Let X be a set, let f be a positive yielding partial function from X to $\mathbb{R}$, and let g be a non-positive yielding partial function from X to $\mathbb{R}$. One can check that $f - g$ is positive yielding.

Let X be a set, let f be a non-positive yielding partial function from X to $\mathbb{R}$, and let g be a positive yielding partial function from X to $\mathbb{R}$. Observe that $f - g$ is negative yielding.

Let X be a set, let f be a negative yielding partial function from X to $\mathbb{R}$, and let g be a non-negative yielding partial function from X to $\mathbb{R}$. Note that $f - g$ is negative yielding.

Let X be a set, let f be a non-negative yielding partial function from X to $\mathbb{R}$, and let g be a negative yielding partial function from X to $\mathbb{R}$. One can verify that $f - g$ is positive yielding.

Let X be a set and let f, g be non-positive yielding partial functions from X to $\mathbb{R}$. One can verify that $f\,g$ is non-negative yielding.

Let X be a set and let f, g be non-negative yielding partial functions from X to $\mathbb{R}$. Note that $f\,g$ is non-negative yielding.

Let X be a set, let f be a non-positive yielding partial function from X to $\mathbb{R}$, and let g be a non-negative yielding partial function from X to $\mathbb{R}$. One can verify that $f\,g$ is non-positive yielding.

Let X be a set, let f be a non-negative yielding partial function from X to $\mathbb{R}$, and let g be a non-positive yielding partial function from X to $\mathbb{R}$. Observe that $f\,g$ is non-positive yielding.

Let X be a set, let f be a positive yielding partial function from X to $\mathbb{R}$, and let g be a negative yielding partial function from X to $\mathbb{R}$. Note that $f\,g$ is negative yielding.

Let X be a set, let f be a negative yielding partial function from X to $\mathbb{R}$, and let g be a positive yielding partial function from X to $\mathbb{R}$. One can verify that $f\,g$ is negative yielding.

Let X be a set and let f, g be positive yielding partial functions from X to $\mathbb{R}$. One can verify that $f\,g$ is positive yielding.

Let X be a set and let f, g be negative yielding partial functions from X to $\mathbb{R}$. One can check that $f\,g$ is positive yielding.

Let X be a set and let f, g be non-empty partial functions from X to $\mathbb{R}$. Observe that $f\,g$ is non-empty.

Let X be a set and let f be a partial function from X to $\mathbb{R}$. Note that $f\,f$ is non-negative yielding.

Let X be a set, let r be a non positive real number, and let f be a non-positive yielding partial function from X to $\mathbb{R}$. One can verify that $r\,f$ is non-negative yielding.

Let X be a set, let r be a non negative real number, and let f be a non-negative yielding partial function from X to $\mathbb{R}$. Observe that $r\,f$ is non-negative yielding.

Let X be a set, let r be a non positive real number, and let f be a non-negative yielding partial function from X to $\mathbb{R}$. One can verify that $r\,f$ is non-positive yielding.

Let X be a set, let r be a non negative real number, and let f be a non-positive yielding partial function from X to $\mathbb{R}$. One can verify that $r\,f$ is non-positive yielding.

Let X be a set, let r be a positive real number, and let f be a negative yielding partial function from X to $\mathbb{R}$. Note that $r\,f$ is negative yielding.

Let X be a set, let r be a negative real number, and let f be a positive yielding partial function from X to $\mathbb{R}$. One can check that $r\,f$ is negative yielding.

Let X be a set, let r be a positive real number, and let f be a positive yielding partial function from X to $\mathbb{R}$. One can verify that $r\,f$ is positive yielding.

Let X be a set, let r be a negative real number, and let f be a negative yielding partial function from X to $\mathbb{R}$. Note that $r\,f$ is positive yielding.

Let X be a set, let r be a non zero real number, and let f be a non-empty partial function from X to $\mathbb{R}$. Observe that $r\,f$ is non-empty.

Let X be a non empty set and let f, g be non-positive yielding partial functions from X to $\mathbb{R}$. Note that $\frac{f}{g}$ is non-negative yielding.

Let X be a non empty set and let f, g be non-negative yielding partial functions from X to $\mathbb{R}$. Observe that $\frac{f}{g}$ is non-negative yielding.

Let X be a non empty set, let f be a non-positive yielding partial function from X to $\mathbb{R}$, and let g be a non-negative yielding partial function from X to $\mathbb{R}$. Note that $\frac{f}{g}$ is non-positive yielding.

Let X be a non empty set, let f be a non-negative yielding partial function from X to $\mathbb{R}$, and let g be a non-positive yielding partial function from X to $\mathbb{R}$. Note that $\frac{f}{g}$ is non-positive yielding.

Let X be a non empty set, let f be a positive yielding partial function from X to $\mathbb{R}$, and let g be a negative yielding partial function from X to $\mathbb{R}$. One can verify that $\frac{f}{g}$ is negative yielding.

Let X be a non empty set, let f be a negative yielding partial function from X to $\mathbb{R}$, and let g be a positive yielding partial function from X to $\mathbb{R}$. Observe that $\frac{f}{g}$ is negative yielding.

Let X be a non empty set and let f, g be positive yielding partial functions from X to $\mathbb{R}$. One can check that $\frac{f}{g}$ is positive yielding.

Let X be a non empty set and let f, g be negative yielding partial functions from X to $\mathbb{R}$. One can check that $\frac{f}{g}$ is positive yielding.

Let X be a non empty set and let f be a partial function from X to $\mathbb{R}$. Observe that $\frac{f}{f}$ is non-negative yielding.

Let X be a non empty set and let f, g be non-empty partial functions from X to $\mathbb{R}$. One can verify that $\frac{f}{g}$ is non-empty.

Let X be a set and let f be a non-positive yielding function from X into $\mathbb{R}$. One can verify that $\mathrm{Inv}\,f$ is non-positive yielding.

Let X be a set and let f be a non-negative yielding function from X into $\mathbb{R}$. Observe that $\operatorname{Inv} f$ is non-negative yielding.

Let X be a set and let f be a positive yielding function from X into $\mathbb{R}$. One can verify that $\operatorname{Inv} f$ is positive yielding.

Let X be a set and let f be a negative yielding function from X into $\mathbb{R}$. Note that $\operatorname{Inv} f$ is negative yielding.

Let X be a set and let f be a non-empty function from X into $\mathbb{R}$. Note that $\operatorname{Inv} f$ is non-empty.

Let X be a set and let f be a non-empty function from X into $\mathbb{R}$. One can verify that $-f$ is non-empty.

Let X be a set and let f be a non-positive yielding function from X into $\mathbb{R}$. Observe that $-f$ is non-negative yielding.

Let X be a set and let f be a non-negative yielding function from X into $\mathbb{R}$. One can check that $-f$ is non-positive yielding.

Let X be a set and let f be a positive yielding function from X into $\mathbb{R}$. Observe that $-f$ is negative yielding.

Let X be a set and let f be a negative yielding function from X into $\mathbb{R}$. Observe that $-f$ is positive yielding.

Let X be a set and let f be a function from X into $\mathbb{R}$. Note that $|f|$ is non-negative yielding.

Let X be a set and let f be a non-empty function from X into $\mathbb{R}$. One can check that $|f|$ is positive yielding.

Let X be a non empty set and let f be a non-positive yielding function from X into $\mathbb{R}$. Observe that $\frac{1}{f}$ is non-positive yielding.

Let X be a non empty set and let f be a non-negative yielding function from X into $\mathbb{R}$. Note that $\frac{1}{f}$ is non-negative yielding.

Let X be a non empty set and let f be a positive yielding function from X into $\mathbb{R}$. One can check that $\frac{1}{f}$ is positive yielding.

Let X be a non empty set and let f be a negative yielding function from X into $\mathbb{R}$. Note that $\frac{1}{f}$ is negative yielding.

Let X be a non empty set and let f be a non-empty function from X into $\mathbb{R}$. One can check that $\frac{1}{f}$ is non-empty.

Let f be a real-yielding function. The functor $\sqrt{f}$ yields a function and is defined as follows:

(Def. 5) $\operatorname{dom}\sqrt{f} = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom}\sqrt{f}$ holds $\sqrt{f}(x) = \sqrt{f(x)}$.

Let f be a real-yielding function. Observe that $\sqrt{f}$ is real-yielding.

Let C be a set, let D be a real-membered set, and let f be a partial function from C to D. Then $\sqrt{f}$ is a partial function from C to $\mathbb{R}$.

Let X be a set and let f be a non-negative yielding function from X into $\mathbb{R}$. One can check that $\sqrt{f}$ is non-negative yielding.

Let X be a set and let f be a positive yielding function from X into $\mathbb{R}$. Note that $\sqrt{f}$ is positive yielding.

Let X be a set and let f, g be functions from X into $\mathbb{R}$. Then $f+g$ is a function from X into $\mathbb{R}$. Then $f-g$ is a function from X into $\mathbb{R}$. Then $f\,g$ is a function from X into $\mathbb{R}$.

Let X be a set and let f be a function from X into $\mathbb{R}$. Then $-f$ is a function from X into $\mathbb{R}$. Then $|f|$ is a function from X into $\mathbb{R}$. Then $\sqrt{f}$ is a function from X into $\mathbb{R}$.

Let X be a set, let f be a function from X into $\mathbb{R}$, and let r be a real number. Then $r\,f$ is a function from X into $\mathbb{R}$.

Let X be a set and let f be a non-empty function from X into $\mathbb{R}$. Then $\frac{1}{f}$ is a function from X into $\mathbb{R}$.

Let X be a non empty set, let f be a function from X into $\mathbb{R}$, and let g be a non-empty function from X into $\mathbb{R}$. Then $\frac{f}{g}$ is a function from X into $\mathbb{R}$.

In the sequel T is a non empty topological space, f, g are continuous real maps of T, and r is a real number.

Let us consider T, f, g. Then $f+g$ is a continuous real map of T. Then $f-g$ is a continuous real map of T. Then $f\,g$ is a continuous real map of T.

Let us consider T, f. Then $-f$ is a continuous real map of T.

Let us consider T, f. Then $|f|$ is a continuous real map of T.

Let us consider T. Observe that there exists a real map of T which is positive yielding and continuous and there exists a real map of T which is negative yielding and continuous.

Let us consider T and let f be a non-negative yielding continuous real map of T. Then $\sqrt{f}$ is a continuous real map of T.

Let us consider T, f, r. Then $r\,f$ is a continuous real map of T.

Let us consider T and let f be a non-empty continuous real map of T. Then $\frac{1}{f}$ is a continuous real map of T.

Let us consider T, f and let g be a non-empty continuous real map of T. Then $\frac{f}{g}$ is a continuous real map of T.

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Received December 10, 2004

FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Formalization of Ortholattices via Orthoposets

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Summary. There are two approaches to lattices used in the Mizar Mathematical Library: on the one hand, these structures are based on the set with two binary operations (with an equational characterization as in [17]). On the other hand, we may look at them as at relational structures (posets – see [12]). As the main result of this article we can state that the Mizar formalization enables us to use both approaches simultaneously (Section 3). This is especially useful because most of lemmas on ortholattices in the literature are stated in the poset setting, so we cannot use equational theorem provers in a straightforward way. We give also short equational characterization of lattices via four axioms (as it was done in [7] with the help of the Otter prover). Some corresponding results about ortholattices are also formalized.

MML Identifier: `ROBBINS3`.

The notation and terminology used here have been introduced in the following papers: [11], [4], [14], [15], [3], [16], [1], [17], [12], [13], [2], [10], [9], [5], [8], and [6].

1. Another Short Axiomatization of Lattices

Let L be a non empty $\sqcup$-semi lattice structure. We say that L is quasi-join-associative if and only if:

(Def. 1) For all elements x, y, z of L holds $x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$.

Let L be a non empty $\sqcap$-semi lattice structure. We say that L is quasi-meet-associative if and only if:

[1]This work has been partially supported by the KBN grant 4 T11C 039 24 and the FP6 IST grant TYPES No. 510996.

ISSN 1426–2630

(Def. 2) For all elements x, y, z of L holds $x \sqcap (y \sqcap z) = y \sqcap (x \sqcap z)$.

Let L be a non empty lattice structure. We say that L is quasi-meet-absorbing if and only if:

(Def. 3) For all elements x, y of L holds $x \sqcup (x \sqcap y) = x$.

One can prove the following propositions:

(1) Let L be a non empty lattice structure. Suppose L is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing. Then L is meet-idempotent and join-idempotent.

(2) Let L be a non empty lattice structure. Suppose L is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing. Then L is meet-commutative and join-commutative.

(3) Let L be a non empty lattice structure. Suppose L is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing. Then L is meet-absorbing.

(4) Let L be a non empty lattice structure. Suppose L is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing. Then L is meet-associative and join-associative.

(5) Let L be a non empty lattice structure. Then L is lattice-like if and only if L is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing.

One can verify that every non empty lattice structure which is lattice-like is also quasi-meet-associative, quasi-join-associative, meet-absorbing, and join-absorbing and every non empty lattice structure which is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing is also lattice-like.

2. Orthoposets

Let us note that every PartialOrdered non empty orthorelational structure which is OrderInvolutive is also Dneg.

The following propositions are true:

(6) For every Dneg non empty orthorelational structure L and for every element x of L holds $(x^{\rm c})^{\rm c} = x$.

(7) Let O be an OrderInvolutive PartialOrdered non empty orthorelational structure and x, y be elements of O. If $x \leq y$, then $y^{\rm c} \leq x^{\rm c}$.

Let us note that there exists a PreOrthoPoset which is strict and has g.l.b.'s and l.u.b.'s.

Let L be a non empty $\sqcup$-semi lattice structure and let x, y be elements of L. We introduce $x \underline{\sqcup} y$ as a synonym of $x \sqcup y$.

Let L be a non empty $\sqcap$-semi lattice structure and let x, y be elements of L. We introduce $x \overline{\sqcap} y$ as a synonym of $x \sqcap y$.

Let L be a non empty relational structure and let x, y be elements of L. We introduce $x \sqcap_{\leq} y$ as a synonym of $x \sqcap y$. We introduce $x \sqcup_{\leq} y$ as a synonym of $x \sqcup y$.

3. Merging Relational Structures and Lattice Structures Together

We introduce $\sqcup$-relational semilattice structures which are extensions of $\sqcup$-semi lattice structure and relational structure and are systems

⟨ a carrier, a join operation, an internal relation ⟩,

where the carrier is a set, the join operation is a binary operation on the carrier, and the internal relation is a binary relation on the carrier.

We introduce $\sqcap$-relational semilattice structures which are extensions of $\sqcap$-semi lattice structure and relational structure and are systems

⟨ a carrier, a meet operation, an internal relation ⟩,

where the carrier is a set, the meet operation is a binary operation on the carrier, and the internal relation is a binary relation on the carrier.

We introduce relational lattice structures which are extensions of $\sqcap$-relational semilattice structure, $\sqcup$-relational semilattice structure, and lattice structure and are systems

⟨ a carrier, a join operation, a meet operation, an internal relation ⟩,

where the carrier is a set, the join operation and the meet operation are binary operations on the carrier, and the internal relation is a binary relation on the carrier.

The relational lattice structure TrivLattRelStr is defined as follows:

(Def. 4) $\text{TrivLattRelStr} = \langle \{\emptyset\}, \text{op}_2, \text{op}_2, \text{id}_{\{\emptyset\}} \rangle$.

Let us note that TrivLattRelStr is non empty and trivial.

One can check the following observations:

* there exists a $\sqcup$-relational semilattice structure which is non empty,
* there exists a $\sqcap$-relational semilattice structure which is non empty, and
* there exists a relational lattice structure which is non empty.

One can prove the following proposition

(8) Let R be a non empty relational structure. Suppose that

(i) the internal relation of R is reflexive in the carrier of R, and

(ii) the internal relation of R is antisymmetric and transitive.

Then R is reflexive, antisymmetric, and transitive.

Let us mention that TrivLattRelStr is reflexive.

Let us note that there exists a relational lattice structure which is antisymmetric, reflexive, and transitive and has l.u.b.'s and g.l.b.'s.

One can verify that TrivLattRelStr is quasi-meet-absorbing.

One can verify that there exists a non empty relational lattice structure which is lattice-like.

Let L be a lattice. Then LattRel(L) is an order in the carrier of L.

4. Binary Approach to Ortholattices

We consider relational ortholattice structures as extensions of relational lattice structure, ortholattice structure, and orthorelational structure as systems

⟨ a carrier, a join operation, a meet operation, an internal relation, a complement operation ⟩,

where the carrier is a set, the join operation and the meet operation are binary operations on the carrier, the internal relation is a binary relation on the carrier, and the complement operation is a unary operation on the carrier.

The relational ortholattice structure TrivCLRelStr is defined by:

(Def. 5) $\mathrm{TrivCLRelStr} = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_2, \mathrm{id}_{\{\emptyset\}}, \mathrm{op}_1 \rangle$.

Let L be a non empty ComplStr. We say that L is involutive if and only if:

(Def. 6) For every element x of L holds $(x^{\mathrm{c}})^{\mathrm{c}} = x$.

Let L be a non empty complemented lattice structure. We say that L has top if and only if:

(Def. 7) For all elements x, y of L holds $x \underline{\sqcup} x^{\mathrm{c}} = y \underline{\sqcup} y^{\mathrm{c}}$.

One can verify that TrivOrtLat is involutive and has top.

One can verify that TrivCLRelStr is non empty and trivial.

One can check that TrivCLRelStr is reflexive.

Let us observe that TrivCLRelStr is involutive and has top.

Let us observe that there exists a non empty ortholattice structure which is involutive, de Morgan, and lattice-like and has top.

An ortholattice is an involutive de Morgan lattice-like non empty ortholattice structure with top.

5. Lemmas

Next we state a number of propositions:

(9) Let K, L be non empty lattice structures. Suppose the lattice structure of K = the lattice structure of L and K is join-commutative. Then L is join-commutative.

(10) Let K, L be non empty lattice structures. Suppose the lattice structure of K = the lattice structure of L and K is meet-commutative. Then L is meet-commutative.

(11) Let K, L be non empty lattice structures. Suppose the lattice structure of K = the lattice structure of L and K is join-associative. Then L is join-associative.

(12) Let K, L be non empty lattice structures. Suppose the lattice structure of K = the lattice structure of L and K is meet-associative. Then L is meet-associative.

(13) Let K, L be non empty lattice structures. Suppose the lattice structure of K = the lattice structure of L and K is join-absorbing. Then L is join-absorbing.

(14) Let K, L be non empty lattice structures. Suppose the lattice structure of K = the lattice structure of L and K is meet-absorbing. Then L is meet-absorbing.

(15) Let K, L be non empty lattice structures. Suppose the lattice structure of K = the lattice structure of L and K is lattice-like. Then L is lattice-like.

(16) Let L_1, L_2 be non empty $\sqcup$-semi lattice structures. Suppose the upper semilattice structure of L_1 = the upper semilattice structure of L_2. Let a_1, b_1 be elements of L_1 and a_2, b_2 be elements of L_2. If $a_1 = a_2$ and $b_1 = b_2$, then $a_1 \sqcup b_1 = a_2 \sqcup b_2$.

(17) Let L_1, L_2 be non empty $\sqcap$-semi lattice structures. Suppose the lower semilattice structure of L_1 = the lower semilattice structure of L_2. Let a_1, b_1 be elements of L_1 and a_2, b_2 be elements of L_2. If $a_1 = a_2$ and $b_1 = b_2$, then $a_1 \sqcap b_1 = a_2 \sqcap b_2$.

(18) Let K, L be non empty ComplStr, x be an element of K, and y be an element of L. Suppose the complement operation of K = the complement operation of L and $x = y$. Then $x^{\mathrm{c}} = y^{\mathrm{c}}$.

(19) Let K, L be non empty complemented lattice structures such that the complemented lattice structure of K = the complemented lattice structure of L and K has top. Then L has top.

(20) Let K, L be non empty ortholattice structures. Suppose the ortholattice structure of K = the ortholattice structure of L and K is de Morgan. Then L is de Morgan.

(21) Let K, L be non empty ortholattice structures. Suppose the ortholattice structure of K = the ortholattice structure of L and K is involutive. Then L is involutive.

6. Structure Extensions

Let R be a relational structure. A relational lattice structure is said to be a relational augmentation of R if:

(Def. 8) The relational structure of it = the relational structure of R.

Let R be a lattice structure. A relational lattice structure is said to be a lattice augmentation of R if:

(Def. 9) The lattice structure of it = the lattice structure of R.

Let L be a non empty lattice structure. Observe that every lattice augmentation of L is non empty.

Let L be a meet-associative non empty lattice structure. Note that every lattice augmentation of L is meet-associative.

Let L be a join-associative non empty lattice structure. One can check that every lattice augmentation of L is join-associative.

Let L be a meet-commutative non empty lattice structure. One can verify that every lattice augmentation of L is meet-commutative.

Let L be a join-commutative non empty lattice structure. Note that every lattice augmentation of L is join-commutative.

Let L be a join-absorbing non empty lattice structure. One can check that every lattice augmentation of L is join-absorbing.

Let L be a meet-absorbing non empty lattice structure. Observe that every lattice augmentation of L is meet-absorbing.

Let L be a non empty $\sqcup$-relational semilattice structure. We say that L is naturally sup-generated if and only if:

(Def. 10) For all elements x, y of L holds $x \leq y$ iff $x \underline{\sqcup} y = y$.

Let L be a non empty $\sqcap$-relational semilattice structure. We say that L is naturally inf-generated if and only if:

(Def. 11) For all elements x, y of L holds $x \leq y$ iff $x \overline{\sqcap} y = x$.

Let L be a lattice. One can verify that there exists a lattice augmentation of L which is naturally sup-generated, naturally inf-generated, and lattice-like.

Let us mention that there exists a relational lattice structure which is trivial, non empty, and reflexive.

Let us mention that there exists a relational ortholattice structure which is trivial, non empty, and reflexive.

Let us note that there exists a orthorelational structure which is trivial, non empty, and reflexive.

One can check that every non empty ortholattice structure which is trivial is also involutive, de Morgan, and well-complemented and has top.

Let us note that every non empty reflexive orthorelational structure which is trivial is also OrderInvolutive, Pure, and PartialOrdered.

One can check that every non empty reflexive relational lattice structure which is trivial is also naturally sup-generated and naturally inf-generated.

Let us note that there exists a non empty relational ortholattice structure which is naturally sup-generated, naturally inf-generated, de Morgan, lattice-like, OrderInvolutive, Pure, and PartialOrdered and has g.l.b.'s and l.u.b.'s.

Let us observe that there exists a non empty relational lattice structure which is naturally sup-generated, naturally inf-generated, and lattice-like and has g.l.b.'s and l.u.b.'s.

Next we state two propositions:

(22) Let L be a naturally sup-generated non empty relational lattice structure and x, y be elements of L. Then $x \leq y$ if and only if $x \sqsubseteq y$.

(23) Let L be a naturally sup-generated lattice-like non empty relational lattice structure. Then the relational structure of $L = \mathrm{Poset}(L)$.

One can check that every non empty relational lattice structure which is naturally sup-generated and lattice-like has also g.l.b.'s and l.u.b.'s.

7. Extending Orthocomplemented Lattice Structure

Let R be an ortholattice structure. A relational ortholattice structure is said to be a complemented lattice augmentation of R if:

(Def. 12) The ortholattice structure of it = the ortholattice structure of R.

Let L be a non empty ortholattice structure. One can check that every complemented lattice augmentation of L is non empty.

Let L be a meet-associative non empty ortholattice structure. Note that every complemented lattice augmentation of L is meet-associative.

Let L be a join-associative non empty ortholattice structure. One can verify that every complemented lattice augmentation of L is join-associative.

Let L be a meet-commutative non empty ortholattice structure. Observe that every complemented lattice augmentation of L is meet-commutative.

Let L be a join-commutative non empty ortholattice structure. Note that every complemented lattice augmentation of L is join-commutative.

Let L be a meet-absorbing non empty ortholattice structure. Note that every complemented lattice augmentation of L is meet-absorbing.

Let L be a join-absorbing non empty ortholattice structure. Note that every complemented lattice augmentation of L is join-absorbing.

Let L be a non empty ortholattice structure with top. Observe that every complemented lattice augmentation of L has top.

Let L be a non empty ortholattice. Note that there exists a complemented lattice augmentation of L which is naturally sup-generated, naturally inf-generated, and lattice-like.

Let us observe that there exists a non empty relational ortholattice structure which is involutive, de Morgan, lattice-like, naturally sup-generated, and well-complemented and has top.

Next we state the proposition

(24) Let L be a PartialOrdered non empty orthorelational structure with g.l.b.'s and l.u.b.'s and x, y be elements of L. If $x \leq y$, then $y = x \sqcup_{\leq} y$ and $x = x \sqcap_{\leq} y$.

Let L be a meet-commutative non empty $\sqcap$-semi lattice structure and let a, b be elements of L. Let us observe that the functor $a \overline{\sqcap} b$ is commutative.

Let L be a join-commutative non empty $\sqcup$-semi lattice structure and let a, b be elements of L. Let us notice that the functor $a \underline{\sqcup} b$ is commutative.

One can check that every non empty relational lattice structure which is meet-absorbing, join-absorbing, meet-commutative, and naturally sup-generated is also reflexive.

Let us observe that every non empty relational lattice structure which is join-associative and naturally sup-generated is also transitive.

One can check that every non empty relational lattice structure which is join-commutative and naturally sup-generated is also antisymmetric.

Next we state three propositions:

(25) Let L be a naturally sup-generated lattice-like non empty relational ortholattice structure with g.l.b.'s and l.u.b.'s and x, y be elements of L. Then $x \sqcup_{\leq} y = x \underline{\sqcup} y$.

(26) Let L be a naturally sup-generated lattice-like non empty relational ortholattice structure with g.l.b.'s and l.u.b.'s and x, y be elements of L. Then $x \sqcap_{\leq} y = x \overline{\sqcap} y$.

(27) Every naturally sup-generated naturally inf-generated lattice-like Order-Involutive PartialOrdered non empty relational ortholattice structure with g.l.b.'s and l.u.b.'s is de Morgan.

Let L be an ortholattice. Note that every complemented lattice augmentation of L is involutive.

Let L be an ortholattice. Observe that every complemented lattice augmentation of L is de Morgan.

The following two propositions are true:

(28) Let L be a non empty relational ortholattice structure. Suppose L is involutive, de Morgan, lattice-like, and naturally sup-generated and has top. Then L is Orthocomplemented and PartialOrdered.

(29) For every ortholattice L holds every naturally sup-generated complemented lattice augmentation of L is Orthocomplemented.

Let L be an ortholattice. Observe that every naturally sup-generated complemented lattice augmentation of L is Orthocomplemented.

We now state the proposition

(30) Let L be a non empty ortholattice structure. Suppose L is Boolean, well-complemented, and lattice-like. Then L is an ortholattice.

Let us observe that every non empty ortholattice structure which is Boolean, well-complemented, and lattice-like is also involutive and de Morgan and has top.

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Received December 28, 2004

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