FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Some Properties of Rectangles on the Plane[1]

Artur Korniłowicz
University of Białystok

Yasunari Shidama
Shinshu University
Nagano

MML Identifier: TOPREALA.

The terminology and notation used in this paper have been introduced in the following articles: [25], [9], [28], [2], [29], [5], [30], [8], [6], [16], [3], [23], [24], [27], [1], [4], [7], [22], [17], [21], [20], [26], [13], [10], [19], [31], [14], [12], [11], [18], and [15].

1. Real Numbers

We adopt the following rules: i is an integer and a, b, r, s are real numbers.

The following propositions are true:

(1) $\operatorname{frac}(r+i) = \operatorname{frac} r$.

(2) If $r \leq a$ and $a < \lfloor r \rfloor + 1$, then $\lfloor a \rfloor = \lfloor r \rfloor$.

(3) If $r \leq a$ and $a < \lfloor r \rfloor + 1$, then $\operatorname{frac} r \leq \operatorname{frac} a$.

(4) If $r < a$ and $a < \lfloor r \rfloor + 1$, then $\operatorname{frac} r < \operatorname{frac} a$.

(5) If $a \geq \lfloor r \rfloor + 1$ and $a \leq r + 1$, then $\lfloor a \rfloor = \lfloor r \rfloor + 1$.

(6) If $a \geq \lfloor r \rfloor + 1$ and $a < r + 1$, then $\operatorname{frac} a < \operatorname{frac} r$.

(7) If $r \leq a$ and $a < r + 1$ and $r \leq b$ and $b < r + 1$ and $\operatorname{frac} a = \operatorname{frac} b$, then $a = b$.

[1]The paper was written during the first author's post-doctoral fellowship granted by the Shinshu University, Japan.

ISSN 1426–2630

2. Subsets of $\mathbb{R}$

Let r be a real number and let s be a positive real number. One can verify the following observations:

* $]r, r+s[$ is non empty,
* $[r, r+s[$ is non empty,
* $]r, r+s]$ is non empty,
* $[r, r+s]$ is non empty,
* $]r-s, r[$ is non empty,
* $[r-s, r[$ is non empty,
* $]r-s, r]$ is non empty, and
* $[r-s, r]$ is non empty.

Let r be a non positive real number and let s be a positive real number. One can verify the following observations:

* $]r, s[$ is non empty,
* $[r, s[$ is non empty,
* $]r, s]$ is non empty, and
* $[r, s]$ is non empty.

Let r be a negative real number and let s be a non negative real number. One can check the following observations:

* $]r, s[$ is non empty,
* $[r, s[$ is non empty,
* $]r, s]$ is non empty, and
* $[r, s]$ is non empty.

We now state a number of propositions:

(8) If $r \leq a$ and $b < s$, then $[a, b] \subseteq [r, s[$.

(9) If $r < a$ and $b \leq s$, then $[a, b] \subseteq]r, s]$.

(10) If $r < a$ and $b < s$, then $[a, b] \subseteq]r, s[$.

(11) If $r \leq a$ and $b \leq s$, then $[a, b[\subseteq [r, s]$.

(12) If $r \leq a$ and $b \leq s$, then $[a, b[\subseteq [r, s[$.

(13) If $r < a$ and $b \leq s$, then $[a, b[\subseteq]r, s]$.

(14) If $r < a$ and $b \leq s$, then $[a, b[\subseteq]r, s[$.

(15) If $r \leq a$ and $b \leq s$, then $]a, b] \subseteq [r, s]$.

(16) If $r \leq a$ and $b < s$, then $]a, b] \subseteq [r, s[$.

(17) If $r \leq a$ and $b \leq s$, then $]a, b] \subseteq]r, s]$.

(18) If $r \leq a$ and $b < s$, then $]a, b] \subseteq]r, s[$.

(19) If $r \leq a$ and $b \leq s$, then $]a, b[\subseteq [r, s]$.

(20) If $r \leq a$ and $b \leq s$, then $]a, b[\subseteq [r, s[$.

(21) If $r \leq a$ and $b \leq s$, then $]a, b[\subseteq]r, s]$.

3. Functions

The following propositions are true:

(22) For every function f and for all sets x, X such that $x \in \operatorname{dom} f$ and $f(x) \in f^\circ X$ and f is one-to-one holds $x \in X$.

(23) For every finite sequence f and for every natural number i such that $i + 1 \in \operatorname{dom} f$ holds $i \in \operatorname{dom} f$ or $i = 0$.

(24) For all sets x, y, X, Y and for every function f such that $x \neq y$ and $f \in \prod[x \longmapsto X, y \longmapsto Y]$ holds $f(x) \in X$ and $f(y) \in Y$.

(25) For all sets a, b holds $\langle a, b \rangle = [1 \longmapsto a, 2 \longmapsto b]$.

4. General Topology

Let us note that there exists a topological space which is constituted finite sequences, non empty, and strict.

Let T be a constituted finite sequences topological space. Note that every subspace of T is constituted finite sequences.

One can prove the following proposition

(26) Let T be a non empty topological space, Z be a non empty subspace of T, t be a point of T, z be a point of Z, N be an open neighbourhood of t, and M be a subset of Z. If $t = z$ and $M = N \cap \Omega_Z$, then M is an open neighbourhood of z.

Let us note that every topological space which is empty is also discrete and anti-discrete.

Let X be a discrete topological space and let Y be a topological space. Note that every map from X into Y is continuous.

The following proposition is true

(27) Let X be a topological space, Y be a topological structure, and f be a map from X into Y. If f is empty, then f is continuous.

Let X be a topological space and let Y be a topological structure. Observe that every map from X into Y which is empty is also continuous.

One can prove the following propositions:

(28) Let X be a topological structure, Y be a non empty topological structure, and Z be a non empty subspace of Y. Then every map from X into Z is a map from X into Y.

(29) Let S, T be non empty topological spaces, X be a subset of S, Y be a subset of T, f be a continuous map from S into T, and g be a map from $S \restriction X$ into $T \restriction Y$. If $g = f \restriction X$, then g is continuous.

(30) Let S, T be non empty topological spaces, Z be a non empty subspace of T, f be a map from S into T, and g be a map from S into Z. If $f = g$ and f is open, then g is open.

(31) Let S, T be non empty topological spaces, S_1 be a subset of S, T_1 be a subset of T, f be a map from S into T, and g be a map from $S \restriction S_1$ into $T \restriction T_1$. If $g = f \restriction S_1$ and g is onto and f is open and one-to-one, then g is open.

(32) Let X, Y, Z be non empty topological spaces, f be a map from X into Y, and g be a map from Y into Z. If f is open and g is open, then $g \cdot f$ is open.

(33) Let X, Y be topological spaces, Z be an open subspace of Y, f be a map from X into Y, and g be a map from X into Z. If $f = g$ and g is open, then f is open.

(34) Let S, T be non empty topological spaces and f be a map from S into T. Suppose f is one-to-one and onto. Then f is continuous if and only if f^{-1} is open.

(35) Let S, T be non empty topological spaces and f be a map from S into T. Suppose f is one-to-one and onto. Then f is open if and only if f^{-1} is continuous.

(36) Let S be a topological space and T be a non empty topological space. Then S and T are homeomorphic if and only if the topological structure of S and the topological structure of T are homeomorphic.

(37) Let S, T be non empty topological spaces and f be a map from S into T. Suppose f is one-to-one, onto, continuous, and open. Then f is a homeomorphism.

5. $\mathbb{R}^{\mathbf{1}}$

One can prove the following propositions:

(38) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that $f = \mathbb{R} \longmapsto r$ holds f is continuous on $\mathbb{R}$.

(39) Let f, f_1, f_2 be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $\operatorname{dom} f = \operatorname{dom} f_1 \cup \operatorname{dom} f_2$ and $\operatorname{dom} f_1$ is open and $\operatorname{dom} f_2$ is open and f_1 is continuous on $\operatorname{dom} f_1$ and f_2 is continuous on $\operatorname{dom} f_2$ and for every set z such that $z \in \operatorname{dom} f_1$ holds $f(z) = f_1(z)$ and for every set z such that $z \in \operatorname{dom} f_2$ holds $f(z) = f_2(z)$. Then f is continuous on $\operatorname{dom} f$.

(40) Let x be a point of $\mathbb{R}^{\mathbf{1}}$, N be a subset of $\mathbb{R}$, and M be a subset of $\mathbb{R}^{\mathbf{1}}$. Suppose $M = N$. If N is a neighbourhood of x, then M is a neighbourhood of x.

(41) For every point x of $\mathbb{R}^{\mathbf{1}}$ and for every neighbourhood M of x there exists a neighbourhood N of x such that $N \subseteq M$.

(42) Let f be a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$, g be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and x be a point of $\mathbb{R}^{\mathbf{1}}$. If $f = g$ and g is continuous in x, then f is continuous at x.

(43) Let f be a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$ and g be a function from $\mathbb{R}$ into $\mathbb{R}$. If $f = g$ and g is continuous on $\mathbb{R}$, then f is continuous.

(44) If $a \leq r$ and $s \leq b$, then $[r, s]$ is a closed subset of $[a,\ b]_{\mathrm{T}}$.

(45) If $a \leq r$ and $s \leq b$, then $]r, s[$ is an open subset of $[a,\ b]_{\mathrm{T}}$.

(46) If $a \leq b$ and $a \leq r$, then $]r, b]$ is an open subset of $[a,\ b]_{\mathrm{T}}$.

(47) If $a \leq b$ and $r \leq b$, then $[a, r[$ is an open subset of $[a,\ b]_{\mathrm{T}}$.

(48) If $a \leq b$ and $r \leq s$, then the carrier of $[\!:[a,\ b]_{\mathrm{T}},\ [r,\ s]_{\mathrm{T}}:\!] = [\!:[a, b],\ [r, s]:\!]$.

6. $\mathcal{E}^2_{\mathrm{T}}$

Next we state four propositions:

(49) $[a, b] = [1 \longmapsto a, 2 \longmapsto b]$.

(50) $[a, b](1) = a$ and $[a, b](2) = b$.

(51) $\mathrm{ClosedInsideOfRectangle}(a, b, r, s) = \prod[1 \longmapsto [a, b], 2 \longmapsto [r, s]]$.

(52) If $a \leq b$ and $r \leq s$, then $[a, r] \in \mathrm{ClosedInsideOfRectangle}(a, b, r, s)$.

Let a, b, c, d be real numbers. The functor $\mathrm{Trectangle}(a, b, c, d)$ yielding a subspace of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 1) $\mathrm{Trectangle}(a, b, c, d) = (\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$.

The following propositions are true:

(53) The carrier of $\mathrm{Trectangle}(a, b, r, s) = \mathrm{ClosedInsideOfRectangle}(a, b, r, s)$.

(54) If $a \leq b$ and $r \leq s$, then $\mathrm{Trectangle}(a, b, r, s)$ is non empty.

Let a, c be non positive real numbers and let b, d be non negative real numbers. Observe that $\mathrm{Trectangle}(a, b, c, d)$ is non empty.

The map R2Homeo from $[\!:\mathbb{R}^{\mathbf{1}},\ \mathbb{R}^{\mathbf{1}}:\!]$ into $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 2) For all real numbers x, y holds $\mathrm{R2Homeo}(\langle\!\langle x,\ y\rangle\!\rangle) = \langle x, y\rangle$.

Next we state several propositions:

(55) For all subsets A, B of $\mathbb{R}$ holds $\mathrm{R2Homeo}^\circ[\!:A,\ B:\!] = \prod[1 \longmapsto A, 2 \longmapsto B]$.

(56) R2Homeo is a homeomorphism.

(57) If $a \leq b$ and $r \leq s$, then R2Homeo $\restriction$the carrier of $[: [a, b]_T, [r, s]_T :]$ is a map from $[: [a, b]_T, [r, s]_T :]$ into Trectangle(a, b, r, s).

(58) Suppose $a \leq b$ and $r \leq s$. Let h be a map from $[: [a, b]_T, [r, s]_T :]$ into Trectangle(a, b, r, s). If $h =$ R2Homeo $\restriction$the carrier of $[: [a, b]_T, [r, s]_T :]$, then h is a homeomorphism.

(59) If $a \leq b$ and $r \leq s$, then $[: [a, b]_T, [r, s]_T :]$ and Trectangle(a, b, r, s) are homeomorphic.

(60) If $a \leq b$ and $r \leq s$, then for every subset A of $[a, b]_T$ and for every subset B of $[r, s]_T$ holds $\prod[1 \longmapsto A, 2 \longmapsto B]$ is a subset of Trectangle(a, b, r, s).

(61) Suppose $a \leq b$ and $r \leq s$. Let A be an open subset of $[a, b]_T$ and B be an open subset of $[r, s]_T$. Then $\prod[1 \longmapsto A, 2 \longmapsto B]$ is an open subset of Trectangle(a, b, r, s).

(62) Suppose $a \leq b$ and $r \leq s$. Let A be a closed subset of $[a, b]_T$ and B be a closed subset of $[r, s]_T$. Then $\prod[1 \longmapsto A, 2 \longmapsto B]$ is a closed subset of Trectangle(a, b, r, s).

References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[10] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[11] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[12] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.

[13] Zbigniew Karno. Separated and weakly separated subspaces of topological spaces. *Formalized Mathematics*, 2(**5**):665–674, 1991.

[14] Zbigniew Karno. Continuity of mappings over the union of subspaces. *Formalized Mathematics*, 3(**1**):1–16, 1992.

[15] Artur Korniłowicz. The fundamental group of convex subspaces of $\mathcal{E}^n_T$. *Formalized Mathematics*, 12(**3**):295–299, 2004.

[16] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[17] Yatsuka Nakamura. Half open intervals in real numbers. *Formalized Mathematics*, 10(**1**):21–22, 2002.

[18] Yatsuka Nakamura. General Fashoda meet theorem for unit circle and square. *Formalized Mathematics*, 11(**3**):213–224, 2003.

[19] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.

[20] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[21] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.
[22] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[23] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[24] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[25] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[26] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[27] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[28] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[29] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[30] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[31] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

Received October 18, 2004