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On the Boundary and Derivative of a Set[1]

Adam Grabowski
University of Białystok

Summary. This is the first Mizar article in a series aiming at a complete formalization of the textbook "General Topology" by Engelking [7]. We cover the first part of Section 1.3, by defining such notions as a derivative of a subset A of a topological space (usually denoted by A^{d}, but Der A in our notation), the derivative and the boundary of families of subsets, points of accumulation and isolated points. We also introduce dense-in-itself, perfect and scattered topological spaces and formulate the notion of the density of a space. Some basic properties are given as well as selected exercises from [7].

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The terminology and notation used in this paper are introduced in the following papers: [13], [15], [1], [2], [12], [3], [5], [10], [16], [9], [14], [4], [6], [8], and [11].

1. Preliminaries

Let T be a set, let A be a subset of T, and let B be a set. Then $A \setminus B$ is a subset of T.

The following three propositions are true:

(1) For every 1-sorted structure T and for all subsets A, B of T holds A meets B^{c} iff $A \setminus B \neq \emptyset$.

(2) For every 1-sorted structure T holds T is countable iff Ω_T is countable.

(3) For every 1-sorted structure T holds T is countable iff $\overline{\overline{\Omega_T}} \leq \aleph_0$.

Let T be a finite 1-sorted structure. Note that Ω_T is finite.

Let us note that every 1-sorted structure which is finite is also countable.

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Let us observe that there exists a 1-sorted structure which is countable and non empty and there exists a topological space which is countable and non empty.

Let T be a countable 1-sorted structure. Observe that Ω_T is countable.

Let us observe that there exists a topological space which is T_1 and non empty.

2. Boundary of a Subset

Next we state two propositions:

(4) For every topological structure T and for every subset A of T holds $A \cup \Omega_T = \Omega_T$.

(5) For every topological space T and for every subset A of T holds $\operatorname{Int} A = \overline{A^{\mathrm{c}}}^{\mathrm{c}}$.

Let T be a topological space and let F be a family of subsets of T. The functor $\operatorname{Fr} F$ yielding a family of subsets of T is defined by:

(Def. 1) For every subset A of T holds $A \in \operatorname{Fr} F$ iff there exists a subset B of T such that $A = \operatorname{Fr} B$ and $B \in F$.

The following propositions are true:

(6) For every topological space T and for every family F of subsets of T such that $F = \emptyset$ holds $\operatorname{Fr} F = \emptyset$.

(7) Let T be a topological space, F be a family of subsets of T, and A be a subset of T. If $F = \{A\}$, then $\operatorname{Fr} F = \{\operatorname{Fr} A\}$.

(8) For every topological space T and for all families F, G of subsets of T such that $F \subseteq G$ holds $\operatorname{Fr} F \subseteq \operatorname{Fr} G$.

(9) For every topological space T and for all families F, G of subsets of T holds $\operatorname{Fr}(F \cup G) = \operatorname{Fr} F \cup \operatorname{Fr} G$.

(10) For every topological structure T and for every subset A of T holds $\operatorname{Fr} A = \overline{A} \setminus \operatorname{Int} A$.

(11) Let T be a non empty topological space, A be a subset of T, and p be a point of T. Then $p \in \operatorname{Fr} A$ if and only if for every subset U of T such that U is open and $p \in U$ holds A meets U and $U \setminus A \neq \emptyset$.

(12) Let T be a non empty topological space, A be a subset of T, and p be a point of T. Then $p \in \operatorname{Fr} A$ if and only if for every basis B of p and for every subset U of T such that $U \in B$ holds A meets U and $U \setminus A \neq \emptyset$.

(13) Let T be a non empty topological space, A be a subset of T, and p be a point of T. Then $p \in \operatorname{Fr} A$ if and only if there exists a basis B of p such that for every subset U of T such that $U \in B$ holds A meets U and $U \setminus A \neq \emptyset$.

(14) For every topological space T and for all subsets A, B of T holds $\mathrm{Fr}(A \cap B) \subseteq \overline{A} \cap \mathrm{Fr}\, B \cup \mathrm{Fr}\, A \cap \overline{B}$.

(15) For every topological space T and for every subset A of T holds the carrier of $T = \mathrm{Int}\, A \cup \mathrm{Fr}\, A \cup \mathrm{Int}(A^{\mathrm{c}})$.

(16) For every topological space T and for every subset A of T holds A is open and closed iff $\mathrm{Fr}\, A = \emptyset$.

3. Accumulation Points and Derivative of a Set

Let T be a topological structure, let A be a subset of T, and let x be a set. We say that x is an accumulation point of A if and only if:

(Def. 2) $x \in \overline{A \setminus \{x\}}$.

We now state the proposition

(17) Let T be a topological space, A be a subset of T, and x be a set. If x is an accumulation point of A, then x is a point of T.

Let T be a topological structure and let A be a subset of T. The functor $\mathrm{Der}\, A$ yielding a subset of T is defined by:

(Def. 3) For every set x such that $x \in$ the carrier of T holds $x \in \mathrm{Der}\, A$ iff x is an accumulation point of A.

Next we state four propositions:

(18) Let T be a non empty topological space, A be a subset of T, and x be a set. Then $x \in \mathrm{Der}\, A$ if and only if x is an accumulation point of A.

(19) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \mathrm{Der}\, A$ if and only if for every open subset U of T such that $x \in U$ there exists a point y of T such that $y \in A \cap U$ and $x \neq y$.

(20) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \mathrm{Der}\, A$ if and only if for every basis B of x and for every subset U of T such that $U \in B$ there exists a point y of T such that $y \in A \cap U$ and $x \neq y$.

(21) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \mathrm{Der}\, A$ if and only if there exists a basis B of x such that for every subset U of T such that $U \in B$ there exists a point y of T such that $y \in A \cap U$ and $x \neq y$.

4. Isolated Points

Let T be a topological space, let A be a subset of T, and let x be a set. We say that x is isolated in A if and only if:

(Def. 4) $x \in A$ and x is not an accumulation point of A.

The following three propositions are true:

(22) Let T be a non empty topological space, A be a subset of T, and p be a set. Then $p \in A \setminus \operatorname{Der} A$ if and only if p is isolated in A.

(23) Let T be a non empty topological space, A be a subset of T, and p be a point of T. Then p is an accumulation point of A if and only if for every open subset U of T such that $p \in U$ there exists a point q of T such that $q \neq p$ and $q \in A$ and $q \in U$.

(24) Let T be a non empty topological space, A be a subset of T, and p be a point of T. Then p is isolated in A if and only if there exists an open subset G of T such that $G \cap A = \{p\}$.

Let T be a topological space and let p be a point of T. We say that p is isolated if and only if:

(Def. 5) p is isolated in Ω_T.

Next we state the proposition

(25) For every non empty topological space T and for every point p of T holds p is isolated iff $\{p\}$ is open.

5. Derivative of a Subset-Family

Let T be a topological space and let F be a family of subsets of T. The functor $\operatorname{Der} F$ yielding a family of subsets of T is defined by:

(Def. 6) For every subset A of T holds $A \in \operatorname{Der} F$ iff there exists a subset B of T such that $A = \operatorname{Der} B$ and $B \in F$.

For simplicity, we follow the rules: T is a non empty topological space, A, B are subsets of T, F, G are families of subsets of T, and x is a set.

One can prove the following propositions:

(26) If $F = \emptyset$, then $\operatorname{Der} F = \emptyset$.

(27) If $F = \{A\}$, then $\operatorname{Der} F = \{\operatorname{Der} A\}$.

(28) If $F \subseteq G$, then $\operatorname{Der} F \subseteq \operatorname{Der} G$.

(29) $\operatorname{Der}(F \cup G) = \operatorname{Der} F \cup \operatorname{Der} G$.

(30) For every non empty topological space T and for every subset A of T holds $\operatorname{Der} A \subseteq \overline{A}$.

(31) For every topological space T and for every subset A of T holds $\overline{A} = A \cup \operatorname{Der} A$.

(32) For every non empty topological space T and for all subsets A, B of T such that $A \subseteq B$ holds $\operatorname{Der} A \subseteq \operatorname{Der} B$.

(33) For every non empty topological space T and for all subsets A, B of T holds $\operatorname{Der}(A \cup B) = \operatorname{Der} A \cup \operatorname{Der} B$.

(34) For every non empty topological space T and for every subset A of T such that T is T_1 holds $\operatorname{Der}\operatorname{Der} A \subseteq \operatorname{Der} A$.

(35) For every topological space T and for every subset A of T such that T is T_1 holds $\overline{\operatorname{Der} A} = \operatorname{Der} A$.

Let T be a T_1 non empty topological space and let A be a subset of T. Observe that $\operatorname{Der} A$ is closed.

One can prove the following two propositions:

(36) For every non empty topological space T and for every family F of subsets of T holds $\bigcup \operatorname{Der} F \subseteq \operatorname{Der} \bigcup F$.

(37) If $A \subseteq B$ and x is an accumulation point of A, then x is an accumulation point of B.

6. Density-in-itself

Let T be a topological space and let A be a subset of T. We say that A is dense-in-itself if and only if:

(Def. 7) $A \subseteq \operatorname{Der} A$.

Let T be a non empty topological space. We say that T is dense-in-itself if and only if:

(Def. 8) Ω_T is dense-in-itself.

Next we state the proposition

(38) If T is T_1 and A is dense-in-itself, then $\overline{A}$ is dense-in-itself.

Let T be a topological space and let F be a family of subsets of T. We say that F is dense-in-itself if and only if:

(Def. 9) For every subset A of T such that $A \in F$ holds A is dense-in-itself.

The following propositions are true:

(39) For every family F of subsets of T such that F is dense-in-itself holds $\bigcup F \subseteq \bigcup \operatorname{Der} F$.

(40) If F is dense-in-itself, then $\bigcup F$ is dense-in-itself.

(41) $\operatorname{Fr}(\emptyset_T) = \emptyset$.

Let T be a topological space and let A be an open closed subset of T. Note that $\operatorname{Fr} A$ is empty.

Let T be a non empty non discrete topological space. Note that there exists a subset of T which is non open and there exists a subset of T which is non closed.

Let T be a non empty non discrete topological space and let A be a non open subset of T. Observe that $\operatorname{Fr} A$ is non empty.

Let T be a non empty non discrete topological space and let A be a non closed subset of T. One can check that $\operatorname{Fr} A$ is non empty.

7. Perfect Sets

Let T be a topological space and let A be a subset of T. We say that A is perfect if and only if:

(Def. 10) A is closed and dense-in-itself.

Let T be a topological space. One can check that every subset of T which is perfect is also closed and dense-in-itself and every subset of T which is closed and dense-in-itself is also perfect.

We now state three propositions:

(42) For every topological space T holds $\mathrm{Der}(\emptyset_T) = \emptyset_T$.

(43) For every topological space T and for every subset A of T holds A is perfect iff $\mathrm{Der}\, A = A$.

(44) For every topological space T holds $\emptyset_T$ is perfect.

Let T be a topological space. Note that every subset of T which is empty is also perfect.

Let T be a topological space. Observe that there exists a subset of T which is perfect.

8. Scattered Subsets

Let T be a topological space and let A be a subset of T. We say that A is scattered if and only if:

(Def. 11) It is not true that there exists a subset B of T such that B is non empty and $B \subseteq A$ and B is dense-in-itself.

Let T be a non empty topological space. Observe that every subset of T which is non empty and scattered is also non dense-in-itself and every subset of T which is dense-in-itself and non empty is also non scattered.

The following proposition is true

(45) For every topological space T holds $\emptyset_T$ is scattered.

Let T be a topological space. Note that every subset of T which is empty is also scattered.

One can prove the following proposition

(46) Let T be a non empty topological space. Suppose T is T_1. Then there exist subsets A, B of T such that $A \cup B = \Omega_T$ and A misses B and A is perfect and B is scattered.

Let T be a discrete topological space and let A be a subset of T. Observe that $\mathrm{Fr}\, A$ is empty.

Let T be a discrete topological space. Observe that every subset of T is closed and open.

The following proposition is true

(47) For every discrete topological space T and for every subset A of T holds $\operatorname{Der} A = \emptyset$.

Let T be a discrete non empty topological space and let A be a subset of T. Note that $\operatorname{Der} A$ is empty.

One can prove the following proposition

(48) For every discrete non empty topological space T and for every subset A of T such that A is dense holds $A = \Omega_T$.

9. Density of a Topological Space and Separable Spaces

Let T be a topological space. The functor density T yielding a cardinal number is defined by:

(Def. 12) There exists a subset A of T such that A is dense and $\operatorname{density} T = \overline{\overline{A}}$ and for every subset B of T such that B is dense holds $\operatorname{density} T \leq \overline{\overline{B}}$.

Let T be a topological space. We say that T is separable if and only if:

(Def. 13) $\operatorname{density} T \leq \aleph_0$.

One can prove the following proposition

(49) Every countable topological space is separable.

Let us observe that every topological space which is countable is also separable.

10. Exercises

The following propositions are true:

(50) For every subset A of $\mathbb{R}^1$ such that $A = \mathbb{Q}$ holds $A^{c} = \mathbb{IQ}$.

(51) For every subset A of $\mathbb{R}^1$ such that $A = \mathbb{IQ}$ holds $A^{c} = \mathbb{Q}$.

(52) For every subset A of $\mathbb{R}^1$ such that $A = \mathbb{Q}$ holds $\operatorname{Int} A = \emptyset$.

(53) For every subset A of $\mathbb{R}^1$ such that $A = \mathbb{IQ}$ holds $\operatorname{Int} A = \emptyset$.

(54) For every subset A of $\mathbb{R}^1$ such that $A = \mathbb{Q}$ holds A is dense.

(55) For every subset A of $\mathbb{R}^1$ such that $A = \mathbb{IQ}$ holds A is dense.

(56) For every subset A of $\mathbb{R}^1$ such that $A = \mathbb{Q}$ holds A is boundary.

(57) For every subset A of $\mathbb{R}^1$ such that $A = \mathbb{IQ}$ holds A is boundary.

(58) For every subset A of $\mathbb{R}^1$ such that $A = \mathbb{R}$ holds A is non boundary.

(59) There exist subsets A, B of $\mathbb{R}^1$ such that A is boundary and B is boundary and $A \cup B$ is non boundary.

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