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Quotient Rings

Artur Korniłowicz
Institute of Computer Science
University of Białystok
Sosnowa 64, 15-887 Białystok, Poland

Summary. The notions of prime ideals and maximal ideals of a ring are introduced. Quotient rings are defined. Characterisation of prime and maximal ideals using quotient rings are proved.

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The articles [18], [10], [22], [17], [2], [19], [6], [23], [24], [7], [9], [8], [25], [15], [3], [4], [5], [14], [20], [16], [13], [21], [11], [12], and [1] provide the terminology and notation for this paper.

1. Preliminaries

Let S be a non empty 1-sorted structure. Note that Ω_S is non proper.

The following propositions are true:

(1) Let L be an add-associative right zeroed right complementable non empty loop structure and a, b be elements of L. Then $(a-b)+b=a$.

(2) Let L be an add-associative right zeroed right complementable Abelian non empty loop structure and b, c be elements of L. Then $c=b-(b-c)$.

(3) Let L be an add-associative right zeroed right complementable Abelian non empty loop structure and a, b, c be elements of L. Then $a-b-(c-b)=a-c$.

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2. Ideals

Let K be a non empty groupoid and let S be a subset of K. We say that S is quasi-prime if and only if:

(Def. 1) For all elements a, b of K such that $a \cdot b \in S$ holds $a \in S$ or $b \in S$.

Let K be a non empty multiplicative loop structure and let S be a subset of K. We say that S is prime if and only if:

(Def. 2) S is proper and quasi-prime.

Let R be a non empty double loop structure and let I be a subset of R. We say that I is quasi-maximal if and only if:

(Def. 3) For every ideal J of R such that $I \subseteq J$ holds $J = I$ or J is non proper.

Let R be a non empty double loop structure and let I be a subset of R. We say that I is maximal if and only if:

(Def. 4) I is proper and quasi-maximal.

Let K be a non empty multiplicative loop structure. Note that every subset of K which is prime is also proper and quasi-prime and every subset of K which is proper and quasi-prime is also prime.

Let R be a non empty double loop structure. One can verify that every subset of R which is maximal is also proper and quasi-maximal and every subset of R which is proper and quasi-maximal is also maximal.

Let R be a non empty loop structure. One can verify that Ω_R is add closed.

Let R be a non empty groupoid. Observe that Ω_R is left ideal and right ideal.

We now state the proposition

(4) For every integral domain R holds $\{0_R\}$ is prime.

3. Equivalence Relation

In the sequel R denotes a ring, I denotes an ideal of R, and a, b denote elements of R.

Let R be a ring and let I be an ideal of R. The functor $\approx_I$ yielding a binary relation on R is defined by:

(Def. 5) For all elements a, b of R holds $\langle a, b\rangle \in \approx_I$ iff $a - b \in I$.

Let R be a ring and let I be an ideal of R. One can verify that $\approx_I$ is non empty, total, symmetric, and transitive.

We now state several propositions:

(5) $a \in [b]_{\approx_I}$ iff $a - b \in I$.

(6) $[a]_{\approx_I} = [b]_{\approx_I}$ iff $a - b \in I$.

(7) $[a]_{\approx_{\Omega_R}} =$ the carrier of R.

(8) $\approx_{\Omega_R} = \{\text{the carrier of } R\}$.

(9) $[a]_{\approx_{\{0_R\}}} = \{a\}$.

(10) $\approx_{\{0_R\}} = \mathrm{rng}(\mathrm{singleton}_{\text{the carrier of } R})$.

4. Quotient Ring

Let R be a ring and let I be an ideal of R. The functor ${}^{R}/_{I}$ yields a strict double loop structure and is defined by the conditions (Def. 6).

(Def. 6)(i) The carrier of ${}^{R}/_{I}$ = Classes($\approx_I$),

(ii) the unity of ${}^{R}/_{I} = [1_R]_{\approx_I}$,

(iii) the zero of ${}^{R}/_{I} = [0_R]_{\approx_I}$,

(iv) for all elements x, y of ${}^{R}/_{I}$ there exist elements a, b of R such that $x = [a]_{\approx_I}$ and $y = [b]_{\approx_I}$ and (the addition of ${}^{R}/_{I}$)$(x, y) = [a+b]_{\approx_I}$, and

(v) for all elements x, y of ${}^{R}/_{I}$ there exist elements a, b of R such that $x = [a]_{\approx_I}$ and $y = [b]_{\approx_I}$ and (the multiplication of ${}^{R}/_{I}$)$(x, y) = [a \cdot b]_{\approx_I}$.

Let R be a ring and let I be an ideal of R. Note that ${}^{R}/_{I}$ is non empty.

In the sequel x, y denote elements of ${}^{R}/_{I}$.

We now state several propositions:

(11) There exists an element a of R such that $x = [a]_{\approx_I}$.

(12) $[a]_{\approx_I}$ is an element of ${}^{R}/_{I}$.

(13) If $x = [a]_{\approx_I}$ and $y = [b]_{\approx_I}$, then $x + y = [a+b]_{\approx_I}$.

(14) If $x = [a]_{\approx_I}$ and $y = [b]_{\approx_I}$, then $x \cdot y = [a \cdot b]_{\approx_I}$.

(15) $[1_R]_{\approx_I} = 1_{{}^{R}/_{I}}$.

Let R be a ring and let I be an ideal of R. Observe that ${}^{R}/_{I}$ is Abelian, add-associative, and right zeroed.

Let R be a commutative ring and let I be an ideal of R. Note that ${}^{R}/_{I}$ is commutative.

The following propositions are true:

(16) I is proper iff ${}^{R}/_{I}$ is non degenerated.

(17) I is quasi-prime iff ${}^{R}/_{I}$ is integral domain-like.

(18) For every commutative ring R and for every ideal I of R holds I is prime iff ${}^{R}/_{I}$ is an integral domain.

(19) If R is commutative and I is quasi-maximal, then ${}^{R}/_{I}$ is field-like.

(20) If ${}^{R}/_{I}$ is field-like, then I is quasi-maximal.

(21) For every commutative ring R and for every ideal I of R holds I is maximal iff ${}^{R}/_{I}$ is a skew field.

Let R be a non degenerated commutative ring. One can check that every ideal of R which is maximal is also prime.

Let R be a non degenerated ring. Note that there exists an ideal of R which is maximal.

Let R be a non degenerated commutative ring and let I be a quasi-prime ideal of R. Observe that ${}^{R}/_{I}$ is integral domain-like.

Let R be a non degenerated commutative ring and let I be a quasi-maximal ideal of R. Observe that ${}^{R}/_{I}$ is field-like.

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