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Limit of Sequence of Subsets

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Summary. A concept of "limit of sequence of subsets" is defined here. This article contains the following items: 1. definition of the superior sequence and the inferior sequence of sets, 2. definition of the superior limit and the inferior limit of sets, and additional properties for the sigma-field of sets, 3. definition of the limit value of a convergent sequence of sets, and additional properties for the sigma-field of sets.

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The notation and terminology used here are introduced in the following papers: [9], [1], [13], [2], [10], [6], [11], [4], [12], [14], [8], [7], [3], and [5].

For simplicity, we adopt the following rules: n, m, k, k_1, k_2 denote natural numbers, x, X, Y, Z denote sets, A denotes a subset of X, B, A_1, A_2, A_3 denote sequences of subsets of X, S_1 denotes a σ-field of subsets of X, and S, S_2, S_3, S_4 denote sequences of subsets of S_1.

Next we state a number of propositions:

(1) For every function f from $\mathbb{N}$ into Y and for every n holds $\{f(k) : n \leq k\} \neq \emptyset$.

(2) For every function f from $\mathbb{N}$ into Y holds $f(n+m) \in \{f(k) : n \leq k\}$.

(3) For every function f from $\mathbb{N}$ into Y holds $\{f(k_1) : n \leq k_1\} = \{f(k_2) : n+1 \leq k_2\} \cup \{f(n)\}$.

(4) Let f be a function from $\mathbb{N}$ into Y. Then for every k_1 holds $x \in f(n+k_1)$ if and only if for every Z such that $Z \in \{f(k_2) : n \leq k_2\}$ holds $x \in Z$.

(5) For every non empty set Y and for every function f from $\mathbb{N}$ into Y holds $x \in \operatorname{rng} f$ iff there exists n such that $x = f(n)$.

(6) For every non empty set Y and for every function f from $\mathbb{N}$ into Y holds $\operatorname{rng} f = \{f(k)\}$.

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(7) For every non empty set Y and for every function f from $\mathbb{N}$ into Y holds $\mathrm{rng}(f \uparrow k) = \{f(n) : k \leq n\}$.

(8) $x \in \bigcap \mathrm{rng}\, B$ iff for every n holds $x \in B(n)$.

(9) Intersection $B = \bigcap \mathrm{rng}\, B$.

(10) Intersection $B \subseteq \bigcup B$.

(11) If for every n holds $B(n) = A$, then $\bigcup B = A$.

(12) If for every n holds $B(n) = A$, then Intersection $B = A$.

(13) If B is constant, then $\bigcup B =$ Intersection B.

(14) If B is constant and the value of $B = A$, then for every n holds $\bigcup\{B(k) : n \leq k\} = A$.

(15) If B is constant and the value of $B = A$, then for every n holds $\bigcap\{B(k) : n \leq k\} = A$.

(16) Let given X, B and f be a function. Suppose $\mathrm{dom}\, f = \mathbb{N}$ and for every n holds $f(n) = \bigcap\{B(k) : n \leq k\}$. Then f is a sequence of subsets of X.

(17) Let X be a set, B be a sequence of subsets of X, and f be a function. Suppose $\mathrm{dom}\, f = \mathbb{N}$ and for every n holds $f(n) = \bigcup\{B(k) : n \leq k\}$. Then f is a function from $\mathbb{N}$ into 2^X.

Let us consider X, B. We say that B is monotone if and only if:

(Def. 1) B is non-decreasing or non-increasing.

Let B be a function. The inferior setsequence B yields a function and is defined by the conditions (Def. 2).

(Def. 2)(i) dom (the inferior setsequence B) $= \mathbb{N}$, and

(ii) for every n holds (the inferior setsequence B)$(n) = \bigcap\{B(k) : n \leq k\}$.

Let X be a set and let B be a sequence of subsets of X. Then the inferior setsequence B is a sequence of subsets of X.

Let B be a function. The superior setsequence B yields a function and is defined by the conditions (Def. 3).

(Def. 3)(i) dom (the superior setsequence B) $= \mathbb{N}$, and

(ii) for every n holds (the superior setsequence B)$(n) = \bigcup\{B(k) : n \leq k\}$.

Let X be a set and let B be a sequence of subsets of X. Then the superior setsequence B is a sequence of subsets of X.

Next we state several propositions:

(18) (The inferior setsequence B)$(0) =$ Intersection B.

(19) (The superior setsequence B)$(0) = \bigcup B$.

(20) $x \in$ (the inferior setsequence B)(n) iff for every k holds $x \in B(n + k)$.

(21) $x \in$ (the superior setsequence B)(n) iff there exists k such that $x \in B(n + k)$.

(22) (The inferior setsequence B)$(n) =$ (the inferior setsequence B)$(n + 1) \cap B(n)$.

(23) (The superior setsequence $B)(n) =$ (the superior setsequence $B)(n+1) \cup B(n)$.

(24) The inferior setsequence B is non-decreasing.

(25) The superior setsequence B is non-increasing.

(26) The inferior setsequence B is monotone and the superior setsequence B is monotone.

Let X be a set and let A be a sequence of subsets of X. Observe that the inferior setsequence A is non-decreasing.

Let X be a set and let A be a sequence of subsets of X. Observe that the superior setsequence A is non-increasing.

The following propositions are true:

(27) Intersection $B \subseteq$ (the inferior setsequence $B)(n)$.

(28) (The superior setsequence $B)(n) \subseteq \bigcup B$.

(29) For all B, n holds $\{B(k) : n \leq k\}$ is a family of subsets of X.

(30) $\bigcup B =$ (Intersection Complement $B)^{\rm c}$.

(31) (The inferior setsequence $B)(n) =$ (the superior setsequence Complement $B)(n)^{\rm c}$.

(32) (The superior setsequence $B)(n) =$ (the inferior setsequence Complement $B)(n)^{\rm c}$.

(33) Complement (the inferior setsequence B) $=$ the superior setsequence Complement B.

(34) Complement (the superior setsequence B) $=$ the inferior setsequence Complement B.

(35) Suppose that for every n holds $A_3(n) = A_1(n) \cup A_2(n)$. Let given n. Then (the inferior setsequence $B)(n) \cup$ (the inferior setsequence $A_2)(n) \subseteq$ (the inferior setsequence $A_3)(n)$.

(36) Suppose that for every n holds $A_3(n) = A_1(n) \cap A_2(n)$. Let given n. Then (the inferior setsequence $A_3)(n) =$ (the inferior setsequence $A_1)(n) \cap$ (the inferior setsequence $A_2)(n)$.

(37) Suppose that for every n holds $A_3(n) = A_1(n) \cup A_2(n)$. Let given n. Then (the superior setsequence $A_3)(n) =$ (the superior setsequence $A_1)(n) \cup$ (the superior setsequence $A_2)(n)$.

(38) Suppose that for every n holds $A_3(n) = A_1(n) \cap A_2(n)$. Let given n. Then (the superior setsequence $A_3)(n) \subseteq$ (the superior setsequence $A_1)(n) \cap$ (the superior setsequence $A_2)(n)$.

(39) If B is constant and the value of $B = A$, then for every n holds (the inferior setsequence $B)(n) = A$.

(40) If B is constant and the value of $B = A$, then for every n holds (the superior setsequence $B)(n) = A$.

(41) If B is non-decreasing, then $B(n) \subseteq$ (the superior setsequence $B)(n+1)$.

(42) If B is non-decreasing, then (the superior setsequence $B)(n)$ = (the superior setsequence $B)(n+1)$.

(43) If B is non-decreasing, then (the superior setsequence $B)(n) = \bigcup B$.

(44) If B is non-decreasing, then Intersection (the superior setsequence B) = $\bigcup B$.

(45) If B is non-decreasing, then $B(n) \subseteq$ (the inferior setsequence $B)(n+1)$.

(46) If B is non-decreasing, then (the inferior setsequence $B)(n) = B(n)$.

(47) If B is non-decreasing, then the inferior setsequence $B = B$.

(48) If B is non-increasing, then (the superior setsequence $B)(n+1) \subseteq B(n)$.

(49) If B is non-increasing, then (the superior setsequence $B)(n) = B(n)$.

(50) If B is non-increasing, then the superior setsequence $B = B$.

(51) If B is non-increasing, then (the inferior setsequence $B)(n+1) \subseteq B(n)$.

(52) If B is non-increasing, then (the inferior setsequence $B)(n)$ = (the inferior setsequence $B)(n+1)$.

(53) If B is non-increasing, then (the inferior setsequence $B)(n)$ = Intersection B.

(54) If B is non-increasing, then $\bigcup$(the inferior setsequence B) = Intersection B.

Let X be a set and let B be a sequence of subsets of X. Then $\liminf B$ can be characterized by the condition:

(Def. 4) $\liminf B = \bigcup$(the inferior setsequence B).

Let X be a set and let B be a sequence of subsets of X. Then $\limsup B$ can be characterized by the condition:

(Def. 5) $\limsup B$ = Intersection (the superior setsequence B).

Let X be a set and let B be a sequence of subsets of X. We introduce $\lim B$ as a synonym of $\limsup B$.

Next we state a number of propositions:

(55) Intersection $B \subseteq \liminf B$.

(56) $\liminf B = \lim$ (the inferior setsequence B).

(57) $\limsup B = \lim$ (the superior setsequence B).

(58) $\limsup B = (\liminf \text{Complement } B)^{\mathrm{c}}$.

(59) If B is constant and the value of $B = A$, then B is convergent and $\lim B = A$ and $\liminf B = A$ and $\limsup B = A$.

(60) If B is non-decreasing, then $\limsup B = \bigcup B$.

(61) If B is non-decreasing, then $\liminf B = \bigcup B$.

(62) If B is non-increasing, then $\limsup B$ = Intersection B.

(63) If B is non-increasing, then $\liminf B$ = Intersection B.

(64) If B is non-decreasing, then B is convergent and $\lim B = \bigcup B$.

(65) If B is non-increasing, then B is convergent and $\lim B = \mathrm{Intersection}\, B$.

(66) If B is monotone, then B is convergent.

Let X be a set, let S_1 be a σ-field of subsets of X, and let S be a sequence of subsets of S_1. Let us observe that S is constant if and only if:

(Def. 6) There exists an element A of S_1 such that for every n holds $S(n) = A$.

Let X be a set, let S_1 be a σ-field of subsets of X, and let S be a sequence of subsets of S_1. Then the inferior setsequence S is a sequence of subsets of S_1.

Let X be a set, let S_1 be a σ-field of subsets of X, and let S be a sequence of subsets of S_1. Then the superior setsequence S is a sequence of subsets of S_1.

The following propositions are true:

(67) $x \in \limsup S$ iff for every n there exists k such that $x \in S(n+k)$.

(68) $x \in \liminf S$ iff there exists n such that for every k holds $x \in S(n+k)$.

(69) $\mathrm{Intersection}\, S \subseteq \liminf S$.

(70) $\limsup S \subseteq \bigcup S$.

(71) $\liminf S \subseteq \limsup S$.

Let X be a set, let S_1 be a σ-field of subsets of X, and let S be a sequence of subsets of S_1. The functor $S^{\mathbf{c}}$ yields a sequence of subsets of S_1 and is defined by:

(Def. 7) $S^{\mathbf{c}} = \mathrm{Complement}\, S$.

Next we state a number of propositions:

(72) $\liminf S = (\limsup(S^{\mathbf{c}}))^{\mathrm{c}}$.

(73) $\limsup S = (\liminf(S^{\mathbf{c}}))^{\mathrm{c}}$.

(74) If for every n holds $S_4(n) = S_2(n) \cup S_3(n)$, then $\liminf S_2 \cup \liminf S_3 \subseteq \liminf S_4$.

(75) If for every n holds $S_4(n) = S_2(n) \cap S_3(n)$, then $\liminf S_4 = \liminf S_2 \cap \liminf S_3$.

(76) If for every n holds $S_4(n) = S_2(n) \cup S_3(n)$, then $\limsup S_4 = \limsup S_2 \cup \limsup S_3$.

(77) If for every n holds $S_4(n) = S_2(n) \cap S_3(n)$, then $\limsup S_4 \subseteq \limsup S_2 \cap \limsup S_3$.

(78) If S is constant and the value of $S = A$, then S is convergent and $\lim S = A$ and $\liminf S = A$ and $\limsup S = A$.

(79) If S is non-decreasing, then $\limsup S = \bigcup S$.

(80) If S is non-decreasing, then $\liminf S = \bigcup S$.

(81) If S is non-decreasing, then S is convergent and $\lim S = \bigcup S$.

(82) If S is non-increasing, then $\limsup S = \mathrm{Intersection}\, S$.

(83) If S is non-increasing, then $\liminf S = \mathrm{Intersection}\, S$.

(84) If S is non-increasing, then S is convergent and $\lim S = \text{Intersection}\, S$.

(85) If S is monotone, then S is convergent.

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