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Uniform Continuity of Functions on Normed Complex Linear Spaces

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The papers [19], [22], [1], [17], [10], [23], [4], [24], [5], [13], [20], [21], [18], [3], [12], [11], [2], [25], [16], [6], [8], [15], [7], [14], and [9] provide the notation and terminology for this paper.

1. Uniform Continuity of Functions on Real and Complex Normed Linear Spaces

For simplicity, we follow the rules: X, X_1 denote sets, r, s denote real numbers, z denotes a complex number, R_1 denotes a real normed space, and C_1, C_2, C_3 denote complex normed spaces.

Let X be a set, let C_2, C_3 be complex normed spaces, and let f be a partial function from C_2 to C_3. We say that f is uniformly continuous on X if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of C_2 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

Let X be a set, let R_1 be a real normed space, let C_1 be a complex normed space, and let f be a partial function from C_1 to R_1. We say that f is uniformly continuous on X if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

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Let X be a set, let R_1 be a real normed space, let C_1 be a complex normed space, and let f be a partial function from R_1 to C_1. We say that f is uniformly continuous on X if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of R_1 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

Let X be a set, let C_1 be a complex normed space, and let f be a partial function from the carrier of C_1 to $\mathbb{C}$. We say that f is uniformly continuous on X if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $|f_{x_1} - f_{x_2}| < r$.

Let X be a set, let C_1 be a complex normed space, and let f be a partial function from the carrier of C_1 to $\mathbb{R}$. We say that f is uniformly continuous on X if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $|f_{x_1} - f_{x_2}| < r$.

Let X be a set, let R_1 be a real normed space, and let f be a partial function from the carrier of R_1 to $\mathbb{C}$. We say that f is uniformly continuous on X if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all points x_1, x_2 of R_1 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $|f_{x_1} - f_{x_2}| < r$.

Next we state a number of propositions:

(1) Let f be a partial function from C_2 to C_3. Suppose f is uniformly continuous on X and $X_1 \subseteq X$. Then f is uniformly continuous on X_1.

(2) Let f be a partial function from C_1 to R_1. Suppose f is uniformly continuous on X and $X_1 \subseteq X$. Then f is uniformly continuous on X_1.

(3) Let f be a partial function from R_1 to C_1. Suppose f is uniformly continuous on X and $X_1 \subseteq X$. Then f is uniformly continuous on X_1.

(4) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 + f_2$ is uniformly continuous on $X \cap X_1$.

(5) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 + f_2$ is

uniformly continuous on $X \cap X_1$.

(6) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 + f_2$ is uniformly continuous on $X \cap X_1$.

(7) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

(8) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

(9) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1. Then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

(10) Let f be a partial function from C_2 to C_3. If f is uniformly continuous on X, then $z\, f$ is uniformly continuous on X.

(11) Let f be a partial function from C_1 to R_1. If f is uniformly continuous on X, then $r\, f$ is uniformly continuous on X.

(12) Let f be a partial function from R_1 to C_1. If f is uniformly continuous on X, then $z\, f$ is uniformly continuous on X.

(13) Let f be a partial function from C_2 to C_3. If f is uniformly continuous on X, then $-f$ is uniformly continuous on X.

(14) Let f be a partial function from C_1 to R_1. If f is uniformly continuous on X, then $-f$ is uniformly continuous on X.

(15) Let f be a partial function from R_1 to C_1. If f is uniformly continuous on X, then $-f$ is uniformly continuous on X.

(16) Let f be a partial function from C_2 to C_3. If f is uniformly continuous on X, then $\|f\|$ is uniformly continuous on X.

(17) Let f be a partial function from C_1 to R_1. If f is uniformly continuous on X, then $\|f\|$ is uniformly continuous on X.

(18) Let f be a partial function from R_1 to C_1. If f is uniformly continuous on X, then $\|f\|$ is uniformly continuous on X.

(19) For every partial function f from C_2 to C_3 such that f is uniformly continuous on X holds f is continuous on X.

(20) For every partial function f from C_1 to R_1 such that f is uniformly continuous on X holds f is continuous on X.

(21) For every partial function f from R_1 to C_1 such that f is uniformly continuous on X holds f is continuous on X.

(22) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$. If f is uniformly continuous on X, then f is continuous on X.

(23) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. If f is uniformly continuous on X, then f is continuous on X.

(24) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$. If f is uniformly continuous on X, then f is continuous on X.

(25) For every partial function f from C_2 to C_3 such that f is Lipschitzian on X holds f is uniformly continuous on X.

(26) For every partial function f from C_1 to R_1 such that f is Lipschitzian on X holds f is uniformly continuous on X.

(27) For every partial function f from R_1 to C_1 such that f is Lipschitzian on X holds f is uniformly continuous on X.

(28) Let f be a partial function from C_2 to C_3 and Y be a subset of C_2. Suppose Y is compact and f is continuous on Y. Then f is uniformly continuous on Y.

(29) Let f be a partial function from C_1 to R_1 and Y be a subset of C_1. Suppose Y is compact and f is continuous on Y. Then f is uniformly continuous on Y.

(30) Let f be a partial function from R_1 to C_1 and Y be a subset of R_1. Suppose Y is compact and f is continuous on Y. Then f is uniformly continuous on Y.

(31) Let f be a partial function from C_2 to C_3 and Y be a subset of C_2. Suppose $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y. Then $f^{\circ}Y$ is compact.

(32) Let f be a partial function from C_1 to R_1 and Y be a subset of C_1. Suppose $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y. Then $f^{\circ}Y$ is compact.

(33) Let f be a partial function from R_1 to C_1 and Y be a subset of R_1. Suppose $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y. Then $f^{\circ}Y$ is compact.

(34) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$ and Y be a subset of C_1. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y. Then there exist points x_1, x_2 of C_1 such that $x_1 \in Y$ and $x_2 \in Y$ and $f_{x_1} = \sup(f^{\circ}Y)$ and $f_{x_2} = \inf(f^{\circ}Y)$.

(35) Let f be a partial function from C_2 to C_3. If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is uniformly continuous on X.

(36) Let f be a partial function from C_1 to R_1. If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is uniformly continuous on X.

(37) Let f be a partial function from R_1 to C_1. If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is uniformly continuous on X.

2. Contraction Mapping Principle on Normed Complex Linear Spaces

Let M be a complex Banach space. A function from the carrier of M into the carrier of M is said to be a contraction of M if:

(Def. 7) There exists a real number L such that $0 < L$ and $L < 1$ and for all points x, y of M holds $\|\mathrm{it}(x) - \mathrm{it}(y)\| \leq L \cdot \|x - y\|$.

One can prove the following four propositions:

(38) For every complex normed space X and for all points x, y of X holds $\|x - y\| > 0$ iff $x \neq y$.

(39) For every complex normed space X and for all points x, y of X holds $\|x - y\| = \|y - x\|$.

(40) Let X be a complex Banach space and f be a function from X into X. Suppose f is a contraction of X. Then there exists a point x_3 of X such that $f(x_3) = x_3$ and for every point x of X such that $f(x) = x$ holds $x_3 = x$.

(41) Let X be a complex Banach space and f be a function from X into X. Given a natural number n_0 such that f^{n_0} is a contraction of X. Then there exists a point x_3 of X such that $f(x_3) = x_3$ and for every point x of X such that $f(x) = x$ holds $x_3 = x$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(**4**):485–492, 1996.
[3] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Noboru Endou. Algebra of complex vector valued functions. *Formalized Mathematics*, 12(**3**):397–401, 2004.
[7] Noboru Endou. Complex Banach space of bounded linear operators. *Formalized Mathematics*, 12(**2**):201–209, 2004.
[8] Noboru Endou. Complex linear space and complex normed space. *Formalized Mathematics*, 12(**2**):93–102, 2004.
[9] Noboru Endou. Continuous functions on real and complex normed linear spaces. *Formalized Mathematics*, 12(**3**):403–419, 2004.
[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[11] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[12] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[13] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.
[14] Takaya Nishiyama, Artur Korniłowicz, and Yasunari Shidama. The uniform continuity of functions on normed linear spaces. *Formalized Mathematics*, 12(**3**):277–279, 2004.

[15] Takaya Nishiyama, Keiji Ohkubo, and Yasunari Shidama. The continuous functions on normed linear spaces. *Formalized Mathematics*, 12(**3**):269–275, 2004.
[16] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[17] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[18] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[19] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[20] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[21] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[24] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[25] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(**2**):171–175, 1992.

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