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The Maclaurin Expansions

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Summary. A concept of the Maclaurin expansions is defined here. This article contains the definition of the Maclaurin expansion and expansions of exp, sin and cos functions.

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The papers [15], [16], [4], [12], [2], [14], [5], [1], [3], [7], [6], [10], [11], [8], [9], [17], and [13] provide the notation and terminology for this paper.

The following proposition is true

(1) For every real number x and for every natural number n holds $|x^n| = |x|^n$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, let Z be a subset of $\mathbb{R}$, and let a be a real number. The functor $\mathrm{Maclaurin}(f, Z, a)$ yields a sequence of real numbers and is defined by:

(Def. 1) $\mathrm{Maclaurin}(f, Z, a) = \mathrm{Taylor}(f, Z, 0, a)$.

The following propositions are true:

(2) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and r be a real number. Suppose $0 < r$ and f is differentiable $n+1$ times on $]-r, r[$. Let x be a real number. Suppose $x \in\ ]-r, r[$. Then there exists a real number s such that $0 < s$ and $s < 1$ and $f(x) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Maclaurin}(f,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(n) + \frac{f'(]-r,r[)(n+1)(s \cdot x) \cdot x^{n+1}}{(n+1)!}$.

(3) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and x_0, r be real numbers. Suppose $0 < r$ and f is differentiable $n+1$ times on $]x_0 - r, x_0 + r[$. Let x be a real number. Suppose $x \in\ ]x_0 - r, x_0 + r[$. Then there exists a real number s such that $0 < s$ and $s < 1$ and $|f(x) - (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(f,]x_0 - r, x_0 + r[, x_0, x))(\alpha))_{\kappa \in \mathbb{N}}(n)| = |\frac{f'(]x_0-r,x_0+r[)(n+1)(x_0+s\cdot(x-x_0))\cdot(x-x_0)^{n+1}}{(n+1)!}|$.

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(4) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and r be a real number. Suppose $0 < r$ and f is differentiable $n+1$ times on $]-r, r[$. Let x be a real number. Suppose $x \in \,]-r, r[$. Then there exists a real number s such that $0 < s$ and $s < 1$ and $|f(x) - (\sum_{\alpha=0}^{\kappa}(\mathrm{Maclaurin}(f,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(n)| = |\frac{f'(]-r,r[)(n+1)(s \cdot x) \cdot x^{n+1}}{(n+1)!}|$.

(5) For every real number r holds $\exp'_{\restriction]-r,r[} = \exp \restriction]-r, r[$ and $\mathrm{dom}(\exp \restriction]-r, r[) = \,]-r, r[$.

(6) For every natural number n and for every real number r holds $\exp'(]-r, r[)(n) = \exp \restriction]-r, r[$.

(7) For every natural number n and for all real numbers r, x such that $x \in \,]-r, r[$ holds $\exp'(]-r, r[)(n)(x) = \exp(x)$.

(8) For every natural number n and for all real numbers r, x such that $0 < r$ holds $(\mathrm{Maclaurin}(\exp,]-r, r[, x))(n) = \frac{x^n}{n!}$.

(9) Let n be a natural number and r, x, s be real numbers. Suppose $x \in \,]-r, r[$ and $0 < s$ and $s < 1$. Then $|\frac{\exp'(]-r,r[)(n+1)(s \cdot x) \cdot x^{n+1}}{(n+1)!}| \leq \frac{|\exp(s \cdot x)| \cdot |x|^{n+1}}{(n+1)!}$.

(10) For every real number r and for every natural number n holds exp is differentiable n times on $]-r, r[$.

(11) Let r be a real number. Suppose $0 < r$. Then there exist real numbers M, L such that
  (i) $0 \leq M$,
  (ii) $0 \leq L$, and
  (iii) for every natural number n and for all real numbers x, s such that $x \in \,]-r, r[$ and $0 < s$ and $s < 1$ holds $|\frac{\exp'(]-r,r[)(n)(s \cdot x) \cdot x^n}{n!}| \leq \frac{M \cdot L^n}{n!}$.

(12) Let M, L be real numbers. Suppose $M \geq 0$ and $L \geq 0$. Let e be a real number. Suppose $e > 0$. Then there exists a natural number n such that for every natural number m if $n \leq m$, then $\frac{M \cdot L^m}{m!} < e$.

(13) Let r, e be real numbers. Suppose $0 < r$ and $0 < e$. Then there exists a natural number n such that for every natural number m if $n \leq m$, then for all real numbers x, s such that $x \in \,]-r, r[$ and $0 < s$ and $s < 1$ holds $|\frac{\exp'(]-r,r[)(m)(s \cdot x) \cdot x^m}{m!}| < e$.

(14) Let r, e be real numbers. Suppose $0 < r$ and $0 < e$. Then there exists a natural number n such that for every natural number m if $n \leq m$, then for every real number x such that $x \in \,]-r, r[$ holds $|\exp(x) - (\sum_{\alpha=0}^{\kappa}(\mathrm{Maclaurin}(\exp,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(m)| < e$.

(15) For every real number x holds $x\,\mathrm{ExpSeq}$ is absolutely summable.

(16) For all real numbers r, x such that $0 < r$ holds $\mathrm{Maclaurin}(\exp,]-r, r[, x) = x\,\mathrm{ExpSeq}$ and $\mathrm{Maclaurin}(\exp,]-r, r[, x)$ is absolutely summable and $\exp(x) = \sum \mathrm{Maclaurin}(\exp,]-r, r[, x)$.

(17) Let r be a real number. Then

(i) (the function sin)$'_{\restriction]-r,r[}$ = (the function cos)$\restriction]-r,r[$,

(ii) (the function cos)$'_{\restriction]-r,r[}$ = ($-$the function sin)$\restriction]-r,r[$,

(iii) dom((the function sin)$\restriction]-r,r[) =]-r,r[$, and

(iv) dom((the function cos)$\restriction]-r,r[) =]-r,r[$.

(18) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and Z be a subset of $\mathbb{R}$. If f is differentiable on Z, then $(-f)'_{\restriction Z} = -f'_{\restriction Z}$.

(19) Let r be a real number and n be a natural number. Then

(i) (the function sin)$'(]-r,r[)(2 \cdot n) = (-1)^n$ ((the function sin)$\restriction]-r,r[$),

(ii) (the function sin)$'(]-r,r[)(2 \cdot n+1) = (-1)^n$ ((the function cos)$\restriction]-r,r[$),

(iii) (the function cos)$'(]-r,r[)(2 \cdot n) = (-1)^n$ ((the function cos)$\restriction]-r,r[$), and

(iv) (the function cos)$'(]-r,r[)(2 \cdot n + 1) = (-1)^{n+1}$ ((the function sin)$\restriction]-r,r[$).

(20) Let n be a natural number and r, x be real numbers. Suppose $r > 0$. Then

(i) (Maclaurin(the function sin, $]-r,r[, x$))$(2 \cdot n) = 0$,

(ii) (Maclaurin(the function sin, $]-r,r[, x$))$(2 \cdot n + 1) = \frac{(-1)^n \cdot x^{2 \cdot n+1}}{(2 \cdot n+1)!}$,

(iii) (Maclaurin(the function cos, $]-r,r[, x$))$(2 \cdot n) = \frac{(-1)^n \cdot x^{2 \cdot n}}{(2 \cdot n)!}$, and

(iv) (Maclaurin(the function cos, $]-r,r[, x$))$(2 \cdot n + 1) = 0$.

(21) Let r be a real number and n be a natural number. Then the function sin is differentiable n times on $]-r,r[$ and the function cos is differentiable n times on $]-r,r[$.

(22) Let r be a real number. Suppose $r > 0$. Then there exist real numbers r_1, r_2 such that

(i) $r_1 \geq 0$,

(ii) $r_2 \geq 0$, and

(iii) for every natural number n and for all real numbers x, s such that $x \in]-r,r[$ and $0 < s$ and $s < 1$ holds $|\frac{(\text{the function sin})'(]-r,r[)(n)(s \cdot x) \cdot x^n}{n!}| \leq \frac{r_1 \cdot r_2{}^n}{n!}$ and $|\frac{(\text{the function cos})'(]-r,r[)(n)(s \cdot x) \cdot x^n}{n!}| \leq \frac{r_1 \cdot r_2{}^n}{n!}$.

(23) Let r, e be real numbers. Suppose $0 < r$ and $0 < e$. Then there exists a natural number n such that for every natural number m if $n \leq m$, then for all real numbers x, s such that $x \in]-r,r[$ and $0 < s$ and $s < 1$ holds $|\frac{(\text{the function sin})'(]-r,r[)(m)(s \cdot x) \cdot x^m}{m!}| < e$ and $|\frac{(\text{the function cos})'(]-r,r[)(m)(s \cdot x) \cdot x^m}{m!}| < e$.

(24) Let r, e be real numbers. Suppose $0 < r$ and $0 < e$. Then there exists a natural number n such that for every natural number m if $n \leq m$, then for every real number x such that $x \in]-r,r[$ holds |(the function sin)(x) $-$ ($\sum_{\alpha=0}^{\kappa}$(Maclaurin(the func-

tion sin, $]-r,r[, x))(\alpha))_{\kappa \in \mathbb{N}}(m)| < e$ and $|$(the function cos)$(x) - (\sum_{\alpha=0}^{\kappa}$(Maclaurin(the function cos, $]-r,r[, x))(\alpha))_{\kappa \in \mathbb{N}}(m)| < e$.

(25) Let r, x be real numbers and m be a natural number. Suppose $0 < r$. Then $(\sum_{\alpha=0}^{\kappa}$(Maclaurin(the function sin, $]-r,r[, x))(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot m+1) = (\sum_{\alpha=0}^{\kappa} x\,\mathrm{P\_sin}(\alpha))_{\kappa \in \mathbb{N}}(m)$ and $(\sum_{\alpha=0}^{\kappa}$(Maclaurin(the function cos, $]-r,r[, x))(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot m+1) = (\sum_{\alpha=0}^{\kappa} x\,\mathrm{P\_cos}(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(26) Let r, x be real numbers and m be a natural number. Suppose $0 < r$ and $m > 0$. Then $(\sum_{\alpha=0}^{\kappa}$(Maclaurin(the function sin, $]-r,r[, x))(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot m) = (\sum_{\alpha=0}^{\kappa} x\,\mathrm{P\_sin}(\alpha))_{\kappa \in \mathbb{N}}(m-1)$ and $(\sum_{\alpha=0}^{\kappa}$(Maclaurin(the function cos, $]-r,r[, x))(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot m) = (\sum_{\alpha=0}^{\kappa} x\,\mathrm{P\_cos}(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(27) Let r, x be real numbers and m be a natural number. If $0 < r$, then $(\sum_{\alpha=0}^{\kappa}$(Maclaurin(the function cos, $]-r,r[, x))(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot m) = (\sum_{\alpha=0}^{\kappa} x\,\mathrm{P\_cos}(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(28) Let r, x be real numbers. Suppose $r > 0$. Then

(i) $(\sum_{\alpha=0}^{\kappa}$(Maclaurin(the function sin, $]-r,r[, x))(\alpha))_{\kappa \in \mathbb{N}}$ is convergent,

(ii) (the function sin)$(x) = \sum$ Maclaurin(the function sin, $]-r,r[, x)$,

(iii) $(\sum_{\alpha=0}^{\kappa}$(Maclaurin(the function cos, $]-r,r[, x))(\alpha))_{\kappa \in \mathbb{N}}$ is convergent, and

(iv) (the function cos)$(x) = \sum$ Maclaurin(the function cos, $]-r,r[, x)$.

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