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Introduction to Real Linear Topological Spaces[1]

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The terminology and notation used in this paper are introduced in the following articles: [20], [7], [23], [10], [15], [19], [1], [4], [24], [5], [6], [3], [13], [18], [17], [25], [9], [16], [8], [14], [2], [21], [22], [12], and [11].

1. Preliminaries

In this paper X is a non empty RLS structure and r, s, t are real numbers.

Let us note that there exists a real number which is non zero.

We now state a number of propositions:

(2)[2] Let T be a non empty topological space, X be a non empty subset of T, and F_1 be a family of subsets of T. Suppose F_1 is a cover of X. Let x be a point of T. If $x \in X$, then there exists a subset W of T such that $x \in W$ and $W \in F_1$.

(4)[3] Let X be a non empty loop structure, M, N be subsets of X, and F be a family of subsets of X. If $F = \{x + N; x$ ranges over points of X: $x \in M\}$, then $M + N = \bigcup F$.

(5) Let X be an add-associative right zeroed right complementable non empty loop structure and M be a subset of X. Then $0_X + M = M$.

(6) Let X be an add-associative non empty loop structure, x, y be points of X, and M be a subset of X. Then $(x + y) + M = x + (y + M)$.

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[2]The proposition (1) has been removed.
[3]The proposition (3) has been removed.

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(7) Let X be an add-associative non empty loop structure, x be a point of X, and M, N be subsets of X. Then $(x+M)+N = x+(M+N)$.

(8) Let X be a non empty loop structure, M, N be subsets of X, and x be a point of X. If $M \subseteq N$, then $x+M \subseteq x+N$.

(9) Let X be a non empty real linear space, M be a subset of X, and x be a point of X. If $x \in M$, then $0_X \in -x+M$.

(10) For every non empty loop structure X and for all subsets M, N, V of X such that $M \subseteq N$ holds $M+V \subseteq N+V$.

(11) For every non empty loop structure X and for all subsets V_1, V_2, W_1, W_2 of X such that $V_1 \subseteq W_1$ and $V_2 \subseteq W_2$ holds $V_1+V_2 \subseteq W_1+W_2$.

(12) For every non empty real linear space X and for all subsets V_1, V_2 of X such that $0_X \in V_2$ holds $V_1 \subseteq V_1+V_2$.

(13) For every non empty real linear space X and for every real number r holds $r \cdot \{0_X\} = \{0_X\}$.

(14) Let X be a non empty real linear space, M be a subset of X, and r be a non zero real number. If $0_X \in r \cdot M$, then $0_X \in M$.

(15) Let X be a non empty real linear space, M, N be subsets of X, and r be a non zero real number. Then $(r \cdot M) \cap (r \cdot N) = r \cdot (M \cap N)$.

(16) Let X be a non empty topological space, x be a point of X, A be a neighbourhood of x, and B be a subset of X. If $A \subseteq B$, then B is a neighbourhood of x.

Let V be a non empty real linear space and let M be a subset of V. Let us observe that M is convex if and only if:

(Def. 1) For all points u, v of V and for every real number r such that $0 \leq r$ and $r \leq 1$ and $u \in M$ and $v \in M$ holds $r \cdot u + (1-r) \cdot v \in M$.

One can prove the following proposition

(17) Let X be a non empty real linear space, M be a convex subset of X, and r_1, r_2 be real numbers. If $0 \leq r_1$ and $0 \leq r_2$, then $r_1 \cdot M + r_2 \cdot M = (r_1+r_2) \cdot M$.

Let X be a non empty real linear space and let M be an empty subset of X. One can check that $\operatorname{conv} M$ is empty.

Next we state several propositions:

(18) For every non empty real linear space X and for every convex subset M of X holds $\operatorname{conv} M = M$.

(19) For every non empty real linear space X and for every subset M of X and for every real number r holds $r \cdot \operatorname{conv} M = \operatorname{conv} r \cdot M$.

(20) For every non empty real linear space X and for all subsets M_1, M_2 of X such that $M_1 \subseteq M_2$ holds Convex-Family $M_2 \subseteq$ Convex-Family M_1.

(21) For every non empty real linear space X and for all subsets M_1, M_2 of X such that $M_1 \subseteq M_2$ holds $\operatorname{conv} M_1 \subseteq \operatorname{conv} M_2$.

(22) Let X be a non empty real linear space, M be a convex subset of X, and r be a real number. If $0 \leq r$ and $r \leq 1$ and $0_X \in M$, then $r \cdot M \subseteq M$.

Let X be a non empty real linear space and let v, w be points of X. The functor $\mathcal{L}(v, w)$ yields a subset of X and is defined as follows:

(Def. 2) $\mathcal{L}(v, w) = \{(1 - r) \cdot v + r \cdot w : 0 \leq r \ \wedge \ r \leq 1\}$.

Let X be a non empty real linear space and let v, w be points of X. Note that $\mathcal{L}(v, w)$ is non empty and convex.

Next we state the proposition

(23) Let X be a non empty real linear space and M be a subset of X. Then M is convex if and only if for all points u, w of X such that $u \in M$ and $w \in M$ holds $\mathcal{L}(u, w) \subseteq M$.

Let V be a non empty RLS structure and let P be a family of subsets of V. We say that P is convex-membered if and only if:

(Def. 3) For every subset M of V such that $M \in P$ holds M is convex.

Let V be a non empty RLS structure. One can verify that there exists a family of subsets of V which is non empty and convex-membered.

We now state the proposition

(24) For every non empty RLS structure V and for every convex-membered family F of subsets of V holds $\bigcap F$ is convex.

Let X be a non empty RLS structure and let A be a subset of X. The functor $-A$ yielding a subset of X is defined by:

(Def. 4) $-A = (-1) \cdot A$.

One can prove the following proposition

(25) Let X be a non empty real linear space, M, N be subsets of X, and v be a point of X. Then $v + M$ meets N if and only if $v \in N + -M$.

Let X be a non empty RLS structure and let A be a subset of X. We say that A is symmetric if and only if:

(Def. 5) $A = -A$.

Let X be a non empty real linear space. Observe that there exists a subset of X which is non empty and symmetric.

One can prove the following proposition

(26) Let X be a non empty real linear space, A be a symmetric subset of X, and x be a point of X. If $x \in A$, then $-x \in A$.

Let X be a non empty RLS structure and let A be a subset of X. We say that A is circled if and only if:

(Def. 6) For every real number r such that $|r| \leq 1$ holds $r \cdot A \subseteq A$.

Let X be a non empty real linear space. Note that $\emptyset_X$ is circled.

We now state the proposition

(27) For every non empty real linear space X holds $\{0_X\}$ is circled.

Let X be a non empty real linear space. Observe that there exists a subset of X which is non empty and circled.

The following proposition is true

(28) For every non empty real linear space X and for every non empty circled subset B of X holds $0_X \in B$.

Let X be a non empty real linear space and let A, B be circled subsets of X. One can verify that $A + B$ is circled.

We now state the proposition

(29) Let X be a non empty real linear space, A be a circled subset of X, and r be a real number. If $|r| = 1$, then $r \cdot A = A$.

Let X be a non empty real linear space. One can check that every subset of X which is circled is also symmetric.

Let X be a non empty real linear space and let M be a circled subset of X. One can check that $\operatorname{conv} M$ is circled.

Let X be a non empty RLS structure and let F be a family of subsets of X. We say that F is circled-membered if and only if:

(Def. 7) For every subset V of X such that $V \in F$ holds V is circled.

Let V be a non empty real linear space. Note that there exists a family of subsets of V which is non empty and circled-membered.

The following two propositions are true:

(30) For every non empty real linear space X and for every circled-membered family F of subsets of X holds $\bigcup F$ is circled.

(31) For every non empty real linear space X and for every circled-membered family F of subsets of X holds $\bigcap F$ is circled.

2. Real Linear Topological Space

We introduce real linear topological structures which are extensions of RLS structure and topological structure and are systems

$\langle$ a carrier, a zero, an addition, an external multiplication, a topology $\rangle$,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, the external multiplication is a function from $[\!: \mathbb{R}, \text{the carrier} :\!]$ into the carrier, and the topology is a family of subsets of the carrier.

Let X be a non empty set, let O be an element of X, let F be a binary operation on X, let G be a function from $[\!: \mathbb{R}, X :\!]$ into X, and let T be a family of subsets of X. Observe that $\langle X, O, F, G, T \rangle$ is non empty.

Let us note that there exists a real linear topological structure which is strict and non empty.

Let X be a non empty real linear topological structure. We say that X is add-continuous if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let x_1, x_2 be points of X and V be a subset of X. Suppose V is open and $x_1 + x_2 \in V$. Then there exist subsets V_1, V_2 of X such that V_1 is open and V_2 is open and $x_1 \in V_1$ and $x_2 \in V_2$ and $V_1 + V_2 \subseteq V$.

We say that X is mult-continuous if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let a be a real number, x be a point of X, and V be a subset of X. Suppose V is open and $a \cdot x \in V$. Then there exists a positive real number r and there exists a subset W of X such that W is open and $x \in W$ and for every real number s such that $|s - a| < r$ holds $s \cdot W \subseteq V$.

Let us note that there exists a non empty real linear topological structure which is non empty, strict, add-continuous, mult-continuous, topological space-like, Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

A linear topological space is an add-continuous mult-continuous topological space-like Abelian add-associative right zeroed right complementable real linear space-like non empty real linear topological structure.

One can prove the following two propositions:

(32) Let X be a non empty linear topological space, x_1, x_2 be points of X, and V be a neighbourhood of $x_1 + x_2$. Then there exists a neighbourhood V_1 of x_1 and there exists a neighbourhood V_2 of x_2 such that $V_1 + V_2 \subseteq V$.

(33) Let X be a non empty linear topological space, a be a real number, x be a point of X, and V be a neighbourhood of $a \cdot x$. Then there exists a positive real number r and there exists a neighbourhood W of x such that for every real number s if $|s - a| < r$, then $s \cdot W \subseteq V$.

Let X be a non empty real linear topological structure and let a be a point of X. The functor $\mathrm{transl}(a, X)$ yields a map from X into X and is defined by:

(Def. 10) For every point x of X holds $(\mathrm{transl}(a, X))(x) = a + x$.

The following propositions are true:

(34) Let X be a non empty real linear topological structure, a be a point of X, and V be a subset of X. Then $(\mathrm{transl}(a, X))^{\circ}V = a + V$.

(35) For every non empty linear topological space X and for every point a of X holds $\mathrm{rng}\,\mathrm{transl}(a, X) = \Omega_X$.

(36) For every non empty linear topological space X and for every point a of X holds $(\mathrm{transl}(a, X))^{-1} = \mathrm{transl}(-a, X)$.

Let X be a non empty linear topological space and let a be a point of X. Note that $\mathrm{transl}(a, X)$ is homeomorphism.

Let X be a non empty linear topological space, let E be an open subset of X, and let x be a point of X. Note that $x + E$ is open.

Let X be a non empty linear topological space, let E be an open subset of X, and let x be a point of X. Observe that $x + E$ is open.

Let X be a non empty linear topological space, let E be an open subset of X, and let K be a subset of X. Observe that $K + E$ is open.

Let X be a non empty linear topological space, let D be a closed subset of X, and let x be a point of X. Note that $x + D$ is closed.

We now state several propositions:

(37) For every non empty linear topological space X and for all subsets V_1, V_2, V of X such that $V_1 + V_2 \subseteq V$ holds $\operatorname{Int} V_1 + \operatorname{Int} V_2 \subseteq \operatorname{Int} V$.

(38) For every non empty linear topological space X and for every point x of X and for every subset V of X holds $x + \operatorname{Int} V = \operatorname{Int}(x + V)$.

(39) For every non empty linear topological space X and for every point x of X and for every subset V of X holds $x + \overline{V} = \overline{x + V}$.

(40) Let X be a non empty linear topological space, x, v be points of X, and V be a neighbourhood of x. Then $v + V$ is a neighbourhood of $v + x$.

(41) Let X be a non empty linear topological space, x be a point of X, and V be a neighbourhood of x. Then $-x + V$ is a neighbourhood of 0_X.

Let X be a non empty real linear topological structure. A local base of X is a generalized basis of 0_X.

Let X be a non empty real linear topological structure. We say that X is locally-convex if and only if:

(Def. 11) There exists a local base of X which is convex-membered.

Let X be a non empty linear topological space and let E be a subset of X. We say that E is bounded if and only if:

(Def. 12) For every neighbourhood V of 0_X there exists s such that $s > 0$ and for every t such that $t > s$ holds $E \subseteq t \cdot V$.

Let X be a non empty linear topological space. Note that $\emptyset_X$ is bounded.

Let X be a non empty linear topological space. Observe that there exists a subset of X which is bounded.

The following propositions are true:

(42) For every non empty linear topological space X and for all bounded subsets V_1, V_2 of X holds $V_1 \cup V_2$ is bounded.

(43) Let X be a non empty linear topological space, P be a bounded subset of X, and Q be a subset of X. If $Q \subseteq P$, then Q is bounded.

(44) Let X be a non empty linear topological space and F be a family of subsets of X. Suppose F is finite and $F = \{P : P$ ranges over bounded subsets of $X\}$. Then $\bigcup F$ is bounded.

(45) Let X be a non empty linear topological space and P be a family of subsets of X. Suppose $P = \{U : U$ ranges over neighbourhoods of $0_X\}$. Then P is a local base of X.

(46) Let X be a non empty linear topological space, O be a local base of X, and P be a family of subsets of X. Suppose $P = \{a + U; a$ ranges over points of X, U ranges over subsets of X: $U \in O\}$. Then P is a generalized basis of X.

Let X be a non empty real linear topological structure and let r be a real number. The functor $r \bullet X$ yielding a map from X into X is defined as follows:

(Def. 13) For every point x of X holds $(r \bullet X)(x) = r \cdot x$.

The following propositions are true:

(47) Let X be a non empty real linear topological structure, V be a subset of X, and r be a non zero real number. Then $(r \bullet X)^{\circ}V = r \cdot V$.

(48) For every non empty linear topological space X and for every non zero real number r holds $\mathrm{rng}(r \bullet X) = \Omega_X$.

(49) For every non empty linear topological space X and for every non zero real number r holds $(r \bullet X)^{-1} = r^{-1} \bullet X$.

Let X be a non empty linear topological space and let r be a non zero real number. One can check that $r \bullet X$ is homeomorphism.

Next we state several propositions:

(50) Let X be a non empty linear topological space, V be an open subset of X, and r be a non zero real number. Then $r \cdot V$ is open.

(51) Let X be a non empty linear topological space, V be a closed subset of X, and r be a non zero real number. Then $r \cdot V$ is closed.

(52) Let X be a non empty linear topological space, V be a subset of X, and r be a non zero real number. Then $r \cdot \mathrm{Int}\, V = \mathrm{Int}(r \cdot V)$.

(53) Let X be a non empty linear topological space, A be a subset of X, and r be a non zero real number. Then $r \cdot \overline{A} = \overline{r \cdot A}$.

(54) Let X be a non empty linear topological space and A be a subset of X. If X is a T_1 space, then $0 \cdot \overline{A} = \overline{0 \cdot A}$.

(55) Let X be a non empty linear topological space, x be a point of X, V be a neighbourhood of x, and r be a non zero real number. Then $r \cdot V$ is a neighbourhood of $r \cdot x$.

(56) Let X be a non empty linear topological space, V be a neighbourhood of 0_X, and r be a non zero real number. Then $r \cdot V$ is a neighbourhood of 0_X.

Let X be a non empty linear topological space, let V be a bounded subset of X, and let r be a real number. Observe that $r \cdot V$ is bounded.

We now state four propositions:

(57) Let X be a non empty linear topological space and W be a neighbourhood of 0_X. Then there exists an open neighbourhood U of 0_X such that U is symmetric and $U + U \subseteq W$.

(58) Let X be a non empty linear topological space, K be a compact subset of X, and C be a closed subset of X. Suppose K misses C. Then there exists a neighbourhood V of 0_X such that $K + V$ misses $C + V$.

(59) Let X be a non empty linear topological space, B be a local base of X, and V be a neighbourhood of 0_X. Then there exists a neighbourhood W of 0_X such that $W \in B$ and $\overline{W} \subseteq V$.

(60) Let X be a non empty linear topological space and V be a neighbourhood of 0_X. Then there exists a neighbourhood W of 0_X such that $\overline{W} \subseteq V$.

Let us observe that every non empty linear topological space which is T_1 is also Hausdorff.

We now state three propositions:

(61) Let X be a non empty linear topological space and A be a subset of X. Then $\overline{A} = \bigcap\{A + V : V$ ranges over neighbourhoods of $0_X\}$.

(62) For every non empty linear topological space X and for all subsets A, B of X holds $\operatorname{Int} A + \operatorname{Int} B \subseteq \operatorname{Int}(A + B)$.

(63) For every non empty linear topological space X and for all subsets A, B of X holds $\overline{A} + \overline{B} \subseteq \overline{A + B}$.

Let X be a non empty linear topological space and let C be a convex subset of X. Note that $\overline{C}$ is convex.

Let X be a non empty linear topological space and let C be a convex subset of X. Note that $\operatorname{Int} C$ is convex.

Let X be a non empty linear topological space and let B be a circled subset of X. One can check that $\overline{B}$ is circled.

One can prove the following proposition

(64) Let X be a non empty linear topological space and B be a circled subset of X. If $0_X \in \operatorname{Int} B$, then $\operatorname{Int} B$ is circled.

Let X be a non empty linear topological space and let E be a bounded subset of X. Note that $\overline{E}$ is bounded.

The following propositions are true:

(65) Let X be a non empty linear topological space and U be a neighbourhood of 0_X. Then there exists a neighbourhood W of 0_X such that W is circled and $W \subseteq U$.

(66) Let X be a non empty linear topological space and U be a neighbourhood of 0_X. Suppose U is convex. Then there exists a neighbourhood W of 0_X such that W is circled and convex and $W \subseteq U$.

(67) For every non empty linear topological space X holds there exists a local base of X which is circled-membered.

(68) For every non empty linear topological space X such that X is locally-convex holds there exists a local base of X which is convex-membered.

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