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Niemytzki Plane - an Example of Tychonoff Space Which Is Not T_4

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Summary. We continue Mizar formalization of General Topology according to the book [20] by Engelking. Niemytzki plane is defined as halfplane $y \geq 0$ with topology introduced by a neighborhood system. Niemytzki plane is not T_4. Next, the definition of Tychonoff space is given. The characterization of Tychonoff space by prebasis and the fact that Tychonoff spaces are between T_3 and T_4 is proved. The final result is that Niemytzki plane is also a Tychonoff space.

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The notation and terminology used here are introduced in the following papers: [38], [34], [15], [41], [17], [40], [35], [42], [11], [14], [12], [8], [13], [33], [10], [37], [4], [2], [1], [3], [5], [32], [39], [22], [25], [23], [29], [27], [26], [28], [43], [18], [31], [30], [36], [19], [24], [9], [16], [21], [7], and [6].

1. Preliminaries

In this paper x, y are elements of $\mathbb{R}$.

One can prove the following propositions:

(1) For all functions f, g such that $f \approx g$ and for every set A holds $(f{+\cdot}g)^{-1}(A) = f^{-1}(A) \cup g^{-1}(A)$.

(2) For all functions f, g such that $\mathrm{dom}\, f$ misses $\mathrm{dom}\, g$ and for every set A holds $(f{+\cdot}g)^{-1}(A) = f^{-1}(A) \cup g^{-1}(A)$.

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Let X be a set and let Y be a non empty real-membered set. Note that every relation between X and Y is real-yielding.

Next we state several propositions:

(3) For all sets x, a and for every function f such that $a \in \mathrm{dom}\, f$ holds $(\mathrm{commute}(x \dot{\longmapsto} f))(a) = x \dot{\longmapsto} f(a)$.

(4) Let b be a set and f be a function. Then $b \in \mathrm{dom}\,\mathrm{commute}(f)$ if and only if there exists a set a and there exists a function g such that $a \in \mathrm{dom}\, f$ and $g = f(a)$ and $b \in \mathrm{dom}\, g$.

(5) Let a, b be sets and f be a function. Then $a \in \mathrm{dom}(\mathrm{commute}(f))(b)$ if and only if there exists a function g such that $a \in \mathrm{dom}\, f$ and $g = f(a)$ and $b \in \mathrm{dom}\, g$.

(6) For all sets a, b and for all functions f, g such that $a \in \mathrm{dom}\, f$ and $g = f(a)$ and $b \in \mathrm{dom}\, g$ holds $(\mathrm{commute}(f))(b)(a) = g(b)$.

(7) For every set a and for all functions f, g, h such that $h = f \cup g$ holds $(\mathrm{commute}(h))(a) = (\mathrm{commute}(f))(a) \cup (\mathrm{commute}(g))(a)$.

Let us note that every finite subset of $\mathbb{R}$ is bounded.

The following propositions are true:

(8) For all real numbers a, b, c, d such that $a < b$ and $c \leq d$ holds $]a, c[\cap [b, d] = [b, c[$.

(9) For all real numbers a, b, c, d such that $a \geq b$ and $c > d$ holds $]a, c[\cap [b, d] =]a, d]$.

(10) For all real numbers a, b, c, d such that $a \leq b$ and $b < c$ and $c \leq d$ holds $[a, c[\cup]b, d] = [a, d]$.

(11) For all real numbers a, b, c, d such that $a \leq b$ and $b < c$ and $c \leq d$ holds $[a, c[\cap]b, d] =]b, c[$.

(12) For all sets X, Y holds $\prod \langle X, Y \rangle \approx [\!: X, Y :\!]$ and $\overline{\overline{\prod \langle X, Y \rangle}} = \overline{\overline{X}} \cdot \overline{\overline{Y}}$.

In this article we present several logical schemes. The scheme *SCH1* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and a unary predicate $\mathcal{P}$, and states that:

> There exists a function f from $\mathcal{C}$ into $\mathcal{B}$ such that for every element a of $\mathcal{A}$ holds
>
> (i) if $\mathcal{P}[a]$, then $f(a) = \mathcal{F}(a)$, and
>
> (ii) if not $\mathcal{P}[a]$, then $f(a) = \mathcal{G}(a)$

provided the parameters meet the following conditions:

- $\mathcal{C} \subseteq \mathcal{A}$, and
- For every element a of $\mathcal{A}$ such that $a \in \mathcal{C}$ holds if $\mathcal{P}[a]$, then $\mathcal{F}(a) \in \mathcal{B}$ and if not $\mathcal{P}[a]$, then $\mathcal{G}(a) \in \mathcal{B}$.

The scheme *SCH2* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, three unary functors $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ yielding sets, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

There exists a function f from $\mathcal{C}$ into $\mathcal{B}$ such that for every element a of $\mathcal{A}$ holds

(i) if $\mathcal{P}[a]$, then $f(a) = \mathcal{F}(a)$,

(ii) if not $\mathcal{P}[a]$ and $\mathcal{Q}[a]$, then $f(a) = \mathcal{G}(a)$, and

(iii) if not $\mathcal{P}[a]$ and not $\mathcal{Q}[a]$, then $f(a) = \mathcal{H}(a)$

provided the parameters meet the following conditions:

- $\mathcal{C} \subseteq \mathcal{A}$, and
- For every element a of $\mathcal{A}$ such that $a \in \mathcal{C}$ holds if $\mathcal{P}[a]$, then $\mathcal{F}(a) \in \mathcal{B}$ and if not $\mathcal{P}[a]$ and $\mathcal{Q}[a]$, then $\mathcal{G}(a) \in \mathcal{B}$ and if not $\mathcal{P}[a]$ and not $\mathcal{Q}[a]$, then $\mathcal{H}(a) \in \mathcal{B}$.

The following four propositions are true:

(13) For all real numbers a, b holds $|[a, b]|^{\mathbf{2}} = a^{\mathbf{2}} + b^{\mathbf{2}}$.

(14) Let X be a topological space, Y be a non empty topological space, A, B be closed subsets of X, f be a continuous function from $X \restriction A$ into Y, and g be a continuous function from $X \restriction B$ into Y. If $f \approx g$, then $f{+\cdot}g$ is a continuous function from $X \restriction (A \cup B)$ into Y.

(15) Let X be a topological space, Y be a non empty topological space, and A, B be closed subsets of X. Suppose A misses B. Let f be a continuous function from $X \restriction A$ into Y and g be a continuous function from $X \restriction B$ into Y. Then $f{+\cdot}g$ is a continuous function from $X \restriction (A \cup B)$ into Y.

(16) Let X be a topological space, Y be a non empty topological space, A be an open closed subset of X, f be a continuous function from $X \restriction A$ into Y, and g be a continuous function from $X \restriction A^{\rm c}$ into Y. Then $f{+\cdot}g$ is a continuous function from X into Y.

2. Niemytzki Plane

One can prove the following proposition

(17) For every natural number n and for every point a of $\mathcal{E}^n_{\rm T}$ and for every positive real number r holds $a \in \mathrm{Ball}(a, r)$.

The subset $(y = 0)$-line of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 1) $(y = 0)\text{-line} = \{[x, 0]\}$.

The subset $(y \geq 0)$-plane of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 2) $(y \geq 0)\text{-plane} = \{[x, y] : y \geq 0\}$.

We now state several propositions:

(18) For all sets a, b holds $\langle a, b\rangle \in (y = 0)$-line iff $a \in \mathbb{R}$ and $b = 0$.

(19) For all real numbers a, b holds $[a, b] \in (y = 0)$-line iff $b = 0$.

(20) $\overline{\overline{(y = 0)\text{-line}}} = \mathfrak{c}$.

(21) For all sets a, b holds $\langle a, b\rangle \in (y \geq 0)$-plane iff $a \in \mathbb{R}$ and there exists y such that $b = y$ and $y \geq 0$.

(22) For all real numbers a, b holds $[a, b] \in (y \geq 0)$-plane iff $b \geq 0$.

Let us note that $(y = 0)$-line is non empty and $(y \geq 0)$-plane is non empty.

We now state several propositions:

(23) $(y = 0)$-line $\subseteq (y \geq 0)$-plane.

(24) For all real numbers a, b, r such that $r > 0$ holds $\mathrm{Ball}([a, b], r) \subseteq (y \geq 0)$-plane iff $r \leq b$.

(25) For all real numbers a, b, r such that $r > 0$ and $b \geq 0$ holds $\mathrm{Ball}([a, b], r)$ misses $(y = 0)$-line iff $r \leq b$.

(26) Let n be a natural number, a, b be elements of $\mathcal{E}^n_{\mathrm{T}}$, and r_1, r_2 be positive real numbers. If $|a - b| \leq r_1 - r_2$, then $\mathrm{Ball}(b, r_2) \subseteq \mathrm{Ball}(a, r_1)$.

(27) For every real number a and for all positive real numbers r_1, r_2 such that $r_1 \leq r_2$ holds $\mathrm{Ball}([a, r_1], r_1) \subseteq \mathrm{Ball}([a, r_2], r_2)$.

(28) Let T_1, T_2 be non empty topological spaces, B_1 be a neighborhood system of T_1, and B_2 be a neighborhood system of T_2. Suppose $B_1 = B_2$. Then the topological structure of $T_1 =$ the topological structure of T_2.

In the sequel r is an element of $\mathbb{R}$.

Niemytzki plane is a strict non empty topological space and is defined by the conditions (Def. 3).

(Def. 3)(i) The carrier of Niemytzki plane $= (y \geq 0)$-plane, and

(ii) there exists a neighborhood system B of Niemytzki plane such that for every x holds $B([x, 0]) = \{\mathrm{Ball}([x, r], r) \cup \{[x, 0]\} : r > 0\}$ and for all x, y such that $y > 0$ holds $B([x, y]) = \{\mathrm{Ball}([x, y], r) \cap (y \geq 0)\text{-plane} : r > 0\}$.

The following propositions are true:

(29) $(y \geq 0)$-plane $\setminus (y = 0)$-line is an open subset of Niemytzki plane.

(30) $(y = 0)$-line is a closed subset of Niemytzki plane.

(31) Let x be a real number and r be a positive real number. Then $\mathrm{Ball}([x, r], r) \cup \{[x, 0]\}$ is an open subset of Niemytzki plane.

(32) Let x be a real number and y, r be positive real numbers. Then $\mathrm{Ball}([x, y], r) \cap (y \geq 0)$-plane is an open subset of Niemytzki plane.

(33) Let x, y be real numbers and r be a positive real number. If $r \leq y$, then $\mathrm{Ball}([x, y], r)$ is an open subset of Niemytzki plane.

(34) Let p be a point of Niemytzki plane and r be a positive real number. Then there exists a point a of $\mathcal{E}^2_{\mathrm{T}}$ and there exists an open subset U of Niemytzki plane such that $p \in U$ and $a \in U$ and for every point b of $\mathcal{E}^2_{\mathrm{T}}$ such that $b \in U$ holds $|b - a| < r$.

(35) Let x, y be real numbers and r be a positive real number. Then there exist rational numbers w, v such that $[w, v] \in \mathrm{Ball}([x, y], r)$ and $[w, v] \neq [x,

y].

(36) Let A be a subset of Niemytzki plane. If $A = ((y \geq 0)$-plane $\setminus (y = 0)$-line$) \cap \prod\langle \mathbb{Q}, \mathbb{Q}\rangle$, then for every set x holds $\overline{A \setminus \{x\}} = \Omega_{\text{Niemytzki plane}}$.

(37) Let A be a subset of Niemytzki plane. If $A = (y \geq 0)$-plane$\setminus(y = 0)$-line, then for every set x holds $\overline{A \setminus \{x\}} = \Omega_{\text{Niemytzki plane}}$.

(38) For every subset A of Niemytzki plane such that $A = (y \geq 0)$-plane$\setminus(y = 0)$-line holds $\overline{A} = \Omega_{\text{Niemytzki plane}}$.

(39) For every subset A of Niemytzki plane such that $A = (y = 0)$-line holds $\overline{A} = A$ and $\operatorname{Int} A = \emptyset$.

(40) $((y \geq 0)$-plane $\setminus (y = 0)$-line$) \cap \prod\langle \mathbb{Q}, \mathbb{Q}\rangle$ is a dense subset of Niemytzki plane.

(41) $((y \geq 0)$-plane $\setminus (y = 0)$-line$) \cap \prod\langle \mathbb{Q}, \mathbb{Q}\rangle$ is a dense-in-itself subset of Niemytzki plane.

(42) $(y \geq 0)$-plane $\setminus (y = 0)$-line is a dense subset of Niemytzki plane.

(43) $(y \geq 0)$-plane$\setminus(y = 0)$-line is a dense-in-itself subset of Niemytzki plane.

(44) $(y = 0)$-line is a nowhere dense subset of Niemytzki plane.

(45) For every subset A of Niemytzki plane such that $A = (y = 0)$-line holds $\operatorname{Der} A$ is empty.

(46) Every subset of $(y = 0)$-line is a closed subset of Niemytzki plane.

(47) $\mathbb{Q}$ is a dense subset of Sorgenfrey line.

(48) Sorgenfrey line is separable.

(49) Niemytzki plane is separable.

(50) Niemytzki plane is a T_1 space.

(51) Niemytzki plane is not T_4.

3. Tychonoff Spaces

Let T be a topological space. We say that T is Tychonoff if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) T is a T_1 space, and

(ii) for every closed subset A of T and for every point a of T such that $a \in A^{\rm c}$ there exists a continuous function f from T into $\mathbb{I}$ such that $f(a) = 0$ and $f^\circ A \subseteq \{1\}$.

Let us observe that every topological space which is Tychonoff is also T_1 and T_3 and every non empty topological space which is T_1 and T_4 is also Tychonoff.

We now state the proposition

(52) Let X be a T_1 topological space. Suppose X is Tychonoff. Let B be a prebasis of X, x be a point of X, and V be a subset of X. Suppose $x \in V$

and $V \in B$. Then there exists a continuous function f from X into $\mathbb{I}$ such that $f(x) = 0$ and $f^{\circ}V^{\mathrm{c}} \subseteq \{1\}$.

Let X be a set and let Y be a non empty real-membered set. Observe that every relation between X and Y is real-yielding.

The following propositions are true:

(53) Let X be a topological space, R be a non empty subspace of $\mathbb{R}^{\mathbf{1}}$, f, g be continuous functions from X into R, and A be a subset of X. Suppose that for every point x of X holds $x \in A$ iff $f(x) \leq g(x)$. Then A is closed.

(54) Let X be a topological space, R be a non empty subspace of $\mathbb{R}^{\mathbf{1}}$, and f, g be continuous functions from X into R. Then there exists a continuous function h from X into R such that for every point x of X holds $h(x) = \max(f(x), g(x))$.

(55) Let X be a non empty topological space, R be a non empty subspace of $\mathbb{R}^{\mathbf{1}}$, A be a finite non empty set, and F be a many sorted function indexed by A. Suppose that for every set a such that $a \in A$ holds $F(a)$ is a continuous function from X into R. Then there exists a continuous function f from X into R such that for every point x of X and for every finite non empty subset S of $\mathbb{R}$ if $S = \mathrm{rng}(\mathrm{commute}(F))(x)$, then $f(x) = \max S$.

(56) Let X be a T_1 non empty topological space and B be a prebasis of X. Suppose that for every point x of X and for every subset V of X such that $x \in V$ and $V \in B$ there exists a continuous function f from X into $\mathbb{I}$ such that $f(x) = 0$ and $f^{\circ}V^{\mathrm{c}} \subseteq \{1\}$. Then X is Tychonoff.

(57) Sorgenfrey line is a T_1 space.

(58) For every real number x holds $]-\infty, x[$ is a closed subset of Sorgenfrey line.

(59) For every real number x holds $]-\infty, x]$ is a closed subset of Sorgenfrey line.

(60) For every real number x holds $[x, +\infty[$ is a closed subset of Sorgenfrey line.

(61) For all real numbers x, y holds $[x, y[$ is a closed subset of Sorgenfrey line.

(62) Let x be a real number and w be a rational number. Suppose $x < w$. Then there exists a continuous function f from Sorgenfrey line into $\mathbb{I}$ such that for every point a of Sorgenfrey line holds

(i) if $a \in [x, w[$, then $f(a) = 0$, and

(ii) if $a \notin [x, w[$, then $f(a) = 1$.

(63) Sorgenfrey line is Tychonoff.

4. Niemytzki Plane is Tychonoff Space

Let x be a real number and let r be a positive real number. The functor $+(x,r)$ yielding a function from Niemytzki plane into $\mathbb{I}$ is defined by the conditions (Def. 5).

(Def. 5)(i) $(+(x,r))([x,0]) = 0$, and

(ii) for every real number a and for every non negative real number b holds if $a \neq x$ or $b \neq 0$ and if $[a,b] \notin \mathrm{Ball}([x,r],r)$, then $(+(x,r))([a,b]) = 1$ and if $[a,b] \in \mathrm{Ball}([x,r],r)$, then $(+(x,r))([a,b]) = \frac{|[x,0]-[a,b]|^{\mathbf{2}}}{2\cdot r\cdot b}$.

One can prove the following propositions:

(64) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \geq 0$. Let x be a real number and r be a positive real number. If $(+(x,r))(p) = 0$, then $p = [x,0]$.

(65) For all real numbers x, y and for every positive real number r such that $x \neq y$ holds $(+(x,r))([y,0]) = 1$.

(66) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$, x be a real number, and a, r be positive real numbers. If $a \leq 1$ and $|p-[x,r\cdot a]| = r\cdot a$ and $p_{\mathbf{2}} \neq 0$, then $(+(x,r))(p) = a$.

(67) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$, x, a be real numbers, and r be a positive real number. If $0 \leq a$ and $a \leq 1$ and $|p-[x,r\cdot a]| < r\cdot a$, then $(+(x,r))(p) < a$.

(68) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \geq 0$. Let x, a be real numbers and r be a positive real number. If $0 \leq a$ and $a < 1$ and $|p - [x, r\cdot a]| > r\cdot a$, then $(+(x,r))(p) > a$.

(69) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \geq 0$. Let x, a, b be real numbers and r be a positive real number. Suppose $0 \leq a$ and $b \leq 1$ and $(+(x,r))(p) \in]a,b[$. Then there exists a positive real number r_1 such that $r_1 \leq p_{\mathbf{2}}$ and $\mathrm{Ball}(p,r_1) \subseteq (+(x,r))^{-1}(]a,b[)$.

(70) For every real number x and for all positive real numbers a, r holds $\mathrm{Ball}([x,r\cdot a],r\cdot a) \subseteq (+(x,r))^{-1}(]0,a[)$.

(71) For every real number x and for all positive real numbers a, r holds $\mathrm{Ball}([x,r\cdot a],r\cdot a) \cup \{[x,0]\} \subseteq (+(x,r))^{-1}([0,a[)$.

(72) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \geq 0$. Let x, a be real numbers and r be a positive real number. If $0 < (+(x,r))(p)$ and $(+(x,r))(p) < a$ and $a \leq 1$, then $p \in \mathrm{Ball}([x,r\cdot a],r\cdot a)$.

(73) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} > 0$. Let x, a be real numbers and r be a positive real number. Suppose $0 \leq a$ and $a < (+(x,r))(p)$. Then there exists a positive real number r_1 such that $r_1 \leq p_{\mathbf{2}}$ and $\mathrm{Ball}(p,r_1) \subseteq (+(x,r))^{-1}(]a,1])$.

(74) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} = 0$. Let x be a real number and r be a positive real number. Suppose $(+(x,r))(p) = 1$. Then there exists a positive real number r_1 such that $\mathrm{Ball}([p_{\mathbf{1}},r_1],r_1) \cup \{p\} \subseteq (+(x,r))^{-1}(\{1\})$.

(75) Let T be a non empty topological space, S be a subspace of T, and B be a basis of T. Then $\{A \cap \Omega_S; A$ ranges over subsets of T: $A \in B \ \wedge \ A$ meets $\Omega_S\}$ is a basis of S.

(76) $\{]a, b[; a$ ranges over real numbers, b ranges over real numbers: $a < b\}$ is a basis of $\mathbb{R}^{\mathbf{1}}$.

(77) Let T be a topological space, U, V be subsets of T, and B be a set. If $U \in B$ and $V \in B$ and $B \cup \{U \cup V\}$ is a basis of T, then B is a basis of T.

(78) $\{[0, a[; a$ ranges over real numbers: $0 < a \ \wedge \ a \leq 1\} \cup \{]a, 1]; a$ ranges over real numbers: $0 \leq a \ \wedge \ a < 1\} \cup \{]a, b[; a$ ranges over real numbers, b ranges over real numbers: $0 \leq a \ \wedge \ a < b \ \wedge \ b \leq 1\}$ is a basis of $\mathbb{I}$.

(79) Let T be a non empty topological space and f be a function from T into $\mathbb{I}$. Then f is continuous if and only if for all real numbers a, b such that $0 \leq a$ and $a < 1$ and $0 < b$ and $b \leq 1$ holds $f^{-1}([0, b[)$ is open and $f^{-1}(]a, 1])$ is open.

Let x be a real number and let r be a positive real number. Note that $+(x, r)$ is continuous.

We now state the proposition

(80) Let U be a subset of Niemytzki plane and given x, r. Suppose $U = \mathrm{Ball}([x, r], r) \cup \{[x, 0]\}$. Then there exists a continuous function f from Niemytzki plane into $\mathbb{I}$ such that

(i) $f([x, 0]) = 0$, and

(ii) for all real numbers a, b holds if $[a, b] \in U^{\mathrm{c}}$, then $f([a, b]) = 1$ and if $[a, b] \in U \setminus \{[x, 0]\}$, then $f([a, b]) = \frac{|[x,0]-[a,b]|^{\mathbf{2}}}{2 \cdot r \cdot b}$.

Let x, y be real numbers and let r be a positive real number. The functor $+(x, y, r)$ yields a function from Niemytzki plane into $\mathbb{I}$ and is defined by the condition (Def. 6).

(Def. 6) Let a be a real number and b be a non negative real number. Then

(i) if $[a, b] \notin \mathrm{Ball}([x, y], r)$, then $(+(x, y, r))([a, b]) = 1$, and

(ii) if $[a, b] \in \mathrm{Ball}([x, y], r)$, then $(+(x, y, r))([a, b]) = \frac{|[x,y]-[a,b]|}{r}$.

The following propositions are true:

(81) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \geq 0$. Let x be a real number, y be a non negative real number, and r be a positive real number. Then $(+(x, y, r))(p) = 0$ if and only if $p = [x, y]$.

(82) Let x be a real number, y be a non negative real number, and r, a be positive real numbers. If $a \leq 1$, then $(+(x, y, r))^{-1}([0, a[) = \mathrm{Ball}([x, y], r \cdot a) \cap (y \geq 0)$-plane.

(83) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} > 0$. Let x be a real number, a be a non negative real number, and y, r be positive real numbers. If $(+(x, y, r))(p) > a$, then $|[x, y] - p| > r \cdot a$ and $\mathrm{Ball}(p, |[x, y] - p| - r \cdot a) \cap (y \geq 0)$-plane $\subseteq (+(x, y, r))^{-1}(]a, 1])$.

(84) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} = 0$. Let x be a real number, a be a non negative real number, and y, r be positive real numbers. Suppose $(+(x, y, r))(p) > a$. Then $|[x, y] - p| > r \cdot a$ and there exists a positive real number r_1 such that $r_1 = \frac{|[x,y]-p|-r \cdot a}{2}$ and $\mathrm{Ball}([p_{\mathbf{1}}, r_1], r_1) \cup \{p\} \subseteq (+(x, y, r))^{-1}(]a, 1])$.

Let x be a real number and let y, r be positive real numbers. One can verify that $+(x, y, r)$ is continuous.

We now state three propositions:

(85) Let U be a subset of Niemytzki plane and given x, y, r. Suppose $y > 0$ and $U = \mathrm{Ball}([x, y], r) \cap (y \geq 0)$-plane. Then there exists a continuous function f from Niemytzki plane into $\mathbb{I}$ such that $f([x, y]) = 0$ and for all real numbers a, b holds if $[a, b] \in U^{\mathrm{c}}$, then $f([a, b]) = 1$ and if $[a, b] \in U$, then $f([a, b]) = \frac{|[x,y]-[a,b]|}{r}$.

(86) Niemytzki plane is a T_1 space.

(87) Niemytzki plane is Tychonoff.

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