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Formulas and Identities of Hyperbolic Functions

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Summary. In this article, we proved formulas of hyperbolic sine, hyperbolic cosine and hyperbolic tangent, and their identities.

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The papers [1], [3], [6], [5], [7], [4], and [2] provide the terminology and notation for this paper.

We follow the rules: x, y, z, w are real numbers and n is a natural number.

One can prove the following propositions:

(1) $\tanh x = \frac{\sinh x}{\cosh x}$ and $\tanh 0 = 0$.

(2) $\sinh x = \frac{1}{\operatorname{cosech} x}$ and $\cosh x = \frac{1}{\operatorname{sech} x}$ and $\tanh x = \frac{1}{\coth x}$.

(3) $\operatorname{sech} x \leq 1$ and $0 < \operatorname{sech} x$ and $\operatorname{sech} 0 = 1$.

(4) If $x \geq 0$, then $\tanh x \geq 0$.

(5) $\cosh x = \frac{1}{\sqrt{1-(\tanh x)^{\mathbf{2}}}}$ and $\sinh x = \frac{\tanh x}{\sqrt{1-(\tanh x)^{\mathbf{2}}}}$.

(6) $(\cosh x + \sinh x)^n = \cosh(n \cdot x) + \sinh(n \cdot x)$ and $(\cosh x - \sinh x)^n = \cosh(n \cdot x) - \sinh(n \cdot x)$.

(7)(i) $\exp x = \cosh x + \sinh x$,

(ii) $\exp(-x) = \cosh x - \sinh x$,

(iii) $\exp x = \frac{\cosh(\frac{x}{2})+\sinh(\frac{x}{2})}{\cosh(\frac{x}{2})-\sinh(\frac{x}{2})}$,

(iv) $\exp(-x) = \frac{\cosh(\frac{x}{2})-\sinh(\frac{x}{2})}{\cosh(\frac{x}{2})+\sinh(\frac{x}{2})}$,

(v) $\exp x = \frac{1+\tanh(\frac{x}{2})}{1-\tanh(\frac{x}{2})}$, and

(vi) $\exp(-x) = \frac{1-\tanh(\frac{x}{2})}{1+\tanh(\frac{x}{2})}$.

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(8) If $x \neq 0$, then $\exp x = \frac{\coth(\frac{x}{2})+1}{\coth(\frac{x}{2})-1}$ and $\exp(-x) = \frac{\coth(\frac{x}{2})-1}{\coth(\frac{x}{2})+1}$.

(9) $\frac{\cosh x+\sinh x}{\cosh x-\sinh x} = \frac{1+\tanh x}{1-\tanh x}$.

(10) If $y \neq 0$, then $\coth y + \tanh z = \frac{\cosh(y+z)}{\sinh y \cdot \cosh z}$ and $\coth y - \tanh z = \frac{\cosh(y-z)}{\sinh y \cdot \cosh z}$.

(11) $\sinh y \cdot \sinh z = \frac{1}{2} \cdot (\cosh(y+z) - \cosh(y-z))$ and $\sinh y \cdot \cosh z = \frac{1}{2} \cdot (\sinh(y+z) + \sinh(y-z))$ and $\cosh y \cdot \sinh z = \frac{1}{2} \cdot (\sinh(y+z) - \sinh(y-z))$ and $\cosh y \cdot \cosh z = \frac{1}{2} \cdot (\cosh(y+z) + \cosh(y-z))$.

(12) $(\sinh y)^{\mathbf{2}} - (\cosh z)^{\mathbf{2}} = \sinh(y+z) \cdot \sinh(y-z) - 1$.

(13) $(\sinh y - \sinh z)^{\mathbf{2}} - (\cosh y - \cosh z)^{\mathbf{2}} = 4 \cdot (\sinh(\frac{y-z}{2}))^{\mathbf{2}}$ and $(\cosh y + \cosh z)^{\mathbf{2}} - (\sinh y + \sinh z)^{\mathbf{2}} = 4 \cdot (\cosh(\frac{y-z}{2}))^{\mathbf{2}}$.

(14) $\frac{\sinh y+\sinh z}{\sinh y-\sinh z} = \tanh(\frac{y+z}{2}) \cdot \coth(\frac{y-z}{2})$.

(15) $\frac{\cosh y+\cosh z}{\cosh y-\cosh z} = \coth(\frac{y+z}{2}) \cdot \coth(\frac{y-z}{2})$.

(16) If $y - z \neq 0$, then $\frac{\sinh y+\sinh z}{\cosh y+\cosh z} = \frac{\cosh y-\cosh z}{\sinh y-\sinh z}$.

(17) If $y + z \neq 0$, then $\frac{\sinh y-\sinh z}{\cosh y+\cosh z} = \frac{\cosh y-\cosh z}{\sinh y+\sinh z}$.

(18) $\frac{\sinh y+\sinh z}{\cosh y+\cosh z} = \tanh(\frac{y}{2} + \frac{z}{2})$ and $\frac{\sinh y-\sinh z}{\cosh y+\cosh z} = \tanh(\frac{y}{2} - \frac{z}{2})$.

(19) $\frac{\tanh y+\tanh z}{\tanh y-\tanh z} = \frac{\sinh(y+z)}{\sinh(y-z)}$.

(20) $\frac{\sinh(y-z)+\sinh y+\sinh(y+z)}{\cosh(y-z)+\cosh y+\cosh(y+z)} = \tanh y$.

(21)(i) $\sinh(y+z+w) = (\tanh y + \tanh z + \tanh w + \tanh y \cdot \tanh z \cdot \tanh w) \cdot \cosh y \cdot \cosh z \cdot \cosh w$,

(ii) $\cosh(y+z+w) = (1 + \tanh y \cdot \tanh z + \tanh z \cdot \tanh w + \tanh w \cdot \tanh y) \cdot \cosh y \cdot \cosh z \cdot \cosh w$, and

(iii) $\tanh(y+z+w) = \frac{\tanh y+\tanh z+\tanh w+\tanh y \cdot \tanh z \cdot \tanh w}{1+\tanh z \cdot \tanh w+\tanh w \cdot \tanh y+\tanh y \cdot \tanh z}$.

(22) $\cosh(2 \cdot y) + \cosh(2 \cdot z) + \cosh(2 \cdot w) + \cosh(2 \cdot (y+z+w)) = 4 \cdot \cosh(z+w) \cdot \cosh(w+y) \cdot \cosh(y+z)$.

(23) $\sinh y \cdot \sinh z \cdot \sinh(z-y) + \sinh z \cdot \sinh w \cdot \sinh(w-z) + \sinh w \cdot \sinh y \cdot \sinh(y-w) + \sinh(z-y) \cdot \sinh(w-z) \cdot \sinh(y-w) = 0$.

(24) If $x \geq 0$, then $\sinh(\frac{x}{2}) = \sqrt{\frac{\cosh x-1}{2}}$.

(25) If $x < 0$, then $\sinh(\frac{x}{2}) = -\sqrt{\frac{\cosh x-1}{2}}$.

(26) $\sinh(2 \cdot x) = 2 \cdot \sinh x \cdot \cosh x$ and $\cosh(2 \cdot x) = 2 \cdot (\cosh x)^{\mathbf{2}} - 1$ and $\tanh(2 \cdot x) = \frac{2 \cdot \tanh x}{1+(\tanh x)^{\mathbf{2}}}$.

(27) $\sinh(2 \cdot x) = \frac{2 \cdot \tanh x}{1-(\tanh x)^{\mathbf{2}}}$ and $\sinh(3 \cdot x) = \sinh x \cdot (4 \cdot (\cosh x)^{\mathbf{2}} - 1)$ and $\sinh(3 \cdot x) = 3 \cdot \sinh x - 2 \cdot \sinh x \cdot (1 - \cosh(2 \cdot x))$ and $\cosh(2 \cdot x) = 1 + 2 \cdot (\sinh x)^{\mathbf{2}}$ and $\cosh(2 \cdot x) = (\cosh x)^{\mathbf{2}} + (\sinh x)^{\mathbf{2}}$ and $\cosh(2 \cdot x) = \frac{1+(\tanh x)^{\mathbf{2}}}{1-(\tanh x)^{\mathbf{2}}}$ and $\cosh(3 \cdot x) = \cosh x \cdot (4 \cdot (\sinh x)^{\mathbf{2}} + 1)$ and $\tanh(3 \cdot x) = \frac{3 \cdot \tanh x+(\tanh x)^{3}}{1+3 \cdot (\tanh x)^{\mathbf{2}}}$.

(28) $\frac{\sinh(5\cdot x)+2\cdot\sinh(3\cdot x)+\sinh x}{\sinh(7\cdot x)+2\cdot\sinh(5\cdot x)+\sinh(3\cdot x)} = \frac{\sinh(3\cdot x)}{\sinh(5\cdot x)}$.

(29) If $x \geq 0$, then $\tanh(\frac{x}{2}) = \sqrt{\frac{\cosh x-1}{\cosh x+1}}$.

(30) If $x < 0$, then $\tanh(\frac{x}{2}) = -\sqrt{\frac{\cosh x-1}{\cosh x+1}}$.

(31)(i) $(\sinh x)^3 = \frac{\sinh(3\cdot x)-3\cdot\sinh x}{4}$,

(ii) $(\sinh x)^4 = \frac{(\cosh(4\cdot x)-4\cdot\cosh(2\cdot x))+3}{8}$,

(iii) $(\sinh x)^5 = \frac{(\sinh(5\cdot x)-5\cdot\sinh(3\cdot x))+10\cdot\sinh x}{16}$,

(iv) $(\sinh x)^6 = \frac{((\cosh(6\cdot x)-6\cdot\cosh(4\cdot x))+15\cdot\cosh(2\cdot x))-10}{32}$,

(v) $(\sinh x)^7 = \frac{((\sinh(7\cdot x)-7\cdot\sinh(5\cdot x))+21\cdot\sinh(3\cdot x))-35\cdot\sinh x}{64}$, and

(vi) $(\sinh x)^8 = \frac{(((\cosh(8\cdot x)-8\cdot\cosh(6\cdot x))+28\cdot\cosh(4\cdot x))-56\cdot\cosh(2\cdot x))+35}{128}$.

(32)(i) $(\cosh x)^3 = \frac{\cosh(3\cdot x)+3\cdot\cosh x}{4}$,

(ii) $(\cosh x)^4 = \frac{\cosh(4\cdot x)+4\cdot\cosh(2\cdot x)+3}{8}$,

(iii) $(\cosh x)^5 = \frac{\cosh(5\cdot x)+5\cdot\cosh(3\cdot x)+10\cdot\cosh x}{16}$,

(iv) $(\cosh x)^6 = \frac{\cosh(6\cdot x)+6\cdot\cosh(4\cdot x)+15\cdot\cosh(2\cdot x)+10}{32}$,

(v) $(\cosh x)^7 = \frac{\cosh(7\cdot x)+7\cdot\cosh(5\cdot x)+21\cdot\cosh(3\cdot x)+35\cdot\cosh x}{64}$, and

(vi) $(\cosh x)^8 = \frac{\cosh(8\cdot x)+8\cdot\cosh(6\cdot x)+28\cdot\cosh(4\cdot x)+56\cdot\cosh(2\cdot x)+35}{128}$.

(33) $\cosh(2 \cdot y) + \cos(2 \cdot z) = 2 + 2 \cdot ((\sinh y)^{\mathbf{2}} - (\sin z)^{\mathbf{2}})$ and $\cosh(2 \cdot y) - \cos(2 \cdot z) = 2 \cdot ((\sinh y)^{\mathbf{2}} + (\sin z)^{\mathbf{2}})$.

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