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On the Characteristic and Weight of a Topological Space[1]

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Summary. We continue Mizar formalization of General Topology according to the book [13] by Engelking. In the article the formalization of Section 1.1 is completed. Namely, the paper includes the formalization of theorems on correspondence of the basis and basis in a point, definitions of the character of a point and a topological space, a neighborhood system, and the weight of a topological space. The formalization is tested with almost discrete topological spaces with infinity.

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The notation and terminology used here are introduced in the following articles: [22], [26], [21], [16], [27], [9], [28], [10], [7], [3], [18], [5], [4], [12], [24], [1], [2], [25], [17], [29], [11], [14], [8], [19], [20], [23], [6], and [15].

1. Characteristic of Topological Spaces

One can prove the following propositions:

(1) Let T be a non empty topological space, B be a basis of T, and x be an element of T. Then $\{U; U$ ranges over subsets of T: $x \in U \ \wedge \ U \in B\}$ is a basis of x.

(2) Let T be a non empty topological space and B be a many sorted set indexed by T. Suppose that for every element x of T holds $B(x)$ is a basis of x. Then $\bigcup B$ is a basis of T.

Let T be a non empty topological structure and let x be an element of T. The functor $\mathrm{Chi}(x,T)$ yielding a cardinal number is defined as follows:

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(Def. 1) There exists a basis B of x such that $\mathrm{Chi}(x,T) = \overline{\overline{B}}$ and for every basis B of x holds $\mathrm{Chi}(x,T) \leq \overline{\overline{B}}$.

One can prove the following proposition

(3) Let X be a set. Suppose that for every set a such that $a \in X$ holds a is a cardinal number. Then $\bigcup X$ is a cardinal number.

Let T be a non empty topological structure. The functor $\mathrm{Chi}\, T$ yields a cardinal number and is defined by the conditions (Def. 2).

(Def. 2)(i) For every point x of T holds $\mathrm{Chi}(x,T) \leq \mathrm{Chi}\, T$, and

(ii) for every cardinal number M such that for every point x of T holds $\mathrm{Chi}(x,T) \leq M$ holds $\mathrm{Chi}\, T \leq M$.

The following three propositions are true:

(4) For every non empty topological structure T holds $\mathrm{Chi}\, T = \bigcup\{\mathrm{Chi}(x,T) : x$ ranges over points of $T\}$.

(5) For every non empty topological structure T and for every point x of T holds $\mathrm{Chi}(x,T) \leq \mathrm{Chi}\, T$.

(6) For every non empty topological space T holds T is first-countable iff $\mathrm{Chi}\, T \leq \aleph_0$.

2. Neighborhood Systems

Let T be a non empty topological space. A many sorted set indexed by T is said to be a neighborhood system of T if:

(Def. 3) For every element x of T holds $\mathrm{it}(x)$ is a basis of x.

Let T be a non empty topological space and let N be a neighborhood system of T. Then $\bigcup N$ is a basis of T. Let x be a point of T. Then $N(x)$ is a basis of x.

We now state several propositions:

(7) Let T be a non empty topological space, N be a neighborhood system of T, and x be an element of T. Then $N(x)$ is non empty and for every set U such that $U \in N(x)$ holds $x \in U$.

(8) Let T be a non empty topological space, x, y be points of T, B_1 be a basis of x, B_2 be a basis of y, and U be a set. If $x \in U$ and $U \in B_2$, then there exists an open subset V of T such that $V \in B_1$ and $V \subseteq U$.

(9) Let T be a non empty topological space, x be a point of T, B be a basis of x, and U_1, U_2 be sets. If $U_1 \in B$ and $U_2 \in B$, then there exists an open subset V of T such that $V \in B$ and $V \subseteq U_1 \cap U_2$.

(10) Let T be a non empty topological space, A be a subset of T, and x be an element of T. Then $x \in \overline{A}$ if and only if for every basis B of x and for every set U such that $U \in B$ holds U meets A.

(11) Let T be a non empty topological space, A be a subset of T, and x be an element of T. Then $x \in \overline{A}$ if and only if there exists a basis B of x such that for every set U such that $U \in B$ holds U meets A.

Let T be a topological space. Note that there exists a family of subsets of T which is open and non empty.

3. Weights of Topological Spaces

Next we state the proposition

(12) Let T be a non empty topological space and S be an open family of subsets of T. Then there exists an open family G of subsets of T such that $G \subseteq S$ and $\bigcup G = \bigcup S$ and $\overline{\overline{G}} \leq \operatorname{weight} T$.

Let T be a topological structure. We say that T is finite-weight if and only if:

(Def. 4) $\operatorname{weight} T$ is finite.

Let T be a topological structure. We introduce T is infinite-weight as an antonym of T is finite-weight.

Let us mention that every topological structure which is finite is also finite-weight and every topological structure which is infinite-weight is also infinite.

Let us note that there exists a topological space which is finite and non empty.

The following propositions are true:

(13) For every set X holds $\overline{\overline{\mathrm{SmallestPartition}(X)}} = \overline{\overline{X}}$.

(14) Let T be a discrete non empty topological structure. Then SmallestPartition(the carrier of T) is a basis of T and for every basis B of T holds SmallestPartition(the carrier of T) $\subseteq B$.

(15) For every discrete non empty topological structure T holds $\operatorname{weight} T = \overline{\overline{\text{the carrier of } T}}$.

One can verify that there exists a topological space which is infinite-weight.

Next we state several propositions:

(16) Let T be an infinite-weight topological space and B be a basis of T. Then there exists a basis B_1 of T such that $B_1 \subseteq B$ and $\overline{\overline{B_1}} = \operatorname{weight} T$.

(17) Let T be a finite-weight non empty topological space. Then there exists a basis B_0 of T and there exists a function f from the carrier of T into the topology of T such that $B_0 = \operatorname{rng} f$ and for every point x of T holds $x \in f(x)$ and for every open subset U of T such that $x \in U$ holds $f(x) \subseteq U$.

(18) Let T be a topological space, B_0, B be bases of T, and f be a function from the carrier of T into the topology of T. Suppose $B_0 = \operatorname{rng} f$ and for every point x of T holds $x \in f(x)$ and for every open subset U of T such that $x \in U$ holds $f(x) \subseteq U$. Then $B_0 \subseteq B$.

(19) Let T be a topological space, B_0 be a basis of T, and f be a function from the carrier of T into the topology of T. Suppose $B_0 = \operatorname{rng} f$ and for every point x of T holds $x \in f(x)$ and for every open subset U of T such that $x \in U$ holds $f(x) \subseteq U$. Then weight $T = \overline{\overline{B_0}}$.

(20) For every non empty topological space T and for every basis B of T there exists a basis B_1 of T such that $B_1 \subseteq B$ and $\overline{\overline{B_1}} = \text{weight}\, T$.

4. Example of Almost Discrete Topological Space with Infinity

Let X, x_0 be sets. The functor $\text{DiscrWithInfin}(X, x_0)$ yielding a strict topological structure is defined by the conditions (Def. 5).

(Def. 5)(i) The carrier of $\text{DiscrWithInfin}(X, x_0) = X$, and

(ii) the topology of $\text{DiscrWithInfin}(X, x_0) = \{U; U$ ranges over subsets of X: $x_0 \notin U\} \cup \{F^{\mathrm{c}}; F$ ranges over subsets of X: F is finite$\}$.

Let X, x_0 be sets. Observe that $\text{DiscrWithInfin}(X, x_0)$ is topological space-like.

Let X be a non empty set and let x_0 be a set. One can verify that $\text{DiscrWithInfin}(X, x_0)$ is non empty.

Next we state a number of propositions:

(21) For all sets X, x_0 and for every subset A of $\text{DiscrWithInfin}(X, x_0)$ holds A is open iff $x_0 \notin A$ or A^{c} is finite.

(22) For all sets X, x_0 and for every subset A of $\text{DiscrWithInfin}(X, x_0)$ holds A is closed iff if $x_0 \in X$, then $x_0 \in A$ or A is finite.

(23) For all sets X, x_0, x such that $x \in X$ holds $\{x\}$ is a closed subset of $\text{DiscrWithInfin}(X, x_0)$.

(24) For all sets X, x_0, x such that $x \in X$ and $x \neq x_0$ holds $\{x\}$ is an open subset of $\text{DiscrWithInfin}(X, x_0)$.

(25) For all sets X, x_0 such that X is infinite and for every subset U of $\text{DiscrWithInfin}(X, x_0)$ such that $U = \{x_0\}$ holds U is not open.

(26) For all sets X, x_0 and for every subset A of $\text{DiscrWithInfin}(X, x_0)$ such that A is finite holds $\overline{A} = A$.

(27) Let T be a non empty topological space and A be a subset of T. Suppose A is not closed. Let a be a point of T. If $A \cup \{a\}$ is closed, then $\overline{A} = A \cup \{a\}$.

(28) For all sets X, x_0 such that $x_0 \in X$ and for every subset A of $\text{DiscrWithInfin}(X, x_0)$ such that A is infinite holds $\overline{A} = A \cup \{x_0\}$.

(29) For all sets X, x_0 and for every subset A of $\text{DiscrWithInfin}(X, x_0)$ such that A^{c} is finite holds $\operatorname{Int} A = A$.

(30) For all sets X, x_0 such that $x_0 \in X$ and for every subset A of $\text{DiscrWithInfin}(X, x_0)$ such that A^{c} is infinite holds $\operatorname{Int} A = A \setminus \{x_0\}$.

(31) For all sets X, x_0 there exists a basis B_0 of DiscrWithInfin(X, x_0) such that $B_0 = (\text{SmallestPartition}(X) \setminus \{\{x_0\}\}) \cup \{F^{\mathrm{c}}; F$ ranges over subsets of X: F is finite$\}$.

In the sequel Z denotes an infinite set.

The following proposition is true

(32) $\overline{\overline{\text{Fin}\, Z}} = \overline{\overline{Z}}$.

In the sequel F is a subset of Z.

One can prove the following propositions:

(33) $\overline{\overline{\{F^{\mathrm{c}} : F \text{ is finite}\}}} = \overline{\overline{Z}}$.

(34) Let X be an infinite set, x_0 be a set, and B_0 be a basis of DiscrWithInfin(X, x_0). If $B_0 = (\text{SmallestPartition}(X) \setminus \{\{x_0\}\}) \cup \{F^{\mathrm{c}}; F$ ranges over subsets of X: F is finite$\}$, then $\overline{\overline{B_0}} = \overline{\overline{X}}$.

(35) For every infinite set X and for every set x_0 and for every basis B of DiscrWithInfin(X, x_0) holds $\overline{\overline{X}} \leq \overline{\overline{B}}$.

(36) For every infinite set X and for every set x_0 holds weight DiscrWithInfin$(X, x_0) = \overline{\overline{X}}$.

(37) Let X be a non empty set and x_0 be a set. Then there exists a prebasis B_0 of DiscrWithInfin(X, x_0) such that $B_0 = (\text{SmallestPartition}(X) \setminus \{\{x_0\}\}) \cup \{\{x\}^{\mathrm{c}} : x$ ranges over elements of $X\}$.

5. Exercises

Next we state four propositions:

(38) Let T be a topological space, F be a family of subsets of T, and I be a non empty family of subsets of F. Suppose that for every set G such that $G \in I$ holds $F \setminus G$ is finite. Then $\overline{\bigcup F} = \bigcup \text{clf}\, F \cup \bigcap \{\overline{\bigcup G}; G$ ranges over families of subsets of T: $G \in I\}$.

(39) Let T be a topological space and F be a family of subsets of T. Then $\overline{\bigcup F} = \bigcup \text{clf}\, F \cup \bigcap \{\overline{\bigcup G}; G$ ranges over families of subsets of T: $G \subseteq F \wedge F \setminus G$ is finite$\}$.

(40) Let X be a set and O be a family of subsets of 2^X. Suppose that for every family o of subsets of X such that $o \in O$ holds $\langle X, o \rangle$ is a topological space. Then there exists a family B of subsets of X such that

(i) $B = \text{Intersect}(O)$,

(ii) $\langle X, B \rangle$ is a topological space,

(iii) for every family o of subsets of X such that $o \in O$ holds $\langle X, o \rangle$ is a topological extension of $\langle X, B \rangle$, and

(iv) for every topological space T such that the carrier of $T = X$ and for every family o of subsets of X such that $o \in O$ holds $\langle X, o \rangle$ is a topological extension of T holds $\langle X, B \rangle$ is a topological extension of T.

(41) Let X be a set and O be a family of subsets of 2^X. Then there exists a family B of subsets of X such that
(i) $B = \mathrm{UniCl}(\mathrm{FinMeetCl}(\bigcup O))$,
(ii) $\langle X, B\rangle$ is a topological space,
(iii) for every family o of subsets of X such that $o \in O$ holds $\langle X, B\rangle$ is a topological extension of $\langle X, o\rangle$, and
(iv) for every topological space T such that the carrier of $T = X$ and for every family o of subsets of X such that $o \in O$ holds T is a topological extension of $\langle X, o\rangle$ holds T is a topological extension of $\langle X, B\rangle$.

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