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Properties of First and Second Order Cutting of Binary Relations

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Summary. This paper introduces some notions concerning binary relations according to [9]. It is also an attempt to complement the knowledge contained in the Mizar Mathematical Library regarding binary relations. We define here an image and inverse image of element of set A under binary relation of two sets A, B as image and inverse image of singleton of the element under this relation, respectively. Next, we define "The First Order Cutting Relation of two sets A, B under a subset of the set A" as the union of images of elements of this subset under the relation. We also define "The Second Order Cutting Subset of the Cartesian Product of two sets A, B under a subset of the set A" as an intersection of images of elements of this subset under the subset of the Cartesian Product. The paper also defines first and second projection of binary relations. The main goal of the article is to prove properties and collocations of definitions introduced in this paper.

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The articles [10], [6], [11], [7], [12], [13], [5], [3], [4], [2], [8], and [1] provide the notation and terminology for this paper.

1. Preliminaries

We adopt the following rules: x, y, X, Y, A, B, C, M are sets and P, Q, R, R_1, R_2 are binary relations.

Let X be a set. We introduce $\{\{*\} : * \in X\}$ as a synonym of SmallestPartition (X).

The following propositions are true:

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(1) $y \in \{\{*\} : * \in X\}$ iff there exists x such that $y = \{x\}$ and $x \in X$.

(2) $X = \emptyset$ iff $\{\{*\} : * \in X\} = \emptyset$.

(3) $\{\{*\} : * \in X \cup Y\} = \{\{*\} : * \in X\} \cup \{\{*\} : * \in Y\}$.

(4) $\{\{*\} : * \in X \cap Y\} = \{\{*\} : * \in X\} \cap \{\{*\} : * \in Y\}$.

(5) $\{\{*\} : * \in X \setminus Y\} = \{\{*\} : * \in X\} \setminus \{\{*\} : * \in Y\}$.

(6) $X \subseteq Y$ iff $\{\{*\} : * \in X\} \subseteq \{\{*\} : * \in Y\}$.

Let M be a set and let X, Y be families of subsets of M. Then $X \cap Y$ is a family of subsets of M.

We now state two propositions:

(7) For all families B_1, B_2 of subsets of M holds $\mathrm{Intersect}(B_1) \cap \mathrm{Intersect}(B_2) \subseteq \mathrm{Intersect}(B_1 \cap B_2)$.

(8) $(P \cap Q) \cdot R \subseteq (P \cdot R) \cap (Q \cdot R)$.

2. The First Order Cutting of Binary Relation of Two Sets A, B under Subset of the Set A

Let X, Y be sets, let R be a relation between X and Y, and let x be an element of X. The functor $R^\circ x$ yielding a subset of Y is defined as follows:

(Def. 1) $R^\circ x = R^\circ\{x\}$.

The following propositions are true:

(9) $y \in R^\circ\{x\}$ iff $\langle x, y\rangle \in R$.

(10) $(R_1 \cup R_2)^\circ\{x\} = R_1{}^\circ\{x\} \cup R_2{}^\circ\{x\}$.

(11) $(R_1 \cap R_2)^\circ\{x\} = R_1{}^\circ\{x\} \cap R_2{}^\circ\{x\}$.

(12) $(R_1 \setminus R_2)^\circ\{x\} = R_1{}^\circ\{x\} \setminus R_2{}^\circ\{x\}$.

(13) $(R_1 \cap R_2)^\circ\{\{*\} : * \in X\} \subseteq R_1{}^\circ\{\{*\} : * \in X\} \cap R_2{}^\circ\{\{*\} : * \in X\}$.

Let X, Y be sets, let R be a relation between X and Y, and let x be an element of X. The functor $R^{-1}(x)$ yields a subset of X and is defined by:

(Def. 2) $R^{-1}(x) = R^{-1}(\{x\})$.

One can prove the following propositions:

(14) Let A be a set, F be a family of subsets of A, and R be a binary relation. Then $R^\circ \bigcup F = \bigcup\{R^\circ X; X$ ranges over subsets of A: $X \in F\}$.

(15) For every non empty set A and for every subset X of A holds $X = \bigcup\{\{x\}; x$ ranges over elements of A: $x \in X\}$.

(16) For every non empty set A and for every subset X of A holds $\{\{x\}; x$ ranges over elements of A: $x \in X\}$ is a family of subsets of A.

(17) Let A be a non empty set, B be a set, X be a subset of A, and R be a relation between A and B. Then $R^\circ X = \bigcup\{R^\circ x; x$ ranges over elements of A: $x \in X\}$.

(18) Let A be a non empty set, B be a set, X be a subset of A, and R be a relation between A and B. Then $\{R^\circ x; x$ ranges over elements of A: $x \in X\}$ is a family of subsets of B.

Let A, B be sets, let R be a subset of $[\!: A, 2^B :\!]$, and let X be a set. Then $R^\circ X$ is a family of subsets of B.

Let A be a set and let R be a binary relation. The functor R^A yields a function and is defined as follows:

(Def. 3) $\mathrm{dom}(R^A) = 2^A$ and for every set X such that $X \subseteq A$ holds $R^A(X) = R^\circ X$.

Let B, A be sets and let R be a subset of $[\!: A, B :\!]$. We introduce $^\circ R$ as a synonym of R^A.

One can prove the following propositions:

(19) For all sets A, B and for every subset R of $[\!: A, B :\!]$ such that $X \in \mathrm{dom}\,^\circ R$ holds $(^\circ R)(X) = R^\circ X$.

(20) For all sets A, B and for every subset R of $[\!: A, B :\!]$ holds $\mathrm{rng}\,^\circ R \subseteq 2^{\mathrm{rng}\,R}$.

(21) For all sets A, B and for every subset R of $[\!: A, B :\!]$ holds $^\circ R$ is a function from 2^A into $2^{\mathrm{rng}\,R}$.

Let B, A be sets and let R be a subset of $[\!: A, B :\!]$. Then $^\circ R$ is a function from 2^A into 2^B.

Next we state the proposition

(22) For all sets A, B and for every subset R of $[\!: A, B :\!]$ holds $\bigcup((^\circ R)^\circ A) \subseteq R^\circ \bigcup A$.

3. The Second Order Cutting of Binary Relation of Two Sets A, B under Subset of the Set A

For simplicity, we adopt the following rules: X, X_1, X_2 are subsets of A, Y is a subset of B, R, R_1, R_2 are subsets of $[\!: A, B :\!]$, F is a family of subsets of A, and F_1 is a family of subsets of $[\!: A, B :\!]$.

Let A, B be sets, let X be a subset of A, and let R be a subset of $[\!: A, B :\!]$. The functor $R[X]$ is defined as follows:

(Def. 4) $R[X] = \mathrm{Intersect}((^\circ R)^\circ\{\{*\} : * \in X\})$.

Let A, B be sets, let X be a subset of A, and let R be a subset of $[\!: A, B :\!]$. Then $R[X]$ is a subset of B.

We now state a number of propositions:

(23) $(^\circ R)^\circ\{\{*\} : * \in X\} = \emptyset$ iff $X = \emptyset$.

(24) If $y \in R[X]$, then for every set x such that $x \in X$ holds $y \in R^\circ\{x\}$.

(25) Let B be a non empty set, A be a set, X be a subset of A, y be an element of B, and R be a subset of $[\!: A, B :\!]$. Then $y \in R[X]$ if and only if for every set x such that $x \in X$ holds $y \in R^\circ\{x\}$.

(26) If $(^\circ R)^\circ\{\{*\} : * \in X_1\} = \emptyset$, then $R[X_1 \cup X_2] = R[X_2]$.

(27) $R[X_1 \cup X_2] = R[X_1] \cap R[X_2]$.

(28) Let A be a non empty set, B be a set, F be a family of subsets of A, and R be a relation between A and B. Then $\{R[X]; X$ ranges over subsets of A: $X \in F\}$ is a family of subsets of B.

(29) If $X = \emptyset$, then $R[X] = B$.

(30) $\bigcup F = \emptyset$ iff for every set X such that $X \in F$ holds $X = \emptyset$.

(31) Let A be a set, B be a non empty set, R be a relation between A and B, F be a family of subsets of A, and G be a family of subsets of B. If $G = \{R[Y]; Y$ ranges over subsets of A: $Y \in F\}$, then $R[\bigcup F] = \mathrm{Intersect}(G)$.

(32) If $X_1 \subseteq X_2$, then $R[X_2] \subseteq R[X_1]$.

(33) $R[X_1] \cup R[X_2] \subseteq R[X_1 \cap X_2]$.

(34) $(R_1 \cap R_2)[X] = R_1[X] \cap R_2[X]$.

(35) $(\bigcup F_1)^\circ X = \bigcup\{R^\circ X; R$ ranges over subsets of $[\!: A, B :\!]$: $R \in F_1\}$.

(36) Let F_1 be a family of subsets of $[\!: A, B :\!]$, A, B be sets, and X be a subset of A. Then $\{R[X]; R$ ranges over subsets of $[\!: A, B :\!]$: $R \in F_1\}$ is a family of subsets of B.

(37) If $R = \emptyset$ and $X \neq \emptyset$, then $R[X] = \emptyset$.

(38) If $R = [\!: A, B :\!]$, then $R[X] = B$.

(39) For every family G of subsets of B such that $G = \{R[X]; R$ ranges over subsets of $[\!: A, B :\!]$: $R \in F_1\}$ holds $(\mathrm{Intersect}(F_1))[X] = \mathrm{Intersect}(G)$.

(40) If $R_1 \subseteq R_2$, then $R_1[X] \subseteq R_2[X]$.

(41) $R_1[X] \cup R_2[X] \subseteq (R_1 \cup R_2)[X]$.

(42) $y \in (R^{\mathrm{c}})^\circ\{x\}$ iff $\langle x, y\rangle \notin R$ and $x \in A$ and $y \in B$.

(43) If $X \neq \emptyset$, then $R[X] \subseteq R^\circ X$.

(44) For all sets X, Y holds X meets $(R^\smile)^\circ Y$ iff there exist sets x, y such that $x \in X$ and $y \in Y$ and $x \in (R^\smile)^\circ\{y\}$.

(45) For all sets X, Y holds there exist sets x, y such that $x \in X$ and $y \in Y$ and $x \in (R^\smile)^\circ\{y\}$ iff Y meets $R^\circ X$.

(46) X misses $(R^\smile)^\circ Y$ iff Y misses $R^\circ X$.

(47) For every set X holds $R^\circ X = R^\circ(X \cap \pi_1(R))$.

(48) For every set Y holds $(R^\smile)^\circ Y = (R^\smile)^\circ(Y \cap \pi_2(R))$.

(49) $(R[X])^{\mathrm{c}} = (R^{\mathrm{c}})^\circ X$.

In the sequel R denotes a relation between A and B and S denotes a relation between B and C.

Let A, B, C be sets, let R be a subset of $[\!: A, B :\!]$, and let S be a subset of $[\!: B, C :\!]$. Then $R \cdot S$ is a relation between A and C.

One can prove the following propositions:

(50) $(R^{\circ}X)^{\mathrm{c}} = R^{\mathrm{c}}[X]$.

(51) $\pi_1(R) = (R^{\smile})^{\circ}B$ and $\pi_2(R) = R^{\circ}A$.

(52) $\pi_1(R \cdot S) = (R^{\smile})^{\circ}\pi_1(S)$ and $\pi_1(R \cdot S) \subseteq \pi_1(R)$.

(53) $\pi_2(R \cdot S) = S^{\circ}\pi_2(R)$ and $\pi_2(R \cdot S) \subseteq \pi_2(S)$.

(54) $X \subseteq \pi_1(R)$ iff $X \subseteq (R \cdot R^{\smile})^{\circ}X$.

(55) $Y \subseteq \pi_2(R)$ iff $Y \subseteq (R^{\smile} \cdot R)^{\circ}Y$.

(56) $\pi_1(R) = (R^{\smile})^{\circ}B$ and $(R^{\smile})^{\circ}R^{\circ}A = (R^{\smile})^{\circ}\pi_2(R)$.

(57) $(R^{\smile})^{\circ}B = (R \cdot R^{\smile})^{\circ}A$.

(58) $R^{\circ}A = (R^{\smile} \cdot R)^{\circ}B$.

(59) $S[R^{\circ}X] = (R \cdot S^{\mathrm{c}})^{\mathrm{c}}[X]$.

(60) $(R^{\mathrm{c}})^{\smile} = (R^{\smile})^{\mathrm{c}}$.

(61) $X \subseteq R^{\smile}[Y]$ iff $Y \subseteq R[X]$.

(62) $R^{\circ}X^{\mathrm{c}} \subseteq Y^{\mathrm{c}}$ iff $(R^{\smile})^{\circ}Y \subseteq X$.

(63) $X \subseteq R^{\smile}[R[X]]$ and $Y \subseteq R[R^{\smile}[Y]]$.

(64) $R[X] = R[R^{\smile}[R[X]]]$ and $R^{\smile}[Y] = R^{\smile}[R[R^{\smile}[Y]]]$.

(65) $\mathrm{id}_A \cdot R = R \cdot \mathrm{id}_B$.

References

[1] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.

[2] Grzegorz Bancerek. Minimal signature for partial algebra. *Formalized Mathematics*, 5(**3**):405–414, 1996.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[7] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[8] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.

[9] Jacques Riguet. Relations binaires, fermetures, correspondances de Galois. *Bulletin de la S.M.F.*, 76:114–155, 1948. On WWW: `http://www.numdam.org/item?id=BSFM_1948__76__114_0`.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[11] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[12] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[13] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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