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Some Properties of Some Special Matrices

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Summary. This article describes definitions of reversible matrix, symmetrical matrix, antisymmetric matrix, orthogonal matrix and their main properties.

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The terminology and notation used in this paper have been introduced in the following articles: [8], [3], [11], [12], [1], [10], [9], [6], [2], [4], [5], [13], and [7].

For simplicity, we adopt the following convention: n denotes a natural number, K denotes a field, a denotes an element of K, and M, M_1, M_2, M_3, M_4 denote matrices over K of dimension n.

Let n be a natural number, let K be a field, and let M_1, M_2 be matrices over K of dimension n. We say that M_1 is permutable with M_2 if and only if:

(Def. 1) $M_1 \cdot M_2 = M_2 \cdot M_1$.

Let us note that the predicate M_1 is permutable with M_2 is symmetric.

Let n be a natural number, let K be a field, and let M_1, M_2 be matrices over K of dimension n. We say that M_1 is reverse of M_2 if and only if:

(Def. 2) $M_1 \cdot M_2 = M_2 \cdot M_1$ and $M_1 \cdot M_2 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

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Let us note that the predicate M_1 is reverse of M_2 is symmetric.

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n. We say that M_1 is reversible if and only if:

(Def. 3) There exists a matrix M_2 over K of dimension n such that M_1 is reverse of M_2.

Let us consider n, K and let M_1 be a matrix over K of dimension n. Then $-M_1$ is a matrix over K of dimension n.

Let us consider n, K and let M_1, M_2 be matrices over K of dimension n. Then $M_1 + M_2$ is a matrix over K of dimension n.

Let us consider n, K and let M_1, M_2 be matrices over K of dimension n. Then $M_1 - M_2$ is a matrix over K of dimension n.

Let us consider n, K and let M_1, M_2 be matrices over K of dimension n. Then $M_1 \cdot M_2$ is a matrix over K of dimension n.

The following propositions are true:

(1) For every field K and for every matrix A over K such that $\operatorname{len} A > 0$ and $\operatorname{width} A > 0$ holds $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} A)\times(\operatorname{len} A)} \cdot A = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} A)\times(\operatorname{width} A)}$.

(2) For every field K and for every matrix A over K such that $\operatorname{len} A > 0$ and $\operatorname{width} A > 0$ holds $A \cdot \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{width} A)\times(\operatorname{width} A)} = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} A)\times(\operatorname{width} A)}$.

(3) If $n > 0$, then M_1 is permutable with $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times n}$.

(4) If M_1 is permutable with M_2 and M_2 is permutable with M_3 and M_1 is permutable with M_3, then M_1 is permutable with $M_2 \cdot M_3$.

(5) If M_1 is permutable with M_2 and permutable with M_3 and $n > 0$, then M_1 is permutable with $M_2 + M_3$.

(6) M_1 is permutable with $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

(7) If M_2 is reverse of M_3 and M_1 is reverse of M_3, then $M_1 = M_2$.

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n. Let us assume that M_1 is reversible. The functor ${M_1}^{\smile}$ yields a matrix over K of dimension n and is defined by:

(Def. 4) ${M_1}^{\smile}$ is reverse of M_1.

We now state a number of propositions:

(8) $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})^{\smile} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ is reversible.

(9) $((\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})^{\smile})^{\smile} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

(10) If $n > 0$, then $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})^{\mathrm{T}} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

(11) Let K be a field, n be a natural number, and M be a matrix over K of dimension n. If $M = (\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})^{\mathrm{T}}$ and $n > 0$, then $M^{\smile} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

(12) If ${M_1}^{\mathrm{T}} = M_2$ and M_3 is reverse of M_1 and $M = {M_3}^{\mathrm{T}}$ and $n > 0$, then M_2 is reverse of M.

(13) If M is reversible and $n > 0$ and $M_1 = M^{\mathrm{T}}$ and $M_2 = (M^{\smile})^{\mathrm{T}}$, then ${M_1}^{\smile} = M_2$.

(14) Let K be a field, n be a natural number, and M_1, M_2, M_3, M_4 be matrices over K of dimension n. If M_3 is reverse of M_1 and M_4 is reverse of M_2, then $M_3 \cdot M_4$ is reverse of $M_2 \cdot M_1$.

(15) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. If M_2 is reverse of M_1, then M_1 is permutable with M_2.

(16) If M is reversible, then $M^{\smile}$ is reversible and $(M^{\smile})^{\smile} = M$.

(17) If $n > 0$ and $M_1 \cdot M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n}$ and M_1 is reversible, then M_1 is permutable with M_2.

(18) If $n > 0$ and $M_1 = M_1 \cdot M_2$ and M_1 is reversible, then M_1 is permutable with M_2.

(19) If $n > 0$ and $M_1 = M_2 \cdot M_1$ and M_1 is reversible, then M_1 is permutable with M_2.

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n. We say that M_1 is symmetrical if and only if:

(Def. 5) ${M_1}^{\mathrm{T}} = M_1$.

The following propositions are true:

(20) If $n > 0$, then $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ is symmetrical.

(21) If $n > 0$, then $(\begin{pmatrix} a & \dots & a \\ \vdots & \ddots & \vdots \\ a & \dots & a \end{pmatrix}^{n \times n})^{\mathrm{T}} = \begin{pmatrix} a & \dots & a \\ \vdots & \ddots & \vdots \\ a & \dots & a \end{pmatrix}^{n \times n}$.

(22) If $n > 0$, then $\begin{pmatrix} a & \dots & a \\ \vdots & \ddots & \vdots \\ a & \dots & a \end{pmatrix}^{n \times n}$ is symmetrical.

(23) If $n > 0$ and M_1 is symmetrical and M_2 is symmetrical, then M_1 is permutable with M_2 iff $M_1 \cdot M_2$ is symmetrical.

(24) If $n > 0$, then $(M_1 + M_2)^{\mathrm{T}} = {M_1}^{\mathrm{T}} + {M_2}^{\mathrm{T}}$.

(25) If $n > 0$ and M_1 is symmetrical and M_2 is symmetrical, then $M_1 + M_2$ is symmetrical.

(26) Suppose that

(i) M_1 is an upper triangular matrix over K of dimension n and a lower triangular matrix over K of dimension n, and

(ii) $n > 0$.

Then M_1 is symmetrical.

(27) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. If $n > 0$, then $(-M_1)^{\mathrm{T}} = -{M_1}^{\mathrm{T}}$.

(28) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. If M_1 is symmetrical and $n > 0$, then $-M_1$ is symmetrical.

(29) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. Suppose $n > 0$ and M_1 is symmetrical and M_2 is symmetrical. Then $M_1 - M_2$ is symmetrical.

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n. We say that M_1 is antisymmetric if and only if:

(Def. 6) $M_1{}^{\mathrm{T}} = -M_1$.

We now state a number of propositions:

(30) Let K be a Fanoian field, n be a natural number, and M_1 be a matrix over K of dimension n. If M_1 is symmetrical and antisymmetric and $n > 0$, then $M_1 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times n}$.

(31) Let K be a Fanoian field, n, i be natural numbers, and M_1 be a matrix over K of dimension n. If M_1 is antisymmetric and $n > 0$ and $i \in \operatorname{Seg} n$, then $M_1 \circ (i, i) = 0_K$.

(32) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. Suppose $n > 0$ and M_1 is antisymmetric and M_2 is antisymmetric. Then $M_1 + M_2$ is antisymmetric.

(33) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. If M_1 is antisymmetric and $n > 0$, then $-M_1$ is antisymmetric.

(34) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. Suppose $n > 0$ and M_1 is antisymmetric and M_2 is antisymmetric. Then $M_1 - M_2$ is antisymmetric.

(35) If $M_2 = M_1 - M_1{}^{\mathrm{T}}$ and $n > 0$, then M_2 is antisymmetric.

(36) If $n > 0$, then M_1 is permutable with M_2 iff $(M_1 + M_2) \cdot (M_1 + M_2) = M_1 \cdot M_1 + M_1 \cdot M_2 + M_1 \cdot M_2 + M_2 \cdot M_2$.

(37) If $n > 0$ and M_1 is reversible and M_2 is reversible, then $M_1 \cdot M_2$ is reversible and $(M_1 \cdot M_2)^{\smile} = M_2{}^{\smile} \cdot M_1{}^{\smile}$.

(38) If $n > 0$ and M_1 is reversible and M_2 is reversible and M_1 is permutable with M_2, then $M_1 \cdot M_2$ is reversible and $(M_1 \cdot M_2)^{\smile} = M_1{}^{\smile} \cdot M_2{}^{\smile}$.

(39) If $n > 0$ and M_1 is reversible and M_2 is reversible and $M_1 \cdot M_2 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$, then M_1 is reverse of M_2.

(40) If $n > 0$ and M_1 is reversible and M_2 is reversible and $M_2 \cdot M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$, then M_1 is reverse of M_2.

(41) If $n > 0$ and M_1 is reversible and permutable with M_2, then $M_1{}^{\smile}$ is permutable with M_2.

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n. We say that M_1 is orthogonal if and only if:

(Def. 7) M_1 is reversible and ${M_1}^{\mathrm{T}} = {M_1}^{\smile}$.

The following propositions are true:

(42) If $n > 0$, then $M_1 \cdot {M_1}^{\mathrm{T}} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and M_1 is reversible iff M_1 is orthogonal.

(43) If $n > 0$, then M_1 is reversible and ${M_1}^{\mathrm{T}} \cdot M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ iff M_1 is orthogonal.

(44) If $n > 0$ and M_1 is orthogonal, then ${M_1}^{\mathrm{T}} \cdot M_1 = M_1 \cdot {M_1}^{\mathrm{T}}$.

(45) If $n > 0$ and M_1 is orthogonal and permutable with M_2 and $M_3 = {M_1}^{\mathrm{T}}$, then M_3 is permutable with M_2.

(46) If $n > 0$ and M_1 is reversible and M_2 is reversible, then $M_1 \cdot M_2$ is reversible and $(M_1 \cdot M_2)^{\smile} = {M_2}^{\smile} \cdot {M_1}^{\smile}$.

(47) If $n > 0$ and M_1 is orthogonal and M_2 is orthogonal, then $M_1 \cdot M_2$ is orthogonal.

(48) If $n > 0$ and M_1 is orthogonal and permutable with M_2 and $M_3 = {M_1}^{\mathrm{T}}$, then M_3 is permutable with M_2.

(49) If $n > 0$ and M_1 is permutable with M_2, then $M_1 + M_1$ is permutable with M_2.

(50) If $n > 0$ and M_1 is permutable with M_2, then $M_1 + M_2$ is permutable with M_2.

(51) If $n > 0$ and M_1 is permutable with M_2, then $M_1 + M_1$ is permutable with $M_2 + M_2$.

(52) If $n > 0$ and M_1 is permutable with M_2, then $M_1 + M_2$ is permutable with $M_2 + M_2$.

(53) If $n > 0$ and M_1 is permutable with M_2, then $M_1 + M_2$ is permutable with $M_1 + M_2$.

(54) If $n > 0$ and M_1 is permutable with M_2, then $M_1 \cdot M_2$ is permutable with M_2.

(55) If $n > 0$ and M_1 is permutable with M_2, then $M_1 \cdot M_1$ is permutable with M_2.

(56) If $n > 0$ and M_1 is permutable with M_2, then $M_1 \cdot M_1$ is permutable with $M_2 \cdot M_2$.

(57) If $n > 0$ and M_1 is permutable with M_2 and $M_3 = {M_1}^{\mathrm{T}}$ and $M_4 = {M_2}^{\mathrm{T}}$,

then M_3 is permutable with M_4.

(58) Suppose $n > 0$ and M_1 is reversible and M_2 is reversible and M_3 is reversible. Then $M_1 \cdot M_2 \cdot M_3$ is reversible and $(M_1 \cdot M_2 \cdot M_3)^{\smile} = {M_3}^{\smile} \cdot {M_2}^{\smile} \cdot {M_1}^{\smile}$.

(59) If $n > 0$ and M_1 is orthogonal and M_2 is orthogonal and M_3 is orthogonal, then $M_1 \cdot M_2 \cdot M_3$ is orthogonal.

(60) If $n > 0$, then $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ is orthogonal.

(61) If $n > 0$ and M_1 is orthogonal and M_2 is orthogonal, then ${M_1}^{\smile} \cdot M_2$ is orthogonal.

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