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Formalization of Ortholattices via Orthoposets

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Summary. There are two approaches to lattices used in the Mizar Mathematical Library: on the one hand, these structures are based on the set with two binary operations (with an equational characterization as in [17]). On the other hand, we may look at them as at relational structures (posets – see [12]). As the main result of this article we can state that the Mizar formalization enables us to use both approaches simultaneously (Section 3). This is especially useful because most of lemmas on ortholattices in the literature are stated in the poset setting, so we cannot use equational theorem provers in a straightforward way. We give also short equational characterization of lattices via four axioms (as it was done in [7] with the help of the Otter prover). Some corresponding results about ortholattices are also formalized.

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The notation and terminology used here have been introduced in the following papers: [11], [4], [14], [15], [3], [16], [1], [17], [12], [13], [2], [10], [9], [5], [8], and [6].

1. Another Short Axiomatization of Lattices

Let L be a non empty $\sqcup$-semi lattice structure. We say that L is quasi-join-associative if and only if:

(Def. 1) For all elements x, y, z of L holds $x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$.

Let L be a non empty $\sqcap$-semi lattice structure. We say that L is quasi-meet-associative if and only if:

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(Def. 2) For all elements x, y, z of L holds $x \sqcap (y \sqcap z) = y \sqcap (x \sqcap z)$.

Let L be a non empty lattice structure. We say that L is quasi-meet-absorbing if and only if:

(Def. 3) For all elements x, y of L holds $x \sqcup (x \sqcap y) = x$.

One can prove the following propositions:

(1) Let L be a non empty lattice structure. Suppose L is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing. Then L is meet-idempotent and join-idempotent.

(2) Let L be a non empty lattice structure. Suppose L is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing. Then L is meet-commutative and join-commutative.

(3) Let L be a non empty lattice structure. Suppose L is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing. Then L is meet-absorbing.

(4) Let L be a non empty lattice structure. Suppose L is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing. Then L is meet-associative and join-associative.

(5) Let L be a non empty lattice structure. Then L is lattice-like if and only if L is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing.

One can verify that every non empty lattice structure which is lattice-like is also quasi-meet-associative, quasi-join-associative, meet-absorbing, and join-absorbing and every non empty lattice structure which is quasi-meet-associative, quasi-join-associative, quasi-meet-absorbing, and join-absorbing is also lattice-like.

2. Orthoposets

Let us note that every PartialOrdered non empty orthorelational structure which is OrderInvolutive is also Dneg.

The following propositions are true:

(6) For every Dneg non empty orthorelational structure L and for every element x of L holds $(x^{\mathrm{c}})^{\mathrm{c}} = x$.

(7) Let O be an OrderInvolutive PartialOrdered non empty orthorelational structure and x, y be elements of O. If $x \leq y$, then $y^{\mathrm{c}} \leq x^{\mathrm{c}}$.

Let us note that there exists a PreOrthoPoset which is strict and has g.l.b.'s and l.u.b.'s.

Let L be a non empty $\sqcup$-semi lattice structure and let x, y be elements of L. We introduce $x \underline{\sqcup} y$ as a synonym of $x \sqcup y$.

Let L be a non empty $\sqcap$-semi lattice structure and let x, y be elements of L. We introduce $x \overline{\sqcap} y$ as a synonym of $x \sqcap y$.

Let L be a non empty relational structure and let x, y be elements of L. We introduce $x \sqcap_{\leq} y$ as a synonym of $x \sqcap y$. We introduce $x \sqcup_{\leq} y$ as a synonym of $x \sqcup y$.

3. Merging Relational Structures and Lattice Structures Together

We introduce $\sqcup$-relational semilattice structures which are extensions of $\sqcup$-semi lattice structure and relational structure and are systems

⟨ a carrier, a join operation, an internal relation ⟩,

where the carrier is a set, the join operation is a binary operation on the carrier, and the internal relation is a binary relation on the carrier.

We introduce $\sqcap$-relational semilattice structures which are extensions of $\sqcap$-semi lattice structure and relational structure and are systems

⟨ a carrier, a meet operation, an internal relation ⟩,

where the carrier is a set, the meet operation is a binary operation on the carrier, and the internal relation is a binary relation on the carrier.

We introduce relational lattice structures which are extensions of $\sqcap$-relational semilattice structure, $\sqcup$-relational semilattice structure, and lattice structure and are systems

⟨ a carrier, a join operation, a meet operation, an internal relation ⟩,

where the carrier is a set, the join operation and the meet operation are binary operations on the carrier, and the internal relation is a binary relation on the carrier.

The relational lattice structure TrivLattRelStr is defined as follows:

(Def. 4) $\mathrm{TrivLattRelStr} = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_2, \mathrm{id}_{\{\emptyset\}} \rangle$.

Let us note that TrivLattRelStr is non empty and trivial.

One can check the following observations:

* there exists a $\sqcup$-relational semilattice structure which is non empty,
* there exists a $\sqcap$-relational semilattice structure which is non empty, and
* there exists a relational lattice structure which is non empty.

One can prove the following proposition

(8) Let R be a non empty relational structure. Suppose that
  (i) the internal relation of R is reflexive in the carrier of R, and
  (ii) the internal relation of R is antisymmetric and transitive.

  Then R is reflexive, antisymmetric, and transitive.

Let us mention that TrivLattRelStr is reflexive.

Let us note that there exists a relational lattice structure which is antisymmetric, reflexive, and transitive and has l.u.b.'s and g.l.b.'s.

One can verify that TrivLattRelStr is quasi-meet-absorbing.

One can verify that there exists a non empty relational lattice structure which is lattice-like.

Let L be a lattice. Then LattRel(L) is an order in the carrier of L.

4. Binary Approach to Ortholattices

We consider relational ortholattice structures as extensions of relational lattice structure, ortholattice structure, and orthorelational structure as systems

⟨ a carrier, a join operation, a meet operation, an internal relation, a complement operation ⟩,

where the carrier is a set, the join operation and the meet operation are binary operations on the carrier, the internal relation is a binary relation on the carrier, and the complement operation is a unary operation on the carrier.

The relational ortholattice structure TrivCLRelStr is defined by:

(Def. 5) $\mathrm{TrivCLRelStr} = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_2, \mathrm{id}_{\{\emptyset\}}, \mathrm{op}_1 \rangle$.

Let L be a non empty ComplStr. We say that L is involutive if and only if:

(Def. 6) For every element x of L holds $(x^{\mathrm{c}})^{\mathrm{c}} = x$.

Let L be a non empty complemented lattice structure. We say that L has top if and only if:

(Def. 7) For all elements x, y of L holds $x \sqcup x^{\mathrm{c}} = y \sqcup y^{\mathrm{c}}$.

One can verify that TrivOrtLat is involutive and has top.

One can verify that TrivCLRelStr is non empty and trivial.

One can check that TrivCLRelStr is reflexive.

Let us observe that TrivCLRelStr is involutive and has top.

Let us observe that there exists a non empty ortholattice structure which is involutive, de Morgan, and lattice-like and has top.

An ortholattice is an involutive de Morgan lattice-like non empty ortholattice structure with top.

5. Lemmas

Next we state a number of propositions:

(9) Let K, L be non empty lattice structures. Suppose the lattice structure of K = the lattice structure of L and K is join-commutative. Then L is join-commutative.

(10) Let K, L be non empty lattice structures. Suppose the lattice structure of K = the lattice structure of L and K is meet-commutative. Then L is meet-commutative.

(11) Let K, L be non empty lattice structures. Suppose the lattice structure of $K =$ the lattice structure of L and K is join-associative. Then L is join-associative.

(12) Let K, L be non empty lattice structures. Suppose the lattice structure of $K =$ the lattice structure of L and K is meet-associative. Then L is meet-associative.

(13) Let K, L be non empty lattice structures. Suppose the lattice structure of $K =$ the lattice structure of L and K is join-absorbing. Then L is join-absorbing.

(14) Let K, L be non empty lattice structures. Suppose the lattice structure of $K =$ the lattice structure of L and K is meet-absorbing. Then L is meet-absorbing.

(15) Let K, L be non empty lattice structures. Suppose the lattice structure of $K =$ the lattice structure of L and K is lattice-like. Then L is lattice-like.

(16) Let L_1, L_2 be non empty $\sqcup$-semi lattice structures. Suppose the upper semilattice structure of $L_1 =$ the upper semilattice structure of L_2. Let a_1, b_1 be elements of L_1 and a_2, b_2 be elements of L_2. If $a_1 = a_2$ and $b_1 = b_2$, then $a_1 \sqcup b_1 = a_2 \sqcup b_2$.

(17) Let L_1, L_2 be non empty $\sqcap$-semi lattice structures. Suppose the lower semilattice structure of $L_1 =$ the lower semilattice structure of L_2. Let a_1, b_1 be elements of L_1 and a_2, b_2 be elements of L_2. If $a_1 = a_2$ and $b_1 = b_2$, then $a_1 \sqcap b_1 = a_2 \sqcap b_2$.

(18) Let K, L be non empty ComplStr, x be an element of K, and y be an element of L. Suppose the complement operation of $K =$ the complement operation of L and $x = y$. Then $x^{\mathrm{c}} = y^{\mathrm{c}}$.

(19) Let K, L be non empty complemented lattice structures such that the complemented lattice structure of $K =$ the complemented lattice structure of L and K has top. Then L has top.

(20) Let K, L be non empty ortholattice structures. Suppose the ortholattice structure of $K =$ the ortholattice structure of L and K is de Morgan. Then L is de Morgan.

(21) Let K, L be non empty ortholattice structures. Suppose the ortholattice structure of $K =$ the ortholattice structure of L and K is involutive. Then L is involutive.

6. Structure Extensions

Let R be a relational structure. A relational lattice structure is said to be a relational augmentation of R if:

(Def. 8) The relational structure of it = the relational structure of R.

Let R be a lattice structure. A relational lattice structure is said to be a lattice augmentation of R if:

(Def. 9) The lattice structure of it = the lattice structure of R.

Let L be a non empty lattice structure. Observe that every lattice augmentation of L is non empty.

Let L be a meet-associative non empty lattice structure. Note that every lattice augmentation of L is meet-associative.

Let L be a join-associative non empty lattice structure. One can check that every lattice augmentation of L is join-associative.

Let L be a meet-commutative non empty lattice structure. One can verify that every lattice augmentation of L is meet-commutative.

Let L be a join-commutative non empty lattice structure. Note that every lattice augmentation of L is join-commutative.

Let L be a join-absorbing non empty lattice structure. One can check that every lattice augmentation of L is join-absorbing.

Let L be a meet-absorbing non empty lattice structure. Observe that every lattice augmentation of L is meet-absorbing.

Let L be a non empty $\sqcup$-relational semilattice structure. We say that L is naturally sup-generated if and only if:

(Def. 10) For all elements x, y of L holds $x \leq y$ iff $x \sqcup y = y$.

Let L be a non empty $\sqcap$-relational semilattice structure. We say that L is naturally inf-generated if and only if:

(Def. 11) For all elements x, y of L holds $x \leq y$ iff $x \sqcap y = x$.

Let L be a lattice. One can verify that there exists a lattice augmentation of L which is naturally sup-generated, naturally inf-generated, and lattice-like.

Let us mention that there exists a relational lattice structure which is trivial, non empty, and reflexive.

Let us mention that there exists a relational ortholattice structure which is trivial, non empty, and reflexive.

Let us note that there exists a orthorelational structure which is trivial, non empty, and reflexive.

One can check that every non empty ortholattice structure which is trivial is also involutive, de Morgan, and well-complemented and has top.

Let us note that every non empty reflexive orthorelational structure which is trivial is also OrderInvolutive, Pure, and PartialOrdered.

One can check that every non empty reflexive relational lattice structure which is trivial is also naturally sup-generated and naturally inf-generated.

Let us note that there exists a non empty relational ortholattice structure which is naturally sup-generated, naturally inf-generated, de Morgan, lattice-like, OrderInvolutive, Pure, and PartialOrdered and has g.l.b.'s and l.u.b.'s.

Let us observe that there exists a non empty relational lattice structure which is naturally sup-generated, naturally inf-generated, and lattice-like and has g.l.b.'s and l.u.b.'s.

Next we state two propositions:

(22) Let L be a naturally sup-generated non empty relational lattice structure and x, y be elements of L. Then $x \leq y$ if and only if $x \sqsubseteq y$.

(23) Let L be a naturally sup-generated lattice-like non empty relational lattice structure. Then the relational structure of $L = \mathrm{Poset}(L)$.

One can check that every non empty relational lattice structure which is naturally sup-generated and lattice-like has also g.l.b.'s and l.u.b.'s.

7. Extending Orthocomplemented Lattice Structure

Let R be an ortholattice structure. A relational ortholattice structure is said to be a complemented lattice augmentation of R if:

(Def. 12) The ortholattice structure of it = the ortholattice structure of R.

Let L be a non empty ortholattice structure. One can check that every complemented lattice augmentation of L is non empty.

Let L be a meet-associative non empty ortholattice structure. Note that every complemented lattice augmentation of L is meet-associative.

Let L be a join-associative non empty ortholattice structure. One can verify that every complemented lattice augmentation of L is join-associative.

Let L be a meet-commutative non empty ortholattice structure. Observe that every complemented lattice augmentation of L is meet-commutative.

Let L be a join-commutative non empty ortholattice structure. Note that every complemented lattice augmentation of L is join-commutative.

Let L be a meet-absorbing non empty ortholattice structure. Note that every complemented lattice augmentation of L is meet-absorbing.

Let L be a join-absorbing non empty ortholattice structure. Note that every complemented lattice augmentation of L is join-absorbing.

Let L be a non empty ortholattice structure with top. Observe that every complemented lattice augmentation of L has top.

Let L be a non empty ortholattice. Note that there exists a complemented lattice augmentation of L which is naturally sup-generated, naturally inf-generated, and lattice-like.

Let us observe that there exists a non empty relational ortholattice structure which is involutive, de Morgan, lattice-like, naturally sup-generated, and well-complemented and has top.

Next we state the proposition

(24) Let L be a PartialOrdered non empty orthorelational structure with g.l.b.'s and l.u.b.'s and x, y be elements of L. If $x \leq y$, then $y = x \sqcup_{\leq} y$ and $x = x \sqcap_{\leq} y$.

Let L be a meet-commutative non empty $\sqcap$-semi lattice structure and let a, b be elements of L. Let us observe that the functor $a \overline{\sqcap} b$ is commutative.

Let L be a join-commutative non empty $\sqcup$-semi lattice structure and let a, b be elements of L. Let us notice that the functor $a \underline{\sqcup} b$ is commutative.

One can check that every non empty relational lattice structure which is meet-absorbing, join-absorbing, meet-commutative, and naturally sup-generated is also reflexive.

Let us observe that every non empty relational lattice structure which is join-associative and naturally sup-generated is also transitive.

One can check that every non empty relational lattice structure which is join-commutative and naturally sup-generated is also antisymmetric.

Next we state three propositions:

(25) Let L be a naturally sup-generated lattice-like non empty relational ortholattice structure with g.l.b.'s and l.u.b.'s and x, y be elements of L. Then $x \sqcup_{\leq} y = x \underline{\sqcup} y$.

(26) Let L be a naturally sup-generated lattice-like non empty relational ortholattice structure with g.l.b.'s and l.u.b.'s and x, y be elements of L. Then $x \sqcap_{\leq} y = x \overline{\sqcap} y$.

(27) Every naturally sup-generated naturally inf-generated lattice-like Order-Involutive PartialOrdered non empty relational ortholattice structure with g.l.b.'s and l.u.b.'s is de Morgan.

Let L be an ortholattice. Note that every complemented lattice augmentation of L is involutive.

Let L be an ortholattice. Observe that every complemented lattice augmentation of L is de Morgan.

The following two propositions are true:

(28) Let L be a non empty relational ortholattice structure. Suppose L is involutive, de Morgan, lattice-like, and naturally sup-generated and has top. Then L is Orthocomplemented and PartialOrdered.

(29) For every ortholattice L holds every naturally sup-generated complemented lattice augmentation of L is Orthocomplemented.

Let L be an ortholattice. Observe that every naturally sup-generated complemented lattice augmentation of L is Orthocomplemented.

We now state the proposition

(30) Let L be a non empty ortholattice structure. Suppose L is Boolean, well-complemented, and lattice-like. Then L is an ortholattice.

Let us observe that every non empty ortholattice structure which is Boolean, well-complemented, and lattice-like is also involutive and de Morgan and has top.

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