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Jordan Curve Theorem

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Summary. This paper formalizes the Jordan curve theorem following [42] and [17].

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The articles [44], [47], [9], [1], [45], [48], [5], [8], [6], [4], [7], [10], [43], [21], [2], [40], [39], [49], [46], [12], [11], [37], [38], [33], [22], [3], [13], [18], [15], [16], [14], [31], [32], [35], [20], [34], [30], [25], [26], [19], [29], [24], [23], [36], [41], [28], and [27] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: a, b, c, d, r, s denote real numbers, n denotes a natural number, p, p_1, p_2 denote points of $\mathcal{E}^2_{\rm T}$, x, y denote points of $\mathcal{E}^n_{\rm T}$, C denotes a simple closed curve, A, B, P denote subsets of $\mathcal{E}^2_{\rm T}$, U, V denote subsets of $(\mathcal{E}^2_{\rm T}) \restriction C^{\rm c}$, and D denotes a compact middle-intersecting subset of $\mathcal{E}^2_{\rm T}$.

Let M be a symmetric triangle Reflexive metric structure and let x, y be points of M. One can verify that $\rho(x,y)$ is non negative.

Let n be a natural number and let x, y be points of $\mathcal{E}^n_{\rm T}$. Note that $\rho(x,y)$ is non negative.

Let n be a natural number and let x, y be points of $\mathcal{E}^n_{\rm T}$. Observe that $|x-y|$ is non negative.

We now state several propositions:

(1) For all points p_1, p_2 of $\mathcal{E}^n_{\rm T}$ such that $p_1 \neq p_2$ holds $\frac{1}{2} \cdot (p_1 + p_2) \neq p_1$.

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(2) If $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$, then $(p_1)_{\mathbf{2}} < (\frac{1}{2} \cdot (p_1 + p_2))_{\mathbf{2}}$.

(3) If $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$, then $(\frac{1}{2} \cdot (p_1 + p_2))_{\mathbf{2}} < (p_2)_{\mathbf{2}}$.

(4) For every vertical subset A of $\mathcal{E}^2_{\mathrm{T}}$ holds $A \cap B$ is vertical.

(5) For every horizontal subset A of $\mathcal{E}^2_{\mathrm{T}}$ holds $A \cap B$ is horizontal.

(6) If $p \in \mathcal{L}(p_1, p_2)$ and $\mathcal{L}(p_1, p_2)$ is vertical, then $\mathcal{L}(p, p_2)$ is vertical.

(7) If $p \in \mathcal{L}(p_1, p_2)$ and $\mathcal{L}(p_1, p_2)$ is horizontal, then $\mathcal{L}(p, p_2)$ is horizontal.

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. One can verify the following observations:

* $\mathcal{L}(\text{SW-corner}(P), \text{SE-corner}(P))$ is horizontal,
* $\mathcal{L}(\text{NW-corner}(P), \text{SW-corner}(P))$ is vertical, and
* $\mathcal{L}(\text{NE-corner}(P), \text{SE-corner}(P))$ is vertical.

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. One can check the following observations:

* $\mathcal{L}(\text{SE-corner}(P), \text{SW-corner}(P))$ is horizontal,
* $\mathcal{L}(\text{SW-corner}(P), \text{NW-corner}(P))$ is vertical, and
* $\mathcal{L}(\text{SE-corner}(P), \text{NE-corner}(P))$ is vertical.

Let us note that every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is vertical, non empty, and compact is also middle-intersecting.

The following propositions are true:

(8) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $X \subseteq Y$ but $\mathrm{W}_{\mathrm{min}}(Y) \in X$ or $\mathrm{W}_{\mathrm{max}}(Y) \in X$ holds W-bound$(X) =$ W-bound(Y).

(9) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $X \subseteq Y$ but $\mathrm{E}_{\mathrm{min}}(Y) \in X$ or $\mathrm{E}_{\mathrm{max}}(Y) \in X$ holds E-bound$(X) =$ E-bound(Y).

(10) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $X \subseteq Y$ but $\mathrm{N}_{\mathrm{min}}(Y) \in X$ or $\mathrm{N}_{\mathrm{max}}(Y) \in X$ holds N-bound$(X) =$ N-bound(Y).

(11) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $X \subseteq Y$ but $\mathrm{S}_{\mathrm{min}}(Y) \in X$ or $\mathrm{S}_{\mathrm{max}}(Y) \in X$ holds S-bound$(X) =$ S-bound(Y).

(12) W-bound$(C) =$ W-bound(NorthArc(C)).

(13) E-bound$(C) =$ E-bound(NorthArc(C)).

(14) W-bound$(C) =$ W-bound(SouthArc(C)).

(15) E-bound$(C) =$ E-bound(SouthArc(C)).

(16) If $(p_1)_{\mathbf{1}} \leq r$ and $r \leq (p_2)_{\mathbf{1}}$, then $\mathcal{L}(p_1, p_2)$ meets VerticalLine(r).

(17) If $(p_1)_{\mathbf{2}} \leq r$ and $r \leq (p_2)_{\mathbf{2}}$, then $\mathcal{L}(p_1, p_2)$ meets HorizontalLine(r).

Let us consider n. One can check that every subset of $\mathcal{E}^n_{\mathrm{T}}$ which is empty is also Bounded and every subset of $\mathcal{E}^n_{\mathrm{T}}$ which is non Bounded is also non empty.

Let n be a non empty natural number. Note that there exists a subset of $\mathcal{E}^n_{\mathrm{T}}$ which is open, closed, non Bounded, and convex.

Next we state several propositions:

(18) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds NorthHalfline UMP $C \setminus \{\text{UMP}\, C\}$ misses C.

(19) For every compact subset C of $\mathcal{E}^2_{\rm T}$ holds SouthHalfline LMP $C \setminus \{$LMP $C\}$ misses C.

(20) For every compact subset C of $\mathcal{E}^2_{\rm T}$ holds NorthHalfline UMP $C \setminus \{$UMP $C\} \subseteq$ UBD C.

(21) For every compact subset C of $\mathcal{E}^2_{\rm T}$ holds SouthHalfline LMP $C \setminus \{$LMP $C\} \subseteq$ UBD C.

(22) If A is an inside component of B, then UBD B misses A.

(23) If A is an outside component of B, then BDD B misses A.

One can prove the following propositions:

(24) For every positive real number r and for every point a of $\mathcal{E}^n_{\rm T}$ holds $a \in {\rm Ball}(a, r)$.

(25) For every non negative real number r holds every point p of $\mathcal{E}^n_{\rm T}$ is a point of ${\rm Tdisk}(p, r)$.

Let r be a positive real number, let n be a non empty natural number, and let p, q be points of $\mathcal{E}^n_{\rm T}$. Observe that $\overline{\rm Ball}(p, r) \setminus \{q\}$ is non empty.

We now state several propositions:

(26) If $r \leq s$, then ${\rm Ball}(x, r) \subseteq {\rm Ball}(x, s)$.

(27) $\overline{\rm Ball}(x, r) \setminus {\rm Ball}(x, r) = {\rm Sphere}(x, r)$.

(28) If $y \in {\rm Sphere}(x, r)$, then $\mathcal{L}(x, y) \setminus \{x, y\} \subseteq {\rm Ball}(x, r)$.

(29) If $r < s$, then $\overline{\rm Ball}(x, r) \subseteq {\rm Ball}(x, s)$.

(30) If $r < s$, then ${\rm Sphere}(x, r) \subseteq {\rm Ball}(x, s)$.

(31) For every non zero real number r holds $\overline{{\rm Ball}(x, r)} = \overline{\rm Ball}(x, r)$.

(32) For every non zero real number r holds ${\rm Fr}\,{\rm Ball}(x, r) = {\rm Sphere}(x, r)$.

Let n be a non empty natural number. Note that every subset of $\mathcal{E}^n_{\rm T}$ which is Bounded is also proper.

Let us consider n. Note that there exists a subset of $\mathcal{E}^n_{\rm T}$ which is non empty, closed, convex, and Bounded and there exists a subset of $\mathcal{E}^n_{\rm T}$ which is non empty, open, convex, and Bounded.

Let n be a natural number and let A be a Bounded subset of $\mathcal{E}^n_{\rm T}$. Observe that $\overline{A}$ is Bounded.

Let n be a natural number and let A be a Bounded subset of $\mathcal{E}^n_{\rm T}$. One can check that ${\rm Fr}\,A$ is Bounded.

The following propositions are true:

(33) Let A be a closed subset of $\mathcal{E}^n_{\rm T}$ and p be a point of $\mathcal{E}^n_{\rm T}$. If $p \notin A$, then there exists a positive real number r such that ${\rm Ball}(p, r)$ misses A.

(34) For every Bounded subset A of $\mathcal{E}^n_{\rm T}$ and for every point a of $\mathcal{E}^n_{\rm T}$ there exists a positive real number r such that $A \subseteq {\rm Ball}(a, r)$.

(35) For all topological structures S, T and for every map f from S into T such that f is a homeomorphism holds f is onto.

(36) Let T be a topological space, S be a subspace of T, A be a subset of T, and B be a subset of S. If $A = B$, then $T \restriction A = S \restriction B$.

Let T be a non empty T_2 topological space. Note that every non empty subspace of T is T_2.

Let us consider p, r. Observe that $\mathrm{Tdisk}(p, r)$ is closed.

Let us consider p, r. Observe that $\mathrm{Tdisk}(p, r)$ is compact.

2. Paths

Next we state a number of propositions:

(37) Let T be a non empty topological space, a, b be points of T, and f be a path from a to b. If a, b are connected, then $\mathrm{rng}\, f$ is connected.

(38) Let X be a non empty topological space, Y be a non empty subspace of X, x_1, x_2 be points of X, y_1, y_2 be points of Y, and f be a path from x_1 to x_2. Suppose $x_1 = y_1$ and $x_2 = y_2$ and x_1, x_2 are connected and $\mathrm{rng}\, f \subseteq$ the carrier of Y. Then y_1, y_2 are connected and f is a path from y_1 to y_2.

(39) Let X be an arcwise connected non empty topological space, Y be a non empty subspace of X, x_1, x_2 be points of X, y_1, y_2 be points of Y, and f be a path from x_1 to x_2. Suppose $x_1 = y_1$ and $x_2 = y_2$ and $\mathrm{rng}\, f \subseteq$ the carrier of Y. Then y_1, y_2 are connected and f is a path from y_1 to y_2.

(40) Let T be a non empty topological space, a, b be points of T, and f be a path from a to b. If a, b are connected, then $\mathrm{rng}\, f = \mathrm{rng}(-f)$.

(41) Let T be an arcwise connected non empty topological space, a, b be points of T, and f be a path from a to b. Then $\mathrm{rng}\, f = \mathrm{rng}(-f)$.

(42) Let T be a non empty topological space, a, b, c be points of T, f be a path from a to b, and g be a path from b to c. If a, b are connected and b, c are connected, then $\mathrm{rng}\, f \subseteq \mathrm{rng}(f + g)$.

(43) Let T be an arcwise connected non empty topological space, a, b, c be points of T, f be a path from a to b, and g be a path from b to c. Then $\mathrm{rng}\, f \subseteq \mathrm{rng}(f + g)$.

(44) Let T be a non empty topological space, a, b, c be points of T, f be a path from b to c, and g be a path from a to b. If a, b are connected and b, c are connected, then $\mathrm{rng}\, f \subseteq \mathrm{rng}(g + f)$.

(45) Let T be an arcwise connected non empty topological space, a, b, c be points of T, f be a path from b to c, and g be a path from a to b. Then $\mathrm{rng}\, f \subseteq \mathrm{rng}(g + f)$.

(46) Let T be a non empty topological space, a, b, c be points of T, f be a path from a to b, and g be a path from b to c. If a, b are connected and b, c are connected, then $\mathrm{rng}(f + g) = \mathrm{rng}\, f \cup \mathrm{rng}\, g$.

(47) Let T be an arcwise connected non empty topological space, a, b, c be points of T, f be a path from a to b, and g be a path from b to c. Then $\operatorname{rng}(f+g) = \operatorname{rng} f \cup \operatorname{rng} g$.

(48) Let T be a non empty topological space, a, b, c, d be points of T, f be a path from a to b, g be a path from b to c, and h be a path from c to d. Suppose a, b are connected and b, c are connected and c, d are connected. Then $\operatorname{rng}(f+g+h) = \operatorname{rng} f \cup \operatorname{rng} g \cup \operatorname{rng} h$.

(49) Let T be an arcwise connected non empty topological space, a, b, c, d be points of T, f be a path from a to b, g be a path from b to c, and h be a path from c to d. Then $\operatorname{rng}(f+g+h) = \operatorname{rng} f \cup \operatorname{rng} g \cup \operatorname{rng} h$.

(50) For every non empty topological space T and for every point a of T holds $\mathbb{I} \longmapsto a$ is a path from a to a.

(51) Let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$ and P be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2. Then there exists a path F from p_1 to p_2 and there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$ such that $\operatorname{rng} f = P$ and $F = f$.

(52) Let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Then there exists a path F from p_1 to p_2 and there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction\mathcal{L}(p_1, p_2)$ such that $\operatorname{rng} f = \mathcal{L}(p_1, p_2)$ and $F = f$.

(53) Let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_2 \in P$ and $q_1 \neq p_1$ and $q_1 \neq p_2$ and $q_2 \neq p_1$ and $q_2 \neq p_2$. Then there exists a path f from q_1 to q_2 such that $\operatorname{rng} f \subseteq P$ and $\operatorname{rng} f$ misses $\{p_1, p_2\}$.

3. Rectangles

Next we state three propositions:

(54) If $a \leq b$ and $c \leq d$, then $\mathrm{Rectangle}(a, b, c, d) \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$.

(55) $\mathrm{InsideOfRectangle}(a, b, c, d) \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$.

(56) $\mathrm{ClosedInsideOfRectangle}(a, b, c, d) = (\mathrm{OutsideOfRectangle}(a, b, c, d))^{\mathrm{c}}$.

Let a, b, c, d be real numbers. Note that $\mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ is closed.

One can prove the following propositions:

(57) $\mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ misses $\mathrm{OutsideOfRectangle}(a, b, c, d)$.

(58) $\mathrm{ClosedInsideOfRectangle}(a, b, c, d) \cap \mathrm{InsideOfRectangle}(a, b, c, d) = \mathrm{InsideOfRectangle}(a, b, c, d)$.

(59) If $a < b$ and $c < d$, then $\mathrm{Int}\,\mathrm{ClosedInsideOfRectangle}(a, b, c, d) = \mathrm{InsideOfRectangle}(a, b, c, d)$.

(60) If $a \leq b$ and $c \leq d$, then $\mathrm{ClosedInsideOfRectangle}(a, b, c, d) \setminus \mathrm{InsideOfRectangle}(a, b, c, d) = \mathrm{Rectangle}(a, b, c, d)$.

(61) If $a < b$ and $c < d$, then $\mathrm{Fr}\,\mathrm{ClosedInsideOfRectangle}(a,b,c,d) = \mathrm{Rectangle}(a,b,c,d)$.

(62) If $a \leq b$ and $c \leq d$, then $\text{W-bound}(\mathrm{ClosedInsideOfRectangle}(a,b,c,d)) = a$.

(63) If $a \leq b$ and $c \leq d$, then $\text{S-bound}(\mathrm{ClosedInsideOfRectangle}(a,b,c,d)) = c$.

(64) If $a \leq b$ and $c \leq d$, then $\text{E-bound}(\mathrm{ClosedInsideOfRectangle}(a,b,c,d)) = b$.

(65) If $a \leq b$ and $c \leq d$, then $\text{N-bound}(\mathrm{ClosedInsideOfRectangle}(a,b,c,d)) = d$.

(66) If $a < b$ and $c < d$ and $p_1 \in \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$ and $p_2 \notin \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$ and P is an arc from p_1 to p_2, then $\mathrm{Segment}(P,p_1,p_2,p_1,\mathrm{FPoint}(P,p_1,p_2,\mathrm{Rectangle}(a,b,c,d))) \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$.

4. Some Useful Functions

Let S, T be non empty topological spaces and let x be a point of $[:S, T:]$. Then $x_{\mathbf{1}}$ is an element of S, and $x_{\mathbf{2}}$ is an element of T.

Let o be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor $(\square_2)_1 - o_1$ yielding a real map of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ is defined as follows:

(Def. 1) For every point x of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ holds $((\square_2)_1 - o_1)(x) = (x_{\mathbf{2}})_{\mathbf{1}} - o_{\mathbf{1}}$.

The functor $(\square_2)_2 - o_2$ yields a real map of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ and is defined as follows:

(Def. 2) For every point x of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ holds $((\square_2)_2 - o_2)(x) = (x_{\mathbf{2}})_{\mathbf{2}} - o_{\mathbf{2}}$.

The real map $(\square_1)_1 - (\square_2)_1$ of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ is defined as follows:

(Def. 3) For every point x of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ holds $((\square_1)_1 - (\square_2)_1)(x) = (x_{\mathbf{1}})_{\mathbf{1}} - (x_{\mathbf{2}})_{\mathbf{1}}$.

The real map $(\square_1)_2 - (\square_2)_2$ of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ is defined as follows:

(Def. 4) For every point x of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ holds $((\square_1)_2 - (\square_2)_2)(x) = (x_{\mathbf{1}})_{\mathbf{2}} - (x_{\mathbf{2}})_{\mathbf{2}}$.

The real map $(\square_2)_1$ of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ is defined as follows:

(Def. 5) For every point x of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ holds $(\square_2)_1(x) = (x_{\mathbf{2}})_{\mathbf{1}}$.

The real map $(\square_2)_2$ of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ is defined by:

(Def. 6) For every point x of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ holds $(\square_2)_2(x) = (x_{\mathbf{2}})_{\mathbf{2}}$.

One can prove the following propositions:

(67) For every point o of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\square_2)_1 - o_1$ is a continuous map from $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ into $\mathbb{R}^{\mathbf{1}}$.

(68) For every point o of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\square_2)_2 - o_2$ is a continuous map from $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ into $\mathbb{R}^{\mathbf{1}}$.

(69) $(\square_1)_1 - (\square_2)_1$ is a continuous map from $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$ into $\mathbb{R}^{\mathbf{1}}$.

(70) $(\square_1)_2 - (\square_2)_2$ is a continuous map from $[: \mathcal{E}^2_{\rm T}, \mathcal{E}^2_{\rm T} :]$ into $\mathbb{R}^{\mathbf{1}}$.

(71) $(\square_2)_1$ is a continuous map from $[: \mathcal{E}^2_{\rm T}, \mathcal{E}^2_{\rm T} :]$ into $\mathbb{R}^{\mathbf{1}}$.

(72) $(\square_2)_2$ is a continuous map from $[: \mathcal{E}^2_{\rm T}, \mathcal{E}^2_{\rm T} :]$ into $\mathbb{R}^{\mathbf{1}}$.

Let o be a point of $\mathcal{E}^2_{\rm T}$. One can check that $(\square_2)_1 - o_1$ is continuous and $(\square_2)_2 - o_2$ is continuous.

One can check the following observations:

* $(\square_1)_1 - (\square_2)_1$ is continuous,
* $(\square_1)_2 - (\square_2)_2$ is continuous,
* $(\square_2)_1$ is continuous, and
* $(\square_2)_2$ is continuous.

Let n be a non empty natural number, let o, p be points of $\mathcal{E}^n_{\rm T}$, and let r be a positive real number. Let us assume that p is a point of $\mathrm{Tdisk}(o,r)$. The functor $\mathrm{DiskProj}(o,r,p)$ yielding a map from $(\mathcal{E}^n_{\rm T}) \restriction (\overline{\mathrm{Ball}}(o,r) \setminus \{p\})$ into $\mathrm{Tcircle}(o,r)$ is defined by:

(Def. 7) For every point x of $(\mathcal{E}^n_{\rm T}) \restriction (\overline{\mathrm{Ball}}(o,r) \setminus \{p\})$ there exists a point y of $\mathcal{E}^n_{\rm T}$ such that $x = y$ and $(\mathrm{DiskProj}(o,r,p))(x) = \mathrm{HC}(p,y,o,r)$.

The following propositions are true:

(73) Let o, p be points of $\mathcal{E}^2_{\rm T}$ and r be a positive real number. If p is a point of $\mathrm{Tdisk}(o,r)$, then $\mathrm{DiskProj}(o,r,p)$ is continuous.

(74) Let n be a non empty natural number, o, p be points of $\mathcal{E}^n_{\rm T}$, and r be a positive real number. If $p \in \mathrm{Ball}(o,r)$, then $\mathrm{DiskProj}(o,r,p) \restriction \mathrm{Sphere}(o,r) = \mathrm{id}_{\mathrm{Sphere}(o,r)}$.

Let n be a non empty natural number, let o, p be points of $\mathcal{E}^n_{\rm T}$, and let r be a positive real number. Let us assume that $p \in \mathrm{Ball}(o,r)$. The functor $\mathrm{RotateCircle}(o,r,p)$ yields a map from $\mathrm{Tcircle}(o,r)$ into $\mathrm{Tcircle}(o,r)$ and is defined by:

(Def. 8) For every point x of $\mathrm{Tcircle}(o,r)$ there exists a point y of $\mathcal{E}^n_{\rm T}$ such that $x = y$ and $(\mathrm{RotateCircle}(o,r,p))(x) = \mathrm{HC}(y,p,o,r)$.

One can prove the following propositions:

(75) For all points o, p of $\mathcal{E}^2_{\rm T}$ and for every positive real number r such that $p \in \mathrm{Ball}(o,r)$ holds $\mathrm{RotateCircle}(o,r,p)$ is continuous.

(76) Let n be a non empty natural number, o, p be points of $\mathcal{E}^n_{\rm T}$, and r be a positive real number. If $p \in \mathrm{Ball}(o,r)$, then $\mathrm{RotateCircle}(o,r,p)$ has no fixpoint.

5. Jordan Curve Theorem

The following propositions are true:

(77) If $U = P$ and U is a component of $(\mathcal{E}^2_{\rm T})\restriction C^{\rm c}$ and V is a component of $(\mathcal{E}^2_{\rm T})\restriction C^{\rm c}$ and $U \neq V$, then $\overline{P}$ misses V.

(78) If U is a component of $(\mathcal{E}^2_{\rm T})\restriction C^{\rm c}$, then $(\mathcal{E}^2_{\rm T})\restriction C^{\rm c}\restriction U$ is arcwise connected.

(79) If $U = P$ and U is a component of $(\mathcal{E}^2_{\rm T})\restriction C^{\rm c}$, then $C = \operatorname{Fr} P$.

One can prove the following propositions:

(80) For every homeomorphism h of $\mathcal{E}^2_{\rm T}$ holds $h^\circ C$ satisfies conditions of simple closed curve.

(81) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then $P \subseteq$ ClosedInsideOfRectangle$(-1, 1, -3, 3)$.

(82) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then P misses $\mathcal{L}([-1, 3], [1, 3])$.

(83) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then P misses $\mathcal{L}([-1, -3], [1, -3])$.

(84) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then $P \cap$ Rectangle$(-1, 1, -3, 3) = \{[-1, 0], [1, 0]\}$.

(85) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then W-bound$(P) = -1$.

(86) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then E-bound$(P) = 1$.

(87) For every compact subset P of $\mathcal{E}^2_{\rm T}$ such that $[-1, 0]$ and $[1, 0]$ realize maximal distance in P holds $\mathrm{W}_{\rm most}(P) = \{[-1, 0]\}$.

(88) For every compact subset P of $\mathcal{E}^2_{\rm T}$ such that $[-1, 0]$ and $[1, 0]$ realize maximal distance in P holds $\mathrm{E}_{\rm most}(P) = \{[1, 0]\}$.

(89) Let P be a compact subset of $\mathcal{E}^2_{\rm T}$. Suppose $[-1, 0]$ and $[1, 0]$ realize maximal distance in P. Then $\mathrm{W}_{\rm min}(P) = [-1, 0]$ and $\mathrm{W}_{\rm max}(P) = [-1, 0]$.

(90) Let P be a compact subset of $\mathcal{E}^2_{\rm T}$. Suppose $[-1, 0]$ and $[1, 0]$ realize maximal distance in P. Then $\mathrm{E}_{\rm min}(P) = [1, 0]$ and $\mathrm{E}_{\rm max}(P) = [1, 0]$.

(91) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then $\mathcal{L}([0, 3], \mathrm{UMP}\, P)$ is vertical.

(92) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then $\mathcal{L}(\mathrm{LMP}\, P, [0, -3])$ is vertical.

(93) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P and $p \in P$, then $p_{\mathbf{2}} < 3$.

(94) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P and $p \in P$, then $-3 < p_{\mathbf{2}}$.

(95) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in D and $p \in \mathcal{L}([0, 3], \mathrm{UMP}\, D)$, then $(\mathrm{UMP}\, D)_{\mathbf{2}} \leq p_{\mathbf{2}}$.

(96) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in D and $p \in \mathcal{L}(\mathrm{LMP}\, D, [0, -3])$, then $p_{\mathbf{2}} \leq (\mathrm{LMP}\, D)_{\mathbf{2}}$.

(97) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in D, then $\mathcal{L}([0, 3], \mathrm{UMP}\, D) \subseteq \mathrm{NorthHalfline}\, \mathrm{UMP}\, D$.

(98) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in D, then $\mathcal{L}(\mathrm{LMP}\, D, [0, -3]) \subseteq \mathrm{SouthHalfline}\, \mathrm{LMP}\, D$.

(99) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in C and P is an inside component of C, then $\mathcal{L}([0, 3], \mathrm{UMP}\, C)$ misses P.

(100) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in C and P is an inside component of C, then $\mathcal{L}(\mathrm{LMP}\, C, [0, -3])$ misses P.

(101) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in D, then $\mathcal{L}([0, 3], \mathrm{UMP}\, D) \cap D = \{\mathrm{UMP}\, D\}$.

(102) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in D, then $\mathcal{L}([0, -3], \mathrm{LMP}\, D) \cap D = \{\mathrm{LMP}\, D\}$.

(103) Suppose P is compact and $[-1, 0]$ and $[1, 0]$ realize maximal distance in P and A is an inside component of P. Then $A \subseteq \mathrm{ClosedInsideOfRectangle}(-1, 1, -3, 3)$.

(104) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in C, then $\mathcal{L}([0, 3], [0, -3])$ meets C.

(105) Suppose $[-1, 0]$ and $[1, 0]$ realize maximal distance in C. Let J_1, J_2 be compact middle-intersecting subsets of T_2. Suppose that J_1 is an arc from $[-1, 0]$ to $[1, 0]$ and J_2 is an arc from $[-1, 0]$ to $[1, 0]$ and $C = J_1 \cup J_2$ and $J_1 \cap J_2 = \{[-1, 0], [1, 0]\}$ and $\mathrm{UMP}\, C \in J_1$ and $\mathrm{LMP}\, C \in J_2$ and $\text{W-bound}(C) = \text{W-bound}(J_1)$ and $\text{E-bound}(C) = \text{E-bound}(J_1)$. Let U_1 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $U_1 = \mathrm{Component}(\mathrm{Down}(\frac{1}{2} \cdot (\mathrm{UMP}(\mathcal{L}(\mathrm{LMP}\, J_1, [0, -3]) \cap J_2) + \mathrm{LMP}\, J_1), C^{\mathrm{c}}))$. Then U_1 is an inside component of C and for every subset V of T_2 such that V is an inside component of C holds $V = U_1$, where $T_2 = \mathcal{E}^2_{\mathrm{T}}$.

(106) Suppose $[-1, 0]$ and $[1, 0]$ realize maximal distance in C. Let J_1, J_2 be compact middle-intersecting subsets of T_2. Suppose that J_1 is an arc from $[-1, 0]$ to $[1, 0]$ and J_2 is an arc from $[-1, 0]$ to $[1, 0]$ and $C = J_1 \cup J_2$ and $J_1 \cap J_2 = \{[-1, 0], [1, 0]\}$ and $\mathrm{UMP}\, C \in J_1$ and $\mathrm{LMP}\, C \in J_2$ and $\text{W-bound}(C) = \text{W-bound}(J_1)$ and $\text{E-bound}(C) = \text{E-bound}(J_1)$. Then $\mathrm{BDD}\, C = \mathrm{Component}(\mathrm{Down}(\frac{1}{2} \cdot (\mathrm{UMP}(\mathcal{L}(\mathrm{LMP}\, J_1, [0, -3]) \cap J_2) + \mathrm{LMP}\, J_1), C^{\mathrm{c}}))$, where $T_2 = \mathcal{E}^2_{\mathrm{T}}$.

(107) Let C be a simple closed curve. Then there exist subsets A_1, A_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that

(i) $C^{\mathrm{c}} = A_1 \cup A_2$,

(ii) A_1 misses A_2,

(iii) $\overline{A_1} \setminus A_1 = \overline{A_2} \setminus A_2$, and

(iv) for all subsets C_1, C_2 of $(\mathcal{E}^2_{\mathrm{T}})\restriction C^{\mathrm{c}}$ such that $C_1 = A_1$ and $C_2 = A_2$ holds C_1 is a component of $(\mathcal{E}^2_{\mathrm{T}})\restriction C^{\mathrm{c}}$ and C_2 is a component of $(\mathcal{E}^2_{\mathrm{T}})\restriction C^{\mathrm{c}}$.

(108) Every simple closed curve is Jordan.

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