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Preliminaries to Mathematical Morphology and Its Properties

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Summary. The article is a translation of chapter 2 of the book *Mathematical Morphological Method and Application* by Changqing Tang, Hongbo Lu, Zheng Huang, Fang Zhang, Science Press, China, 1990. In this article, the basic mathematical morphological operators such as Erosion, Dilation, Adjunction Opening, Adjunction Closing and their properties are given. And these operators are usually used in processing and analysing the images.

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The terminology and notation used here are introduced in the following articles: [5], [1], [2], [6], [4], and [3].

1. The Definition of Erosion and Dilation and Their Algebraic Properties

In this paper n denotes a natural number and q, y, b denote points of $\mathcal{E}^n_{\rm T}$.

Let us consider n, let p be a point of $\mathcal{E}^n_{\rm T}$, and let X be a subset of $\mathcal{E}^n_{\rm T}$. The functor $X + p$ yielding a subset of $\mathcal{E}^n_{\rm T}$ is defined by:

(Def. 1) $X + p = \{q + p : q \in X\}$.

Let us consider n and let X be a subset of $\mathcal{E}^n_{\rm T}$. The functor $X!$ yielding a subset of $\mathcal{E}^n_{\rm T}$ is defined as follows:

(Def. 2) $X! = \{-q : q \in X\}$.

Let us consider n and let X, B be subsets of $\mathcal{E}^n_{\rm T}$. The functor $X \ominus B$ yields a subset of $\mathcal{E}^n_{\rm T}$ and is defined as follows:

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(Def. 3) $X \ominus B = \{y : B + y \subseteq X\}$.

Let us consider n and let X, B be subsets of $\mathcal{E}^n_{\rm T}$. The functor $X \oplus B$ yields a subset of $\mathcal{E}^n_{\rm T}$ and is defined as follows:

(Def. 4) $X \oplus B = \{y + b : y \in X \ \wedge \ b \in B\}$.

We follow the rules: n is a natural number, X, Y, Z, B, C, B_1, B_2 are subsets of $\mathcal{E}^n_{\rm T}$, and x, y, p are points of $\mathcal{E}^n_{\rm T}$.

One can prove the following propositions:

(1) $B!! = B$.

(2) $\{0_{\mathcal{E}^n_{\rm T}}\} + x = \{x\}$.

(3) If $B_1 \subseteq B_2$, then $B_1 + p \subseteq B_2 + p$.

(4) For every X such that $X = \emptyset$ holds $X + x = \emptyset$.

(5) $X \ominus \{0_{\mathcal{E}^n_{\rm T}}\} = X$.

(6) $X \oplus \{0_{\mathcal{E}^n_{\rm T}}\} = X$.

(7) $X \oplus \{x\} = X + x$.

(8) For all X, Y such that $Y = \emptyset$ holds $X \ominus Y = \mathcal{R}^n$.

(9) If $X \subseteq Y$, then $X \ominus B \subseteq Y \ominus B$ and $X \oplus B \subseteq Y \oplus B$.

(10) If $B_1 \subseteq B_2$, then $X \ominus B_2 \subseteq X \ominus B_1$ and $X \oplus B_1 \subseteq X \oplus B_2$.

(11) If $0_{\mathcal{E}^n_{\rm T}} \in B$, then $X \ominus B \subseteq X$ and $X \subseteq X \oplus B$.

(12) $X \oplus Y = Y \oplus X$.

(13) $Y + y \subseteq X + x$ iff $Y + (y - x) \subseteq X$.

(14) $(X + p) \ominus Y = X \ominus Y + p$.

(15) $(X + p) \oplus Y = X \oplus Y + p$.

(16) $(X + x) + y = X + (x + y)$.

(17) $X \ominus (Y + p) = X \ominus Y + -p$.

(18) $X \oplus (Y + p) = X \oplus Y + p$.

(19) If $x \in X$, then $B + x \subseteq B \oplus X$.

(20) $X \subseteq (X \oplus B) \ominus B$.

(21) $X + 0_{\mathcal{E}^n_{\rm T}} = X$.

(22) $X \ominus \{x\} = X + -x$.

(23) $X \ominus (Y \oplus Z) = X \ominus Y \ominus Z$.

(24) $X \ominus (Y \oplus Z) = X \ominus Z \ominus Y$.

(25) $X \oplus (Y \ominus Z) \subseteq (X \oplus Y) \ominus Z$.

(26) $X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z$.

(27) $(B \cup C) + y = (B + y) \cup (C + y)$.

(28) $B \cap C + y = (B + y) \cap (C + y)$.

(29) $X \ominus (B \cup C) = (X \ominus B) \cap (X \ominus C)$.

(30) $X \oplus (B \cup C) = X \oplus B \cup X \oplus C$.

(31) $X \ominus B \cup Y \ominus B \subseteq (X \cup Y) \ominus B$.
(32) $(X \cup Y) \oplus B = X \oplus B \cup Y \oplus B$.
(33) $X \cap Y \ominus B = (X \ominus B) \cap (Y \ominus B)$.
(34) $X \cap Y \oplus B \subseteq (X \oplus B) \cap (Y \oplus B)$.
(35) $B \oplus X \cap Y \subseteq (B \oplus X) \cap (B \oplus Y)$.
(36) $B \ominus X \cup B \ominus Y \subseteq B \ominus X \cap Y$.
(37) $(X^{\mathrm{c}} \ominus B)^{\mathrm{c}} = X \oplus B!$.
(38) $(X \ominus B)^{\mathrm{c}} = X^{\mathrm{c}} \oplus B!$.

2. The Definition of Adjunction Opening and Closing and Their Algebraic Properties

Let n be a natural number and let X, B be subsets of $\mathcal{E}^n_{\mathrm{T}}$. The functor $X \bigcirc B$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 5) $X \bigcirc B = (X \ominus B) \oplus B$.

Let n be a natural number and let X, B be subsets of $\mathcal{E}^n_{\mathrm{T}}$. The functor $X \circledcirc B$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 6) $X \circledcirc B = (X \oplus B) \ominus B$.

We now state a number of propositions:

(39) $(X^{\mathrm{c}} \bigcirc B!)^{\mathrm{c}} = X \circledcirc B$.
(40) $(X^{\mathrm{c}} \circledcirc B!)^{\mathrm{c}} = X \bigcirc B$.
(41) $X \bigcirc B \subseteq X$ and $X \subseteq X \circledcirc B$.
(42) $X \bigcirc X = X$.
(43) $X \bigcirc B \ominus B \subseteq X \ominus B$ and $X \bigcirc B \oplus B \subseteq X \oplus B$.
(44) $X \ominus B \subseteq X \circledcirc B \ominus B$ and $X \oplus B \subseteq X \circledcirc B \oplus B$.
(45) If $X \subseteq Y$, then $X \bigcirc B \subseteq Y \bigcirc B$ and $X \circledcirc B \subseteq Y \circledcirc B$.
(46) $(X + p) \bigcirc Y = X \bigcirc Y + p$.
(47) $(X + p) \circledcirc Y = X \circledcirc Y + p$.
(48) If $C \subseteq B$, then $X \bigcirc B \subseteq (X \ominus C) \oplus B$.
(49) If $B \subseteq C$, then $X \circledcirc B \subseteq (X \oplus C) \ominus B$.
(50) $X \oplus Y = X \circledcirc Y \oplus Y$ and $X \ominus Y = X \bigcirc Y \ominus Y$.
(51) $X \oplus Y = (X \oplus Y) \bigcirc Y$ and $X \ominus Y = (X \ominus Y) \circledcirc Y$.
(52) $X \bigcirc B \bigcirc B = X \bigcirc B$.
(53) $X \circledcirc B \circledcirc B = X \circledcirc B$.
(54) $X \bigcirc B \subseteq (X \cup Y) \bigcirc B$.
(55) If $B = B \bigcirc B_1$, then $X \bigcirc B \subseteq X \bigcirc B_1$.

3. The Definition of Scaling Transformation and Its Algebraic Properties

In the sequel a is a point of $\mathcal{E}^n_{\mathrm{T}}$.

Let t be a real number, let us consider n, and let A be a subset of $\mathcal{E}^n_{\mathrm{T}}$. The functor $t \odot A$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 7) $t \odot A = \{t \cdot a : a \in A\}$.

In the sequel t, s denote real numbers.

One can prove the following propositions:

(56) For every subset X of $\mathcal{E}^n_{\mathrm{T}}$ such that $X = \emptyset$ holds $0 \odot X = \emptyset$.

(57) For every non empty subset X of $\mathcal{E}^n_{\mathrm{T}}$ holds $0 \odot X = \{0_{\mathcal{E}^n_{\mathrm{T}}}\}$.

(58) $1 \odot X = X$.

(59) $2 \odot X \subseteq X \oplus X$.

(60) $(t \cdot s) \odot X = t \odot (s \odot X)$.

(61) If $X \subseteq Y$, then $t \odot X \subseteq t \odot Y$.

(62) $t \odot (X + x) = t \odot X + t \cdot x$.

(63) $t \odot (X \oplus Y) = t \odot X \oplus t \odot Y$.

(64) If $t \neq 0$, then $t \odot (X \ominus Y) = t \odot X \ominus t \odot Y$.

(65) If $t \neq 0$, then $t \odot (X \bigcirc Y) = (t \odot X) \bigcirc (t \odot Y)$.

(66) If $t \neq 0$, then $t \odot (X \circledcirc Y) = (t \odot X) \circledcirc (t \odot Y)$.

4. The Definition of Thinning and Thickening and Their Algebraic Properties

Let n be a natural number and let X, B_1, B_2 be subsets of $\mathcal{E}^n_{\mathrm{T}}$. The functor $X \circledast (B_1, B_2)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 8) $X \circledast (B_1, B_2) = (X \ominus B_1) \cap (X^{\mathrm{c}} \ominus B_2)$.

Let n be a natural number and let X, B_1, B_2 be subsets of $\mathcal{E}^n_{\mathrm{T}}$. The functor $X \otimes (B_1, B_2)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 9) $X \otimes (B_1, B_2) = X \cup (X \circledast (B_1, B_2))$.

Let n be a natural number and let X, B_1, B_2 be subsets of $\mathcal{E}^n_{\mathrm{T}}$. The functor $X \circledast (B_1, B_2)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 10) $X \circledast (B_1, B_2) = X \setminus (X \circledast (B_1, B_2))$.

The following propositions are true:

(67) If $B_1 = \emptyset$, then $X \circledast (B_1, B_2) = X^{\mathrm{c}} \ominus B_2$.

(68) If $B_2 = \emptyset$, then $X \circledast (B_1, B_2) = X \ominus B_1$.

(69) If $0_{\mathcal{E}^n_{\mathrm{T}}} \in B_1$, then $X \circledast (B_1, B_2) \subseteq X$.

(70) If $0_{\mathcal{E}^n_{\mathrm{T}}} \in B_2$, then $(X \circledast (B_1, B_2)) \cap X = \emptyset$.

(71) If $0_{\mathcal{E}^n_{\mathrm{T}}} \in B_1$, then $X \otimes (B_1, B_2) = X$.

(72) If $0_{\mathcal{E}^n_{\mathrm{T}}} \in B_2$, then $X \circledast (B_1, B_2) = X$.

(73) $X \otimes (B_2, B_1) = (X^{\mathrm{c}} \circledast (B_1, B_2))^{\mathrm{c}}$.

(74) $X \circledast (B_2, B_1) = (X^{\mathrm{c}} \otimes (B_1, B_2))^{\mathrm{c}}$.

5. Properties of Erosion, Dilation, Adjunction Opening, Adjunction Closing on Convex Sets

One can prove the following proposition

(75) Let n be a natural number and B be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then B is convex if and only if for all points x, y of $\mathcal{E}^n_{\mathrm{T}}$ and for every real number r such that $0 \leq r$ and $r \leq 1$ and $x \in B$ and $y \in B$ holds $r \cdot x + (1 - r) \cdot y \in B$.

Let n be a natural number and let B be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Let us observe that B is convex if and only if:

(Def. 11) For all points x, y of $\mathcal{E}^n_{\mathrm{T}}$ and for every real number r such that $0 \leq r$ and $r \leq 1$ and $x \in B$ and $y \in B$ holds $r \cdot x + (1 - r) \cdot y \in B$.

One can prove the following propositions:

(76) If X is convex, then $X!$ is convex.

(77) If X is convex and B is convex, then $X \oplus B$ is convex and $X \ominus B$ is convex.

(78) If X is convex and B is convex, then $X \bigcirc B$ is convex and $X \circledcirc B$ is convex.

(79) If B is convex and $0 < t$ and $0 < s$, then $(s + t) \odot B = s \odot B \oplus t \odot B$.

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