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Multiplication of Polynomials using Discrete Fourier Transformation

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Summary. In this article we define the Discrete Fourier Transformation for univariate polynomials and show that multiplication of polynomials can be carried out by two Fourier Transformations with a vector multiplication inbetween. Our proof follows the standard one found in the literature and uses Vandermonde matrices, see e.g. [27].

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The articles [20], [26], [28], [5], [6], [19], [12], [3], [18], [13], [25], [2], [4], [23], [8], [24], [14], [10], [11], [16], [7], [29], [22], [1], [15], [9], [21], and [17] provide the notation and terminology for this paper.

1. Preliminaries

The following proposition is true

(1) Let n be an element of $\mathbb{N}$, L be a unital integral domain-like non degenerated non empty double loop structure, and x be an element of L. If $x \neq 0_L$, then $x^n \neq 0_L$.

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One can verify that every associative right unital add-associative right zeroed right complementable left distributive non empty double loop structure which is field-like is also integral domain-like.

The following four propositions are true:

(2) Let L be an add-associative right zeroed right complementable associative commutative left unital field-like distributive non empty double loop structure and x, y be elements of L. If $x \neq 0_L$ and $y \neq 0_L$, then $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$.

(3) Let L be an associative commutative left unital distributive field-like non empty double loop structure and z, z_1 be elements of L. If $z \neq 0_L$, then $z_1 = \frac{z_1 \cdot z}{z}$.

(4) Let L be a left zeroed right zeroed add-associative right complementable non empty double loop structure, m be an element of $\mathbb{N}$, and s be a finite sequence of elements of L. Suppose $\operatorname{len} s = m$ and for every element k of $\mathbb{N}$ such that $1 \leq k$ and $k \leq m$ holds $s_k = 1_L$. Then $\sum s = m \cdot 1_L$.

(5) Let L be an add-associative right zeroed right complementable associative commutative left unital distributive field-like non empty double loop structure, s be a finite sequence of elements of L, and q be an element of L. Suppose $q \neq 1_L$ and for every natural number i such that $1 \leq i$ and $i \leq \operatorname{len} s$ holds $s(i) = q^{i -' 1}$. Then $\sum s = \frac{1_L - q^{\operatorname{len} s}}{1_L - q}$.

Let L be a unital non empty double loop structure and let m be an element of $\mathbb{N}$. The functor m_L yielding an element of L is defined as follows:

(Def. 1) $m_L = m \cdot 1_L$.

Next we state several propositions:

(6) Let L be a field and m, n, k be elements of $\mathbb{N}$. Suppose $m > 0$ and $n > 0$. Let M_1 be a matrix over L of dimension $m \times n$ and M_2 be a matrix over L of dimension $n \times k$. Then $(m_L \cdot M_1) \cdot M_2 = m_L \cdot (M_1 \cdot M_2)$.

(7) Let L be a non empty zero structure, p be an algebraic sequence of L, and i be an element of $\mathbb{N}$. If $p(i) \neq 0_L$, then $\operatorname{len} p \geq i + 1$.

(8) For every non empty zero structure L and for every algebraic sequence s of L such that $\operatorname{len} s > 0$ holds $s(\operatorname{len} s - 1) \neq 0_L$.

(9) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital integral domain-like non empty double loop structure and p, q be polynomials of L. If $\operatorname{len} p > 0$ and $\operatorname{len} q > 0$, then $\operatorname{len}(p * q) \leq \operatorname{len} p + \operatorname{len} q$.

(10) Let L be an associative non empty double loop structure, k, l be elements of L, and s_1 be a sequence of L. Then $k \cdot (l \cdot s_1) = (k \cdot l) \cdot s_1$.

2. Multiplication of Algebraic Sequences

Let L be a non empty double loop structure and let m_1, m_2 be sequences of L. The functor $m_1 \cdot m_2$ yields a sequence of L and is defined as follows:

(Def. 2) For every element i of $\mathbb{N}$ holds $(m_1 \cdot m_2)(i) = m_1(i) \cdot m_2(i)$.

Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure and let m_1, m_2 be algebraic sequences of L. Observe that $m_1 \cdot m_2$ is finite-Support.

We now state two propositions:

(11) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and m_1, m_2 be algebraic sequences of L. Then $\operatorname{len}(m_1 \cdot m_2) \leq \min(\operatorname{len} m_1, \operatorname{len} m_2)$.

(12) Let L be an add-associative right zeroed right complementable distributive integral domain-like non empty double loop structure and m_1, m_2 be algebraic sequences of L. If $\operatorname{len} m_1 = \operatorname{len} m_2$, then $\operatorname{len}(m_1 \cdot m_2) = \operatorname{len} m_1$.

3. Powers in Double Loop Structures

Let L be an associative commutative left unital distributive field-like non empty double loop structure, let a be an element of L, and let i be an integer. The functor a^i yielding an element of L is defined as follows:

(Def. 3) $a^i = \begin{cases} \operatorname{power}_L(a,\, i), & \text{if } 0 \leq i, \\ \operatorname{power}_L(a,\, |i|)^{-1}, & \text{otherwise.} \end{cases}$

Next we state a number of propositions:

(13) Let L be an associative commutative left unital distributive field-like non empty double loop structure and x be an element of L. Then $x^0 = 1_L$.

(14) Let L be an associative commutative left unital distributive field-like non empty double loop structure and x be an element of L. Then $x^1 = x$.

(15) Let L be an associative commutative left unital distributive field-like non empty double loop structure and x be an element of L. Then $x^{-1} = x^{-1}$.

(16) Let L be an associative commutative left unital distributive field-like non degenerated non empty double loop structure and i be an integer. Then $(1_L)^i = 1_L$.

(17) Let L be an associative commutative left unital distributive field-like non empty double loop structure, x be an element of L, and n be an element of $\mathbb{N}$. Then $x^{n+1} = x^n \cdot x$ and $x^{n+1} = x \cdot x^n$.

(18) Let L be an add-associative right zeroed right complementable associative commutative left unital distributive field-like non degenerated non empty double loop structure, i be an integer, and x be an element of L. If $x \neq 0_L$, then $(x^i)^{-1} = x^{-i}$.

(19) For every field L and for every integer j and for every element x of L such that $x \neq 0_L$ holds $x^{j+1} = x^j \cdot x^1$.

(20) For every field L and for every integer j and for every element x of L such that $x \neq 0_L$ holds $x^{j-1} = x^j \cdot x^{-1}$.

(21) For every field L and for all integers i, j and for every element x of L such that $x \neq 0_L$ holds $x^i \cdot x^j = x^{i+j}$.

(22) Let L be a field-like associative unital add-associative right zeroed right complementable left distributive commutative non degenerated non empty double loop structure, k be an element of $\mathbb{N}$, and x be an element of L. If $x \neq 0_L$, then $(x^{-1})^k = x^{-k}$.

(23) Let L be a field and x be an element of L. Suppose $x \neq 0_L$. Let i, j, k be natural numbers. Then $x^{(i-1)\cdot(k-1)} \cdot x^{-(j-1)\cdot(k-1)} = x^{(i-j)\cdot(k-1)}$.

(24) Let L be an associative commutative left unital distributive field-like non empty double loop structure, x be an element of L, and n, m be elements of $\mathbb{N}$. Then $x^{n\cdot m} = (x^n)^m$.

(25) For every field L and for every element x of L such that $x \neq 0_L$ and for every integer i holds $(x^{-1})^i = (x^i)^{-1}$.

(26) For every field L and for every element x of L such that $x \neq 0_L$ and for all integers i, j holds $x^{i\cdot j} = (x^i)^j$.

(27) Let L be an associative commutative left unital distributive field-like non empty double loop structure, x be an element of L, and i, k be elements of $\mathbb{N}$. If $1 \leq k$, then $x^{i\cdot(k-1)} = (x^i)^{k-1}$.

4. Conversion between Algebraic Sequences and Matrices

Let m be a natural number, let L be a non empty zero structure, and let p be an algebraic sequence of L. The functor $\mathrm{mConv}(p, m)$ yielding a matrix over L of dimension $m \times 1$ is defined as follows:

(Def. 4) For every natural number i such that $1 \leq i$ and $i \leq m$ holds $(\mathrm{mConv}(p, m))_{i,1} = p(i-1)$.

We now state two propositions:

(28) Let m be a natural number. Suppose $m > 0$. Let L be a non empty zero structure and p be an algebraic sequence of L. Then $\mathrm{len}\, \mathrm{mConv}(p, m) = m$ and $\mathrm{width}\, \mathrm{mConv}(p, m) = 1$ and for every natural number i such that $i < m$ holds $(\mathrm{mConv}(p, m))_{i+1,1} = p(i)$.

(29) Let m be a natural number. Suppose $m > 0$. Let L be a non empty zero structure, a be an algebraic sequence of L, and M be a matrix over L of dimension $m \times 1$. Suppose that for every natural number i such that $i < m$ holds $M_{i+1,1} = a(i)$. Then $\mathrm{mConv}(a, m) = M$.

Let L be a non empty zero structure and let M be a matrix over L. The functor aConv M yielding an algebraic sequence of L is defined by the conditions (Def. 5).

(Def. 5)(i) For every natural number i such that $i < \operatorname{len} M$ holds $(\text{aConv}\, M)(i) = M_{i+1,1}$, and

(ii) for every natural number i such that $i \geq \operatorname{len} M$ holds $(\text{aConv}\, M)(i) = 0_L$.

5. Primitive Roots, DFT and Vandermonde Matrix

Let L be a unital non empty double loop structure, let x be an element of L, and let n be an element of $\mathbb{N}$. We say that x is primitive root of degree n if and only if:

(Def. 6) $n \neq 0$ and $x^n = 1_L$ and for every element i of $\mathbb{N}$ such that $0 < i$ and $i < n$ holds $x^i \neq 1_L$.

We now state three propositions:

(30) Let L be a unital add-associative right zeroed right complementable right distributive non degenerated non empty double loop structure and n be an element of $\mathbb{N}$. Then 0_L is !not primitive root of degree n.

(31) Let L be an add-associative right zeroed right complementable associative commutative unital distributive field-like non degenerated non empty double loop structure, m be an element of $\mathbb{N}$, and x be an element of L. If x is primitive root of degree m, then x^{-1} is primitive root of degree m.

(32) Let L be an add-associative right zeroed right complementable associative commutative left unital distributive field-like non degenerated non empty double loop structure, m be an element of $\mathbb{N}$, and x be an element of L. Suppose x is primitive root of degree m. Let i, j be natural numbers. If $1 \leq i$ and $i \leq m$ and $1 \leq j$ and $j \leq m$ and $i \neq j$, then $x^{i-j} \neq 1_L$.

Let m be a natural number, let L be a unital non empty double loop structure, let p be a polynomial of L, and let x be an element of L. The functor $\text{DFT}(p, x, m)$ yielding an algebraic sequence of L is defined by the conditions (Def. 7).

(Def. 7)(i) For every element i of $\mathbb{N}$ such that $i < m$ holds $(\text{DFT}(p, x, m))(i) = \text{eval}(p, x^i)$, and

(ii) for every element i of $\mathbb{N}$ such that $i \geq m$ holds $(\text{DFT}(p, x, m))(i) = 0_L$.

The following propositions are true:

(33) Let m be a natural number, L be a unital non empty double loop structure, and x be an element of L. Then $\text{DFT}(\mathbf{0}.\, L, x, m) = \mathbf{0}.\, L$.

(34) Let m be a natural number, L be a field, p, q be polynomials of L, and x be an element of L. Then $\text{DFT}(p, x, m) \cdot \text{DFT}(q, x, m) = \text{DFT}(p * q, x, m)$.

Let L be an associative commutative left unital distributive field-like non empty double loop structure, let m be a natural number, and let x be an element of L. The functor Vandermonde(x, m) yielding a matrix over L of dimension m is defined as follows:

(Def. 8) For all natural numbers i, j such that $1 \leq i$ and $i \leq m$ and $1 \leq j$ and $j \leq m$ holds $(\mathrm{Vandermonde}(x, m))_{i,j} = x^{(i-1)\cdot(j-1)}$.

Let L be an associative commutative left unital distributive field-like non empty double loop structure, let m be a natural number, and let x be an element of L. We introduce VM(x, m) as a synonym of Vandermonde(x, m).

One can prove the following propositions:

(35) Let L be a field and m, n be natural numbers. Suppose $m > 0$. Let M be a matrix over L of dimension $m \times n$. Then $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_L^{m \times m} \cdot M = M$.

(36) Let L be a field and m be an element of $\mathbb{N}$. Suppose $0 < m$. Let u, v, u_1 be matrices over L of dimension m. Suppose that for all natural numbers i, j such that $1 \leq i$ and $i \leq m$ and $1 \leq j$ and $j \leq m$ holds $(u \cdot v)_{i,j} = m_L \cdot (u_1)_{i,j}$. Then $u \cdot v = m_L \cdot u_1$.

(37) Let L be a field, x be an element of L, s be a finite sequence of elements of L, and i, j, m be elements of $\mathbb{N}$. Suppose that x is primitive root of degree m and $1 \leq i$ and $i \leq m$ and $1 \leq j$ and $j \leq m$ and $\mathrm{len}\, s = m$ and for every natural number k such that $1 \leq k$ and $k \leq m$ holds $s_k = x^{(i-j)\cdot(k-1)}$. Then $(\mathrm{VM}(x, m) \cdot \mathrm{VM}(x^{-1}, m))_{i,j} = \sum s$.

(38) Let L be a field, m, i, j be elements of $\mathbb{N}$, and x be an element of L. Suppose $i \neq j$ and $1 \leq i$ and $i \leq m$ and $1 \leq j$ and $j \leq m$ and x is primitive root of degree m. Then $(\mathrm{VM}(x, m) \cdot \mathrm{VM}(x^{-1}, m))_{i,j} = 0_L$.

(39) Let L be a field and m be an element of $\mathbb{N}$. Suppose $m > 0$. Let x be an element of L. If x is primitive root of degree m, then $\mathrm{VM}(x, m) \cdot \mathrm{VM}(x^{-1}, m) = m_L \cdot \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_L^{m \times m}$.

(40) Let L be a field, m be an element of $\mathbb{N}$, and x be an element of L. If $m > 0$ and x is primitive root of degree m, then $\mathrm{VM}(x, m) \cdot \mathrm{VM}(x^{-1}, m) = \mathrm{VM}(x^{-1}, m) \cdot \mathrm{VM}(x, m)$.

6. DFT-Multiplication of Polynomials

We now state four propositions:

(41) Let L be a field, p be a polynomial of L, and m be an element of $\mathbb{N}$. Suppose $m > 0$ and $\operatorname{len} p \leq m$. Let x be an element of L and i be an element of $\mathbb{N}$. If $i < m$, then $(\mathrm{DFT}(p, x, m))(i) = (\mathrm{VM}(x, m) \cdot \mathrm{mConv}(p, m))_{i+1,1}$.

(42) Let L be a field, p be a polynomial of L, and m be a natural number. If $0 < m$ and $\operatorname{len} p \leq m$, then for every element x of L holds $\mathrm{DFT}(p, x, m) = \mathrm{aConv}(\mathrm{VM}(x, m) \cdot \mathrm{mConv}(p, m))$.

(43) Let L be a field, p, q be polynomials of L, and m be an element of $\mathbb{N}$. Suppose $m > 0$ and $\operatorname{len} p \leq m$ and $\operatorname{len} q \leq m$. Let x be an element of L. If x is primitive root of degree $2 \cdot m$, then $\mathrm{DFT}(\mathrm{DFT}(p * q, x, 2 \cdot m), x^{-1}, 2 \cdot m) = (2 \cdot m)_L \cdot (p * q)$.

(44) Let L be a field, p, q be polynomials of L, and m be an element of $\mathbb{N}$. Suppose $m > 0$ and $\operatorname{len} p \leq m$ and $\operatorname{len} q \leq m$. Let x be an element of L. Suppose x is primitive root of degree $2 \cdot m$. If $(2 \cdot m)_L \neq 0_L$, then $((2 \cdot m)_L)^{-1} \cdot \mathrm{DFT}(\mathrm{DFT}(p, x, 2 \cdot m) \cdot \mathrm{DFT}(q, x, 2 \cdot m), x^{-1}, 2 \cdot m) = p * q$.

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Some Special Matrices of Real Elements and Their Properties

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Summary. This article describes definitions of positive matrix, negative matrix, nonpositive matrix, nonnegative matrix, nonzero matrix, module matrix of real elements and their main properties, and we also give the basic inequalities in matrices of real elements.

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The terminology and notation used here are introduced in the following articles: [2], [9], [3], [12], [1], [5], [8], [4], [7], [11], [6], and [10].

1. Some Special Matrices of Real Elements

We use the following convention: a, b are elements of $\mathbb{R}$, i, j, n are natural numbers, and M, M_1, M_2, M_3, M_4 are matrices over $\mathbb{R}$ of dimension n.

Let M be a matrix over $\mathbb{R}$. We say that M is positive if and only if:

(Def. 1) For all i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} > 0$.

We say that M is negative if and only if:

(Def. 2) For all i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} < 0$.

We say that M is nonpositive if and only if:

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(Def. 3) For all i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} \leq 0$.

We say that M is nonnegative if and only if:

(Def. 4) For all i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} \geq 0$.

Let M_1, M_2 be matrices over $\mathbb{R}$. The predicate $M_1 \sqsubseteq M_2$ is defined as follows:

(Def. 5) For all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $(M_1)_{i,j} < (M_2)_{i,j}$.

We say that M_1 is less or equal with M_2 if and only if:

(Def. 6) For all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $(M_1)_{i,j} \leq (M_2)_{i,j}$.

Let M be a matrix over $\mathbb{R}$. The functor $|{:}M{:}|$ yielding a matrix over $\mathbb{R}$ is defined by:

(Def. 7) $\operatorname{len}|{:}M{:}| = \operatorname{len} M$ and $\operatorname{width}|{:}M{:}| = \operatorname{width} M$ and for all i, j such that $\langle i, j\rangle \in$ the indices of M holds $|{:}M{:}|_{i,j} = |M_{i,j}|$.

Let us consider n and let us consider M. Then $-M$ is a matrix over $\mathbb{R}$ of dimension n.

Let us consider n and let us consider M_1, M_2. Then $M_1 + M_2$ is a matrix over $\mathbb{R}$ of dimension n.

Let us consider n and let us consider M_1, M_2. Then $M_1 - M_2$ is a matrix over $\mathbb{R}$ of dimension n.

Let us consider n, let a be an element of $\mathbb{R}$, and let us consider M. Then $a \cdot M$ is a matrix over $\mathbb{R}$ of dimension n.

Let us observe that there exists a matrix over $\mathbb{R}$ which is positive and nonnegative and there exists a matrix over $\mathbb{R}$ which is negative and nonpositive.

Let M be a positive matrix over $\mathbb{R}$. One can check that M^{T} is positive.

Let M be a negative matrix over $\mathbb{R}$. Note that M^{T} is negative.

Let M be a nonpositive matrix over $\mathbb{R}$. One can verify that M^{T} is nonpositive.

Let M be a nonnegative matrix over $\mathbb{R}$. Observe that M^{T} is nonnegative.

Let us consider n. Observe that $\begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}^{n\times n}$ is positive and nonnegative and $\begin{pmatrix} -1 & \dots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \dots & -1 \end{pmatrix}^{n\times n}$ is negative and nonpositive.

Let us consider n. One can verify that there exists a matrix over $\mathbb{R}$ of dimension n which is positive and nonnegative and there exists a matrix over $\mathbb{R}$ of dimension n which is negative and nonpositive.

We now state a number of propositions:

(1) For every element x_1 of $\mathbb{R}_{\mathrm{F}}$ and for every real number x_2 such that $x_1 = x_2$ holds $-x_1 = -x_2$.

(2) For every matrix M over $\mathbb{R}$ such that $\langle i, j\rangle \in$ the indices of M holds $(-M)_{i,j} = -M_{i,j}$.

(3) For all matrices M_1, M_2 over $\mathbb{R}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\langle i, j\rangle \in$ the indices of M_1 holds $(M_1 - M_2)_{i,j} = (M_1)_{i,j} - (M_2)_{i,j}$.

(4) For every matrix M over $\mathbb{R}$ such that $\operatorname{len}(a \cdot M) = \operatorname{len} M$ and $\operatorname{width}(a \cdot M) = \operatorname{width} M$ and $\langle i, j\rangle \in$ the indices of M holds $(a \cdot M)_{i,j} = a \cdot M_{i,j}$.

(5) The indices of $M =$ the indices of $|{:}M{:}|$.

(6) $|{:}a \cdot M{:}| = |a| \cdot |{:}M{:}|$.

(7) If M is negative, then $-M$ is positive.

(8) If M_1 is positive and M_2 is positive, then $M_1 + M_2$ is positive.

(9) If $-M_2 \sqsubseteq M_1$, then $M_1 + M_2$ is positive.

(10) If M_1 is nonnegative and M_2 is positive, then $M_1 + M_2$ is positive.

(11) If M_1 is positive and M_2 is negative and $|{:}M_2{:}| \sqsubseteq |{:}M_1{:}|$, then $M_1 + M_2$ is positive.

(12) If M_1 is positive and M_2 is negative, then $M_1 - M_2$ is positive.

(13) If $M_2 \sqsubseteq M_1$, then $M_1 - M_2$ is positive.

(14) If $a > 0$ and M is positive, then $a \cdot M$ is positive.

(15) If $a < 0$ and M is negative, then $a \cdot M$ is positive.

(16) If M is positive, then $-M$ is negative.

(17) If M_1 is negative and M_2 is negative, then $M_1 + M_2$ is negative.

(18) If $M_1 \sqsubseteq -M_2$, then $M_1 + M_2$ is negative.

(19) If M_1 is positive and M_2 is negative and $|{:}M_1{:}| \sqsubseteq |{:}M_2{:}|$, then $M_1 + M_2$ is negative.

(20) If $M_1 \sqsubseteq M_2$, then $M_1 - M_2$ is negative.

(21) If M_1 is positive and M_2 is negative, then $M_2 - M_1$ is negative.

(22) If $a < 0$ and M is positive, then $a \cdot M$ is negative.

(23) If $a > 0$ and M is negative, then $a \cdot M$ is negative.

(24) If M is nonnegative, then $-M$ is nonpositive.

(25) If M is negative, then M is nonpositive.

(26) If M_1 is nonpositive and M_2 is nonpositive, then $M_1 + M_2$ is nonpositive.

(27) If M_1 is less or equal with $-M_2$, then $M_1 + M_2$ is nonpositive.

(28) If M_1 is less or equal with M_2, then $M_1 - M_2$ is nonpositive.

(29) If $a \leq 0$ and M is positive, then $a \cdot M$ is nonpositive.

(30) If $a \geq 0$ and M is negative, then $a \cdot M$ is nonpositive.

(31) If $a \geq 0$ and M is nonpositive, then $a \cdot M$ is nonpositive.

(32) If $a \leq 0$ and M is nonnegative, then $a \cdot M$ is nonpositive.

(33) $|{:}M{:}|$ is nonnegative.

(34) If M_1 is positive, then M_1 is nonnegative.

(35) If M is nonpositive, then $-M$ is nonnegative.

(36) If M_1 is nonnegative and M_2 is nonnegative, then $M_1 + M_2$ is nonnegative.

(37) If $-M_1$ is less or equal with M_2, then $M_1 + M_2$ is nonnegative.

(38) If M_2 is less or equal with M_1, then $M_1 - M_2$ is nonnegative.

(39) If $a \geq 0$ and M is positive, then $a \cdot M$ is nonnegative.

(40) If $a \leq 0$ and M is negative, then $a \cdot M$ is nonnegative.

(41) If $a \leq 0$ and M is nonpositive, then $a \cdot M$ is nonnegative.

(42) If $a \geq 0$ and M is nonnegative, then $a \cdot M$ is nonnegative.

(43) If $a \geq 0$ and $b \geq 0$ and M_1 is nonnegative and M_2 is nonnegative, then $a \cdot M_1 + b \cdot M_2$ is nonnegative.

2. Some Basic Inequalities in Matrices of Real Elements

Next we state a number of propositions:

(44) If $M_1 \sqsubseteq M_2$, then M_1 is less or equal with M_2.

(45) If $M_1 \sqsubseteq M_2$ and $M_2 \sqsubseteq M_3$, then $M_1 \sqsubseteq M_3$.

(46) If $M_1 \sqsubseteq M_2$ and $M_3 \sqsubseteq M_4$, then $M_1 + M_3 \sqsubseteq M_2 + M_4$.

(47) If $M_1 \sqsubseteq M_2$, then $M_1 + M_3 \sqsubseteq M_2 + M_3$.

(48) If $M_1 \sqsubseteq M_2$, then $M_3 - M_2 \sqsubseteq M_3 - M_1$.

(49) $|{:}M_1 + M_2{:}|$ is less or equal with $|{:}M_1{:}| + |{:}M_2{:}|$.

(50) If M_1 is less or equal with M_2, then $M_1 - M_3$ is less or equal with $M_2 - M_3$.

(51) If $M_1 - M_3$ is less or equal with $M_2 - M_3$, then M_1 is less or equal with M_2.

(52) If M_1 is less or equal with $M_2 - M_3$, then M_3 is less or equal with $M_2 - M_1$.

(53) If $M_1 - M_2$ is less or equal with M_3, then $M_1 - M_3$ is less or equal with M_2.

(54) If $M_1 \sqsubseteq M_2$ and M_3 is less or equal with M_4, then $M_1 - M_4 \sqsubseteq M_2 - M_3$.

(55) If M_1 is less or equal with M_2 and $M_3 \sqsubseteq M_4$, then $M_1 - M_4 \sqsubseteq M_2 - M_3$.

(56) If $M_1 - M_2$ is less or equal with $M_3 - M_4$, then $M_1 - M_3$ is less or equal with $M_2 - M_4$.

(57) If $M_1 - M_2$ is less or equal with $M_3 - M_4$, then $M_4 - M_2$ is less or equal with $M_3 - M_1$.

(58) If $M_1 - M_2$ is less or equal with $M_3 - M_4$, then $M_4 - M_3$ is less or equal with $M_2 - M_1$.

(59) If $M_1 + M_2$ is less or equal with M_3, then M_1 is less or equal with $M_3 - M_2$.

(60) If $M_1 + M_2$ is less or equal with $M_3 + M_4$, then $M_1 - M_3$ is less or equal with $M_4 - M_2$.

(61) If $M_1 + M_2$ is less or equal with $M_3 - M_4$, then $M_1 + M_4$ is less or equal with $M_3 - M_2$.

(62) If $M_1 - M_2$ is less or equal with $M_3 + M_4$, then $M_1 - M_4$ is less or equal with $M_3 + M_2$.

(63) If M_1 is less or equal with M_2, then $-M_2$ is less or equal with $-M_1$.

(64) If M_1 is less or equal with $-M_2$, then M_2 is less or equal with $-M_1$.

(65) If $-M_2$ is less or equal with M_1, then $-M_1$ is less or equal with M_2.

(66) If M_1 is positive, then $M_2 \sqsubseteq M_2 + M_1$.

(67) If M_1 is negative, then $M_1 + M_2 \sqsubseteq M_2$.

(68) If M_1 is nonnegative, then M_2 is less or equal with $M_1 + M_2$.

(69) If M_1 is nonpositive, then $M_1 + M_2$ is less or equal with M_2.

(70) If M_1 is nonpositive and M_3 is less or equal with M_2, then $M_3 + M_1$ is less or equal with M_2.

(71) If M_1 is nonpositive and $M_3 \sqsubseteq M_2$, then $M_3 + M_1 \sqsubseteq M_2$.

(72) If M_1 is negative and M_3 is less or equal with M_2, then $M_3 + M_1 \sqsubseteq M_2$.

(73) If M_1 is nonnegative and M_2 is less or equal with M_3, then M_2 is less or equal with $M_1 + M_3$.

(74) If M_1 is positive and M_2 is less or equal with M_3, then $M_2 \sqsubseteq M_1 + M_3$.

(75) If M_1 is nonnegative and $M_2 \sqsubseteq M_3$, then $M_2 \sqsubseteq M_1 + M_3$.

(76) If M_1 is nonnegative, then $M_2 - M_1$ is less or equal with M_2.

(77) If M_1 is positive, then $M_2 - M_1 \sqsubseteq M_2$.

(78) If M_1 is nonpositive, then M_2 is less or equal with $M_2 - M_1$.

(79) If M_1 is negative, then $M_2 \sqsubseteq M_2 - M_1$.

(80) If M_1 is less or equal with M_2, then $M_2 - M_1$ is nonnegative.

(81) If M_1 is nonnegative and $M_2 \sqsubseteq M_3$, then $M_2 - M_1 \sqsubseteq M_3$.

(82) If M_1 is nonpositive and M_2 is less or equal with M_3, then M_2 is less or equal with $M_3 - M_1$.

(83) If M_1 is nonpositive and $M_2 \sqsubseteq M_3$, then $M_2 \sqsubseteq M_3 - M_1$.

(84) If M_1 is negative and M_2 is less or equal with M_3, then $M_2 \sqsubseteq M_3 - M_1$.

(85) If $M_1 \sqsubseteq M_2$ and $a > 0$, then $a \cdot M_1 \sqsubseteq a \cdot M_2$.

(86) If $M_1 \sqsubseteq M_2$ and $a \geq 0$, then $a \cdot M_1$ is less or equal with $a \cdot M_2$.

(87) If $M_1 \sqsubseteq M_2$ and $a < 0$, then $a \cdot M_2 \sqsubseteq a \cdot M_1$.

(88) If $M_1 \sqsubseteq M_2$ and $a \leq 0$, then $a \cdot M_2$ is less or equal with $a \cdot M_1$.

(89) If M_1 is less or equal with M_2 and $a \geq 0$, then $a \cdot M_1$ is less or equal with $a \cdot M_2$.

(90) If M_1 is less or equal with M_2 and $a \leq 0$, then $a \cdot M_2$ is less or equal with $a \cdot M_1$.

(91) If $a \geq 0$ and $a \leq b$ and M_1 is nonnegative and less or equal with M_2, then $a \cdot M_1$ is less or equal with $b \cdot M_2$.

(92) If $a \leq 0$ and $b \leq a$ and M_1 is nonpositive and M_2 is less or equal with M_1, then $a \cdot M_1$ is less or equal with $b \cdot M_2$.

(93) If $a < 0$ and $b \leq a$ and M_1 is negative and $M_2 \sqsubseteq M_1$, then $a \cdot M_1 \sqsubseteq b \cdot M_2$.

(94) If $a \geq 0$ and $a < b$ and M_1 is nonnegative and $M_1 \sqsubseteq M_2$, then $a \cdot M_1 \sqsubseteq b \cdot M_2$.

(95) If $a \geq 0$ and $a < b$ and M_1 is positive and less or equal with M_2, then $a \cdot M_1 \sqsubseteq b \cdot M_2$.

(96) If $a > 0$ and $a \leq b$ and M_1 is positive and $M_1 \sqsubseteq M_2$, then $a \cdot M_1 \sqsubseteq b \cdot M_2$.

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Schur's Theorem on the Stability of Networks

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Summary. A complex polynomial is called a Hurwitz polynomial if all its roots have a real part smaller than zero. This kind of polynomial plays an all-dominant role in stability checks of electrical networks.

In this article we prove Schur's criterion [17] that allows to decide whether a polynomial $p(x)$ is Hurwitz without explicitly computing its roots: Schur's recursive algorithm successively constructs polynomials $p_i(x)$ of lesser degree by division with $x - c$, $\Re\{c\} < 0$, such that $p_i(x)$ is Hurwitz if and only if $p(x)$ is.

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The articles [20], [25], [26], [18], [13], [5], [6], [1], [22], [23], [21], [19], [24], [16], [4], [9], [2], [3], [15], [14], [7], [12], [10], [27], [11], and [8] provide the terminology and notation for this paper.

1. Preliminaries

One can prove the following propositions:

(1) Let L be an add-associative right zeroed right complementable associative commutative left unital distributive field-like non empty double loop structure and x be an element of L. If $x \neq 0_L$, then $-x^{-1} = (-x)^{-1}$.

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(2) Let L be an add-associative right zeroed right complementable associative commutative left unital field-like distributive non degenerated non empty double loop structure and k be an element of $\mathbb{N}$. Then $\mathrm{power}_L(-1_L, k) \neq 0_L$.

(3) Let L be an associative right unital non empty multiplicative loop structure, x be an element of L, and k_1, k_2 be elements of $\mathbb{N}$. Then $\mathrm{power}_L(x, k_1) \cdot \mathrm{power}_L(x, k_2) = \mathrm{power}_L(x, k_1 + k_2)$.

(4) Let L be an add-associative right zeroed right complementable left unital distributive non empty double loop structure and k be an element of $\mathbb{N}$. Then $\mathrm{power}_L(-1_L, 2 \cdot k) = 1_L$ and $\mathrm{power}_L(-1_L, 2 \cdot k + 1) = -1_L$.

(5) For every element z of $\mathbb{C}_{\mathrm{F}}$ and for every element k of $\mathbb{N}$ holds $\overline{\mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(z, k)} = \mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(\overline{z}, k)$.

(6) Let F, G be finite sequences of elements of $\mathbb{C}_{\mathrm{F}}$. Suppose $\mathrm{len}\, G = \mathrm{len}\, F$ and for every element i of $\mathbb{N}$ such that $i \in \mathrm{dom}\, G$ holds $G_i = \overline{F_i}$. Then $\sum G = \overline{\sum F}$.

(7) Let L be an add-associative right zeroed right complementable Abelian non empty loop structure and F_1, F_2 be finite sequences of elements of L. Suppose $\mathrm{len}\, F_1 = \mathrm{len}\, F_2$ and for every element i of $\mathbb{N}$ such that $i \in \mathrm{dom}\, F_1$ holds $(F_1)_i = -(F_2)_i$. Then $\sum F_1 = -\sum F_2$.

(8) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, x be an element of L, and F be a finite sequence of elements of L. Then $x \cdot \sum F = \sum(x \cdot F)$.

2. More on Polynomials

We now state four propositions:

(9) For every add-associative right zeroed right complementable non empty loop structure L holds $-\mathbf{0}.\, L = \mathbf{0}.\, L$.

(10) Let L be an add-associative right zeroed right complementable non empty loop structure and p be a polynomial of L. Then $--p = p$.

(11) Let L be an add-associative right zeroed right complementable Abelian distributive non empty double loop structure and p_1, p_2 be polynomials of L. Then $-(p_1 + p_2) = -p_1 + -p_2$.

(12) Let L be an add-associative right zeroed right complementable distributive Abelian non empty double loop structure and p_1, p_2 be polynomials of L. Then $-p_1 * p_2 = (-p_1) * p_2$ and $-p_1 * p_2 = p_1 * -p_2$.

Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, let F be a finite sequence of elements of Polynom-Ring L, and let i be an element of $\mathbb{N}$. The functor $\mathrm{Coeff}(F, i)$ yielding a finite sequence of elements of L is defined by the conditions (Def. 1).

(Def. 1)(i) $\operatorname{len}\operatorname{Coeff}(F, i) = \operatorname{len} F$, and

(ii) for every element j of $\mathbb{N}$ such that $j \in \operatorname{dom}\operatorname{Coeff}(F, i)$ there exists a polynomial p of L such that $p = F(j)$ and $(\operatorname{Coeff}(F, i))(j) = p(i)$.

One can prove the following propositions:

(13) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, p be a polynomial of L, and F be a finite sequence of elements of Polynom-Ring L. If $p = \sum F$, then for every element i of $\mathbb{N}$ holds $p(i) = \sum \operatorname{Coeff}(F, i)$.

(14) Let L be an associative non empty double loop structure, p be a polynomial of L, and x_1, x_2 be elements of L. Then $x_1 \cdot (x_2 \cdot p) = (x_1 \cdot x_2) \cdot p$.

(15) Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure, p be a polynomial of L, and x be an element of L. Then $-x \cdot p = (-x) \cdot p$.

(16) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure, p be a polynomial of L, and x be an element of L. Then $-x \cdot p = x \cdot -p$.

(17) Let L be a left distributive non empty double loop structure, p be a polynomial of L, and x_1, x_2 be elements of L. Then $(x_1 + x_2) \cdot p = x_1 \cdot p + x_2 \cdot p$.

(18) Let L be a right distributive non empty double loop structure, p_1, p_2 be polynomials of L, and x be an element of L. Then $x \cdot (p_1 + p_2) = x \cdot p_1 + x \cdot p_2$.

(19) Let L be an add-associative right zeroed right complementable distributive commutative associative non empty double loop structure, p_1, p_2 be polynomials of L, and x be an element of L. Then $p_1 * (x \cdot p_2) = x \cdot (p_1 * p_2)$.

Let L be a non empty zero structure and let p be a polynomial of L. The functor degree(p) yields an integer and is defined by:

(Def. 2) $\operatorname{degree}(p) = \operatorname{len} p - 1$.

Let L be a non empty zero structure and let p be a polynomial of L. We introduce $\deg p$ as a synonym of degree(p).

We now state several propositions:

(20) For every non empty zero structure L and for every polynomial p of L holds $\deg p = -1$ iff $p = \mathbf{0}.L$.

(21) Let L be an add-associative right zeroed right complementable non empty loop structure and p_1, p_2 be polynomials of L. If $\deg p_1 \neq \deg p_2$, then $\deg(p_1 + p_2) = \max(\deg p_1, \deg p_2)$.

(22) Let L be an add-associative right zeroed right complementable Abelian non empty loop structure and p_1, p_2 be polynomials of L. Then $\deg(p_1 + p_2) \leq \max(\deg p_1, \deg p_2)$.

(23) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital integral domain-like non empty

double loop structure and p_1, p_2 be polynomials of L. If $p_1 \neq \mathbf{0}.\,L$ and $p_2 \neq \mathbf{0}.\,L$, then $\deg(p_1 * p_2) = \deg p_1 + \deg p_2$.

(24) Let L be an add-associative right zeroed right complementable unital non empty double loop structure and p be a polynomial of L such that $\deg p = 0$. Then p does not have roots.

Let L be a unital non empty double loop structure, let z be an element of L, and let k be an element of $\mathbb{N}$. The functor $\mathrm{rpoly}(k, z)$ yields a polynomial of L and is defined by:

(Def. 3) $\mathrm{rpoly}(k, z) = \mathbf{0}.\,L{+\cdot}[0 \longmapsto -\mathrm{power}_L(z,\, k), k \longmapsto 1_L]$.

One can prove the following propositions:

(25) Let L be a unital non empty double loop structure, z be an element of L, and k be an element of $\mathbb{N}$. If $k \neq 0$, then $(\mathrm{rpoly}(k, z))(0) = -\mathrm{power}_L(z,\, k)$ and $(\mathrm{rpoly}(k, z))(k) = 1_L$.

(26) Let L be a unital non empty double loop structure, z be an element of L, and i, k be elements of $\mathbb{N}$. If $i \neq 0$ and $i \neq k$, then $(\mathrm{rpoly}(k, z))(i) = 0_L$.

(27) Let L be a unital non degenerated non empty double loop structure, z be an element of L, and k be an element of $\mathbb{N}$. Then $\deg \mathrm{rpoly}(k, z) = k$.

(28) Let L be an add-associative right zeroed right complementable left unital commutative associative distributive field-like non degenerated non empty double loop structure and p be a polynomial of L. Then $\deg p = 1$ if and only if there exist elements x, z of L such that $x \neq 0_L$ and $p = x \cdot \mathrm{rpoly}(1, z)$.

(29) Let L be an add-associative right zeroed right complementable Abelian unital non degenerated non empty double loop structure and x, z be elements of L. Then $\mathrm{eval}(\mathrm{rpoly}(1, z), x) = x - z$.

(30) Let L be an add-associative right zeroed right complementable unital Abelian non degenerated non empty double loop structure and z be an element of L. Then z is a root of $\mathrm{rpoly}(1, z)$.

Let L be a unital non empty double loop structure, let z be an element of L, and let k be an element of $\mathbb{N}$. The functor $\mathrm{qpoly}(k, z)$ yielding a polynomial of L is defined by the conditions (Def. 4).

(Def. 4)(i) For every element i of $\mathbb{N}$ such that $i < k$ holds $(\mathrm{qpoly}(k, z))(i) = \mathrm{power}_L(z,\, k - i - 1)$, and

(ii) for every element i of $\mathbb{N}$ such that $i \geq k$ holds $(\mathrm{qpoly}(k, z))(i) = 0_L$.

Next we state three propositions:

(31) Let L be a unital non degenerated non empty double loop structure, z be an element of L, and k be an element of $\mathbb{N}$. If $k \geq 1$, then $\deg \mathrm{qpoly}(k, z) = k - 1$.

(32) Let L be an add-associative right zeroed right complementable left distributive unital commutative non empty double loop structure, z be an

element of L, and k be an element of $\mathbb{N}$. If $k > 1$, then $\mathrm{rpoly}(1, z) * \mathrm{qpoly}(k, z) = \mathrm{rpoly}(k, z)$.

(33) Let L be an Abelian add-associative right zeroed right complementable unital associative distributive commutative non empty double loop structure, p be a polynomial of L, and z be an element of L. If z is a root of p, then there exists a polynomial s of L such that $p = \mathrm{rpoly}(1, z) * s$.

3. Division of Polynomials

Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let p, s be polynomials of L. Let us assume that $s \neq \mathbf{0}.\,L$. The functor $p \div s$ yields a polynomial of L and is defined by:

(Def. 5) There exists a polynomial t of L such that $p = (p \div s) * s + t$ and $\deg t < \deg s$.

Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let p, s be polynomials of L. The functor $p \bmod s$ yielding a polynomial of L is defined by:

(Def. 6) $p \bmod s = p - (p \div s) * s$.

Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let p, s be polynomials of L. The predicate $s \mid p$ is defined by:

(Def. 7) $p \bmod s = \mathbf{0}.\,L$.

One can prove the following three propositions:

(34) Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and p, s be polynomials of L. Suppose $s \neq \mathbf{0}.\,L$. Then $s \mid p$ if and only if there exists a polynomial t of L such that $t * s = p$.

(35) Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non degenerated non empty double loop structure, p be a polynomial of L, and z be an element of L. If z is a root of p, then $\mathrm{rpoly}(1, z) \mid p$.

(36) Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non degenerated non empty double loop structure, p be a polynomial of L, and z be an element of L. If $p \neq \mathbf{0}.\,L$ and z is a root of p, then $\deg(p \div \mathrm{rpoly}(1, z)) = \deg p - 1$.

4. Schur's Theorem

Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$. We say that f is Hurwitz if and only if:

(Def. 8) For every element z of $\mathbb{C}_{\mathrm{F}}$ such that z is a root of f holds $\Re(z) < 0$.

We now state several propositions:

(37) $\mathbf{0}.(\mathbb{C}_{\mathrm{F}})$ is non Hurwitz.

(38) For every element x of $\mathbb{C}_{\mathrm{F}}$ such that $x \neq 0_{\mathbb{C}_{\mathrm{F}}}$ holds $x \cdot \mathbf{1}.(\mathbb{C}_{\mathrm{F}})$ is Hurwitz.

(39) For all elements x, z of $\mathbb{C}_{\mathrm{F}}$ such that $x \neq 0_{\mathbb{C}_{\mathrm{F}}}$ holds $x \cdot \mathrm{rpoly}(1, z)$ is Hurwitz iff $\Re(z) < 0$.

(40) Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$ and z be an element of $\mathbb{C}_{\mathrm{F}}$. If $z \neq 0_{\mathbb{C}_{\mathrm{F}}}$, then f is Hurwitz iff $z \cdot f$ is Hurwitz.

(41) For all polynomials f, g of $\mathbb{C}_{\mathrm{F}}$ holds $f * g$ is Hurwitz iff f is Hurwitz and g is Hurwitz.

Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$. The functor $\overline{f}$ yielding a polynomial of $\mathbb{C}_{\mathrm{F}}$ is defined by:

(Def. 9) For every element i of $\mathbb{N}$ holds $\overline{f}(i) = \mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(-1_{\mathbb{C}_{\mathrm{F}}}, i) \cdot \overline{f(i)}$.

We now state several propositions:

(42) For every polynomial f of $\mathbb{C}_{\mathrm{F}}$ holds $\deg \overline{f} = \deg f$.

(43) For every polynomial f of $\mathbb{C}_{\mathrm{F}}$ holds $\overline{\overline{f}} = f$.

(44) For every polynomial f of $\mathbb{C}_{\mathrm{F}}$ and for every element z of $\mathbb{C}_{\mathrm{F}}$ holds $\overline{z \cdot f} = \overline{z} \cdot \overline{f}$.

(45) For every polynomial f of $\mathbb{C}_{\mathrm{F}}$ holds $\overline{-f} = -\overline{f}$.

(46) For all polynomials f, g of $\mathbb{C}_{\mathrm{F}}$ holds $\overline{f + g} = \overline{f} + \overline{g}$.

(47) For all polynomials f, g of $\mathbb{C}_{\mathrm{F}}$ holds $\overline{f * g} = \overline{f} * \overline{g}$.

(48) For all elements x, z of $\mathbb{C}_{\mathrm{F}}$ holds $\mathrm{eval}(\overline{\mathrm{rpoly}(1, z)}, x) = -x - \overline{z}$.

(49) For every polynomial f of $\mathbb{C}_{\mathrm{F}}$ such that f is Hurwitz and for every element x of $\mathbb{C}_{\mathrm{F}}$ such that $\Re(x) \geq 0$ holds $0 < |\mathrm{eval}(f, x)|$.

(50) Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$. Suppose $\deg f \geq 1$ and f is Hurwitz. Let x be an element of $\mathbb{C}_{\mathrm{F}}$. Then

(i) if $\Re(x) < 0$, then $|\mathrm{eval}(f, x)| < |\mathrm{eval}(\overline{f}, x)|$,

(ii) if $\Re(x) > 0$, then $|\mathrm{eval}(f, x)| > |\mathrm{eval}(\overline{f}, x)|$, and

(iii) if $\Re(x) = 0$, then $|\mathrm{eval}(f, x)| = |\mathrm{eval}(\overline{f}, x)|$.

Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$ and let z be an element of $\mathbb{C}_{\mathrm{F}}$. The functor $F * (f, z)$ yields a polynomial of $\mathbb{C}_{\mathrm{F}}$ and is defined as follows:

(Def. 10) $F * (f, z) = \mathrm{eval}(\overline{f}, z) \cdot f - \mathrm{eval}(f, z) \cdot \overline{f}$.

We now state four propositions:

(51) Let a, b be elements of $\mathbb{C}_{\mathrm{F}}$. Suppose $|a| > |b|$. Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$. If $\deg f \geq 1$, then f is Hurwitz iff $a \cdot f - b \cdot \overline{f}$ is Hurwitz.

(52) Let f be a polynomial of $\mathbb{C}_{\rm F}$. Suppose $\deg f \geq 1$. Let r_1 be an element of $\mathbb{C}_{\rm F}$. If $\Re(r_1) < 0$, then if f is Hurwitz, then $F * (f, r_1) \div {\rm rpoly}(1, r_1)$ is Hurwitz.

(53) Let f be a polynomial of $\mathbb{C}_{\rm F}$. Suppose $\deg f \geq 1$. Given an element r_1 of $\mathbb{C}_{\rm F}$ such that $\Re(r_1) < 0$ and $|\,{\rm eval}(f, r_1)| \geq |\,{\rm eval}(\overline{f}, r_1)|$. Then f is non Hurwitz.

(54) Let f be a polynomial of $\mathbb{C}_{\rm F}$. Suppose $\deg f \geq 1$. Let r_1 be an element of $\mathbb{C}_{\rm F}$. Suppose $\Re(r_1) < 0$ and $|\,{\rm eval}(f, r_1)| < |\,{\rm eval}(\overline{f}, r_1)|$. Then f is Hurwitz if and only if $F * (f, r_1) \div {\rm rpoly}(1, r_1)$ is Hurwitz.

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Integral of Real-Valued Measurable Function[1]

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Summary. Based on [16], authors formalized the integral of an extended real valued measurable function in [12] before. However, the integral argued in [12] cannot be applied to real-valued functions unconditionally. Therefore, in this article we have formalized the integral of a real-value function.

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The papers [25], [11], [26], [1], [23], [24], [17], [18], [8], [27], [10], [2], [19], [7], [20], [6], [9], [3], [4], [5], [13], [14], [15], [22], [21], and [12] provide the terminology and notation for this paper.

1. The Measurability of Real-Valued Functions

For simplicity, we follow the rules: X denotes a non empty set, Y denotes a set, S denotes a σ-field of subsets of X, F denotes a function from $\mathbb{N}$ into S, f, g denote partial functions from X to $\mathbb{R}$, A, B denote elements of S, r, s denote real numbers, a denotes a real number, and n denotes a natural number.

Let X be a non empty set, let f be a partial function from X to $\mathbb{R}$, and let a be a real number. The functor LE-dom(f,a) yields a subset of X and is defined as follows:

(Def. 1) $\text{LE-dom}(f,a) = \text{LE-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$.

The following three propositions are true:

(1) $|\overline{\mathbb{R}}(f)| = \overline{\mathbb{R}}(|f|)$.

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(2) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and r be a real number. Suppose $\mathrm{dom}\, f \in S$ and for every set x such that $x \in \mathrm{dom}\, f$ holds $f(x) = r$. Then f is simple function in S.

(3) For every set x holds $x \in \text{LE-dom}(f, a)$ iff $x \in \mathrm{dom}\, f$ and there exists a real number y such that $y = f(x)$ and $y < a$.

Let us consider X, f, a. The functor LEQ-dom(f, a) yields a subset of X and is defined as follows:

(Def. 2) $\text{LEQ-dom}(f, a) = \text{LEQ-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$.

We now state the proposition

(4) For every set x holds $x \in \text{LEQ-dom}(f, a)$ iff $x \in \mathrm{dom}\, f$ and there exists a real number y such that $y = f(x)$ and $y \leq a$.

Let us consider X, f, a. The functor GT-dom(f, a) yielding a subset of X is defined as follows:

(Def. 3) $\text{GT-dom}(f, a) = \text{GT-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$.

We now state the proposition

(5) For every set x holds $x \in \text{GT-dom}(f, r)$ iff $x \in \mathrm{dom}\, f$ and there exists a real number y such that $y = f(x)$ and $r < y$.

Let us consider X, f, a. The functor GTE-dom(f, a) yields a subset of X and is defined as follows:

(Def. 4) $\text{GTE-dom}(f, a) = \text{GTE-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$.

Next we state the proposition

(6) For every set x holds $x \in \text{GTE-dom}(f, r)$ iff $x \in \mathrm{dom}\, f$ and there exists a real number y such that $y = f(x)$ and $r \leq y$.

Let us consider X, f, a. The functor EQ-dom(f, a) yielding a subset of X is defined by:

(Def. 5) $\text{EQ-dom}(f, a) = \text{EQ-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$.

The following propositions are true:

(7) For every set x holds $x \in \text{EQ-dom}(f, r)$ iff $x \in \mathrm{dom}\, f$ and there exists a real number y such that $y = f(x)$ and $r = y$.

(8) If for every n holds $F(n) = Y \cap \text{GT-dom}(f, r - \frac{1}{n+1})$, then $Y \cap \text{GTE-dom}(f, r) = \bigcap \mathrm{rng}\, F$.

(9) If for every n holds $F(n) = Y \cap \text{LE-dom}(f, r + \frac{1}{n+1})$, then $Y \cap \text{LEQ-dom}(f, r) = \bigcap \mathrm{rng}\, F$.

(10) If for every n holds $F(n) = Y \cap \text{LEQ-dom}(f, r - \frac{1}{n+1})$, then $Y \cap \text{LE-dom}(f, r) = \bigcup \mathrm{rng}\, F$.

(11) If for every n holds $F(n) = Y \cap \text{GTE-dom}(f, r + \frac{1}{n+1})$, then $Y \cap \text{GT-dom}(f, r) = \bigcup \mathrm{rng}\, F$.

Let X be a non empty set, let S be a σ-field of subsets of X, let f be a partial function from X to $\mathbb{R}$, and let A be an element of S. We say that f is measurable on A if and only if:

(Def. 6) $\overline{\mathbb{R}}(f)$ is measurable on A.

The following propositions are true:

(12) f is measurable on A iff for every real number r holds $A \cap \text{LE-dom}(f, r)$ is measurable on S.

(13) Suppose $A \subseteq \text{dom} f$. Then f is measurable on A if and only if for every real number r holds $A \cap \text{GTE-dom}(f, r)$ is measurable on S.

(14) f is measurable on A iff for every real number r holds $A \cap \text{LEQ-dom}(f, r)$ is measurable on S.

(15) Suppose $A \subseteq \text{dom} f$. Then f is measurable on A if and only if for every real number r holds $A \cap \text{GT-dom}(f, r)$ is measurable on S.

(16) If $B \subseteq A$ and f is measurable on A, then f is measurable on B.

(17) If f is measurable on A and f is measurable on B, then f is measurable on $A \cup B$.

(18) If f is measurable on A and $A \subseteq \text{dom} f$, then $A \cap \text{GT-dom}(f, r) \cap \text{LE-dom}(f, s)$ is measurable on S.

(19) If f is measurable on A and g is measurable on A and $A \subseteq \text{dom} g$, then $A \cap \text{LE-dom}(f, r) \cap \text{GT-dom}(g, r)$ is measurable on S.

(20) $\overline{\mathbb{R}}(r\, f) = r\, \overline{\mathbb{R}}(f)$.

(21) If f is measurable on A and $A \subseteq \text{dom} f$, then $r\, f$ is measurable on A.

2. The Measurability of $f + g$ and $f - g$ for Real-Valued Functions f, g

For simplicity, we adopt the following rules: X denotes a non empty set, S denotes a σ-field of subsets of X, f, g denote partial functions from X to $\mathbb{R}$, A denotes an element of S, r denotes a real number, and p denotes a rational number.

Next we state several propositions:

(22) $\overline{\mathbb{R}}(f)$ is finite.

(23) $\overline{\mathbb{R}}(f + g) = \overline{\mathbb{R}}(f) + \overline{\mathbb{R}}(g)$ and $\overline{\mathbb{R}}(f - g) = \overline{\mathbb{R}}(f) - \overline{\mathbb{R}}(g)$ and $\text{dom}\, \overline{\mathbb{R}}(f + g) = \text{dom}\, \overline{\mathbb{R}}(f) \cap \text{dom}\, \overline{\mathbb{R}}(g)$ and $\text{dom}\, \overline{\mathbb{R}}(f - g) = \text{dom}\, \overline{\mathbb{R}}(f) \cap \text{dom}\, \overline{\mathbb{R}}(g)$ and $\text{dom}\, \overline{\mathbb{R}}(f + g) = \text{dom} f \cap \text{dom} g$ and $\text{dom}\, \overline{\mathbb{R}}(f - g) = \text{dom} f \cap \text{dom} g$.

(24) For every function F from $\mathbb{Q}$ into S such that for every p holds $F(p) = A \cap \text{LE-dom}(f, p) \cap (A \cap \text{LE-dom}(g, r - p))$ holds $A \cap \text{LE-dom}(f + g, r) = \bigcup \text{rng} F$.

(25) Suppose f is measurable on A and g is measurable on A. Then there exists a function F from $\mathbb{Q}$ into S such that for every rational number p holds $F(p) = A \cap \text{LE-dom}(f, p) \cap (A \cap \text{LE-dom}(g, r - p))$.

(26) If f is measurable on A and g is measurable on A, then $f+g$ is measurable on A.

(27) $\overline{\mathbb{R}}(f) - \overline{\mathbb{R}}(g) = \overline{\mathbb{R}}(f) + \overline{\mathbb{R}}(-g)$.

(28) $-\overline{\mathbb{R}}(f) = \overline{\mathbb{R}}((-1)\, f)$ and $-\overline{\mathbb{R}}(f) = \overline{\mathbb{R}}(-f)$.

(29) If f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom} g$, then $f - g$ is measurable on A.

3. Basic Properties of Real-Valued Functions, $\max_+ f$ and $\max_- f$

In the sequel X denotes a non empty set, f denotes a partial function from X to $\mathbb{R}$, and r denotes a real number.

Next we state a number of propositions:

(30) $\max_+(\overline{\mathbb{R}}(f)) = \max_+(f)$ and $\max_-(\overline{\mathbb{R}}(f)) = \max_-(f)$.

(31) For every element x of X holds $0 \leq (\max_+(f))(x)$.

(32) For every element x of X holds $0 \leq (\max_-(f))(x)$.

(33) $\max_-(f) = \max_+(-f)$.

(34) For every set x such that $x \in \operatorname{dom} f$ and $0 < (\max_+(f))(x)$ holds $(\max_-(f))(x) = 0$.

(35) For every set x such that $x \in \operatorname{dom} f$ and $0 < (\max_-(f))(x)$ holds $(\max_+(f))(x) = 0$.

(36) $\operatorname{dom} f = \operatorname{dom}(\max_+(f) - \max_-(f))$ and $\operatorname{dom} f = \operatorname{dom}(\max_+(f) + \max_-(f))$.

(37) For every set x such that $x \in \operatorname{dom} f$ holds $(\max_+(f))(x) = f(x)$ or $(\max_+(f))(x) = 0$ but $(\max_-(f))(x) = -f(x)$ or $(\max_-(f))(x) = 0$.

(38) For every set x such that $x \in \operatorname{dom} f$ and $(\max_+(f))(x) = f(x)$ holds $(\max_-(f))(x) = 0$.

(39) For every set x such that $x \in \operatorname{dom} f$ and $(\max_+(f))(x) = 0$ holds $(\max_-(f))(x) = -f(x)$.

(40) For every set x such that $x \in \operatorname{dom} f$ and $(\max_-(f))(x) = -f(x)$ holds $(\max_+(f))(x) = 0$.

(41) For every set x such that $x \in \operatorname{dom} f$ and $(\max_-(f))(x) = 0$ holds $(\max_+(f))(x) = f(x)$.

(42) $f = \max_+(f) - \max_-(f)$.

(43) $|r| = |\overline{\mathbb{R}}(r)|$.

(44) $\overline{\mathbb{R}}(|f|) = |\overline{\mathbb{R}}(f)|$.

(45) $|f| = \max_{+}(f) + \max_{-}(f)$.

4. The Measurability of $\max_{+} f, \max_{-} f$ and $|f|$

In the sequel X denotes a non empty set, S denotes a σ-field of subsets of X, f denotes a partial function from X to $\mathbb{R}$, and A denotes an element of S.

The following propositions are true:

(46) If f is measurable on A, then $\max_{+}(f)$ is measurable on A.

(47) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $\max_{-}(f)$ is measurable on A.

(48) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $|f|$ is measurable on A.

5. The Definition and the Measurability of a Real-Valued Simple Function

For simplicity, we adopt the following rules: X is a non empty set, Y is a set, S is a σ-field of subsets of X, f, g, h are partial functions from X to $\mathbb{R}$, A is an element of S, and r is a real number.

Let us consider X, S, f. We say that f is simple function in S if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists a finite sequence F of separated subsets of S such that

(i) $\operatorname{dom} f = \bigcup \operatorname{rng} F$, and

(ii) for every natural number n and for all elements x, y of X such that $n \in \operatorname{dom} F$ and $x \in F(n)$ and $y \in F(n)$ holds $f(x) = f(y)$.

Next we state a number of propositions:

(49) f is simple function in S iff $\overline{\mathbb{R}}(f)$ is simple function in S.

(50) If f is simple function in S, then f is measurable on A.

(51) Let X be a set and f be a partial function from X to $\mathbb{R}$. Then f is non-negative if and only if for every set x holds $0 \leq f(x)$.

(52) Let X be a set and f be a partial function from X to $\mathbb{R}$. If for every set x such that $x \in \operatorname{dom} f$ holds $0 \leq f(x)$, then f is non-negative.

(53) Let X be a set and f be a partial function from X to $\mathbb{R}$. Then f is non-positive if and only if for every set x holds $f(x) \leq 0$.

(54) If for every set x such that $x \in \operatorname{dom} f$ holds $f(x) \leq 0$, then f is non-positive.

(55) If f is non-negative, then $f \restriction Y$ is non-negative.

(56) If f is non-negative and g is non-negative, then $f + g$ is non-negative.

(57) If f is non-negative, then if $0 \leq r$, then $r\,f$ is non-negative and if $r \leq 0$, then $r\,f$ is non-positive.

(58) If for every set x such that $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$ holds $g(x) \leq f(x)$, then $f - g$ is non-negative.

(59) If f is non-negative and g is non-negative and h is non-negative, then $f + g + h$ is non-negative.

(60) For every set x such that $x \in \mathrm{dom}(f + g + h)$ holds $(f + g + h)(x) = f(x) + g(x) + h(x)$.

(61) $\max_+(f)$ is non-negative and $\max_-(f)$ is non-negative.

(62)(i) $\mathrm{dom}(\max_+(f + g) + \max_-(f)) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$,

(ii) $\mathrm{dom}(\max_-(f + g) + \max_+(f)) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$,

(iii) $\mathrm{dom}(\max_+(f + g) + \max_-(f) + \max_-(g)) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$,

(iv) $\mathrm{dom}(\max_-(f + g) + \max_+(f) + \max_+(g)) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$,

(v) $\max_+(f + g) + \max_-(f)$ is non-negative, and

(vi) $\max_-(f + g) + \max_+(f)$ is non-negative.

(63) $\max_+(f+g)+\max_-(f)+\max_-(g) = \max_-(f+g)+\max_+(f)+\max_+(g)$.

(64) If $0 \leq r$, then $\max_+(r\, f) = r \max_+(f)$ and $\max_-(r\, f) = r \max_-(f)$.

(65) If $0 \leq r$, then $\max_+((-r)\, f) = r \max_-(f)$ and $\max_-((-r)\, f) = r \max_+(f)$.

(66) $\max_+(f{\restriction}Y) = \max_+(f){\restriction}Y$ and $\max_-(f{\restriction}Y) = \max_-(f){\restriction}Y$.

(67) If $Y \subseteq \mathrm{dom}(f+g)$, then $\mathrm{dom}((f+g){\restriction}Y) = Y$ and $\mathrm{dom}(f{\restriction}Y+g{\restriction}Y) = Y$ and $(f + g){\restriction}Y = f{\restriction}Y + g{\restriction}Y$.

(68) $\text{EQ-dom}(f, r) = f^{-1}(\{r\})$.

6. Lemmas for a Real-Valued Measurable Function and a Simple Function

For simplicity, we use the following convention: X is a non empty set, S is a σ-field of subsets of X, f, g are partial functions from X to $\mathbb{R}$, A, B are elements of S, and r, s are real numbers.

We now state a number of propositions:

(69) If f is measurable on A and $A \subseteq \mathrm{dom}\, f$, then $A \cap \text{GTE-dom}(f, r) \cap \text{LE-dom}(f, s)$ is measurable on S.

(70) If f is simple function in S, then $f{\restriction}A$ is simple function in S.

(71) If f is simple function in S, then $\mathrm{dom}\, f$ is an element of S.

(72) If f is simple function in S and g is simple function in S, then $f + g$ is simple function in S.

(73) If f is simple function in S, then $r\, f$ is simple function in S.

(74) If for every set x such that $x \in \mathrm{dom}(f - g)$ holds $g(x) \leq f(x)$, then $f - g$ is non-negative.

(75) There exists a partial function f from X to $\mathbb{R}$ such that f is simple function in S and $\operatorname{dom} f = A$ and for every set x such that $x \in A$ holds $f(x) = r$.

(76) If f is measurable on B and $A = \operatorname{dom} f \cap B$, then $f \restriction B$ is measurable on A.

(77) If $A \subseteq \operatorname{dom} f$ and f is measurable on A and g is measurable on A, then $\max_{+}(f + g) + \max_{-}(f)$ is measurable on A.

(78) If $A \subseteq \operatorname{dom} f \cap \operatorname{dom} g$ and f is measurable on A and g is measurable on A, then $\max_{-}(f + g) + \max_{+}(f)$ is measurable on A.

(79) If $\operatorname{dom} f \in S$ and $\operatorname{dom} g \in S$, then $\operatorname{dom}(f + g) \in S$.

(80) If $\operatorname{dom} f = A$, then f is measurable on B iff f is measurable on $A \cap B$.

(81) Given an element A of S such that $\operatorname{dom} f = A$. Let c be a real number and B be an element of S. If f is measurable on B, then $c\, f$ is measurable on B.

7. The Integral of a Real-Valued Function

For simplicity, we follow the rules: X is a non empty set, S is a σ-field of subsets of X, M is a σ-measure on S, f, g are partial functions from X to $\mathbb{R}$, r is a real number, and E, A, B are elements of S.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{R}$. The functor $\int f \, \mathrm{d}M$ yields an element of $\overline{\mathbb{R}}$ and is defined by:

(Def. 8) $\int f \, \mathrm{d}M = \int \overline{\mathbb{R}}(f) \, \mathrm{d}M$.

The following propositions are true:

(82) If there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative, then $\int f \, \mathrm{d}M = \int^{+} \overline{\mathbb{R}}(f) \, \mathrm{d}M$.

(83) If f is simple function in S and f is non-negative, then $\int f \, \mathrm{d}M = \int^{+} \overline{\mathbb{R}}(f) \, \mathrm{d}M$ and $\int f \, \mathrm{d}M = \int' \overline{\mathbb{R}}(f) \, \mathrm{d}M$.

(84) If there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative, then $0 \leq \int f \, \mathrm{d}M$.

(85) Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and A misses B. Then $\int f \restriction (A \cup B) \, \mathrm{d}M = \int f \restriction A \, \mathrm{d}M + \int f \restriction B \, \mathrm{d}M$.

(86) If there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative, then $0 \leq \int f \restriction A \, \mathrm{d}M$.

(87) Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and $A \subseteq B$. Then $\int f \restriction A \, \mathrm{d}M \leq \int f \restriction B \, \mathrm{d}M$.

(88) If there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$, then $\int f{\upharpoonright}A \, \mathrm{d}M = 0$.

(89) If $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$, then $\int f{\upharpoonright}(E \setminus A) \, \mathrm{d}M = \int f \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{R}$. We say that f is integrable on M if and only if:

(Def. 9) $\overline{\mathbb{R}}(f)$ is integrable on M.

We now state a number of propositions:

(90) If f is integrable on M, then $-\infty < \int f \, \mathrm{d}M$ and $\int f \, \mathrm{d}M < +\infty$.

(91) If f is integrable on M, then $f{\upharpoonright}A$ is integrable on M.

(92) If f is integrable on M and A misses B, then $\int f{\upharpoonright}(A \cup B) \, \mathrm{d}M = \int f{\upharpoonright}A \, \mathrm{d}M + \int f{\upharpoonright}B \, \mathrm{d}M$.

(93) If f is integrable on M and $B = \operatorname{dom} f \setminus A$, then $f{\upharpoonright}A$ is integrable on M and $\int f \, \mathrm{d}M = \int f{\upharpoonright}A \, \mathrm{d}M + \int f{\upharpoonright}B \, \mathrm{d}M$.

(94) Given an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A. Then f is integrable on M if and only if $|f|$ is integrable on M.

(95) If f is integrable on M, then $|\int f \, \mathrm{d}M| \leq \int |f| \, \mathrm{d}M$.

(96) Suppose that

(i) there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A,

(ii) $\operatorname{dom} f = \operatorname{dom} g$,

(iii) g is integrable on M, and

(iv) for every element x of X such that $x \in \operatorname{dom} f$ holds $|f(x)| \leq g(x)$.

Then f is integrable on M and $\int |f| \, \mathrm{d}M \leq \int g \, \mathrm{d}M$.

(97) If $\operatorname{dom} f \in S$ and $0 \leq r$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) = r$, then $\int f \, \mathrm{d}M = \overline{\mathbb{R}}(r) \cdot M(\operatorname{dom} f)$.

(98) Suppose f is integrable on M and g is integrable on M and f is non-negative and g is non-negative. Then $f + g$ is integrable on M.

(99) If f is integrable on M and g is integrable on M, then $\operatorname{dom}(f + g) \in S$.

(100) If f is integrable on M and g is integrable on M, then $f + g$ is integrable on M.

(101) Suppose f is integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int f + g \, \mathrm{d}M = \int f{\upharpoonright}E \, \mathrm{d}M + \int g{\upharpoonright}E \, \mathrm{d}M$.

(102) If f is integrable on M, then $r f$ is integrable on M and $\int r f \, \mathrm{d}M = \overline{\mathbb{R}}(r) \cdot \int f \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, let f be a partial function from X to $\mathbb{R}$, and let B be an element of S. The functor $\int_B f\,\mathrm{d}M$ yielding an element of $\overline{\mathbb{R}}$ is defined by:

(Def. 10) $\int_B f\,\mathrm{d}M = \int f{\restriction}B\,\mathrm{d}M$.

Next we state two propositions:

(103) Suppose f is integrable on M and g is integrable on M and $B \subseteq \mathrm{dom}(f+g)$. Then $f+g$ is integrable on M and $\int_B f+g\,\mathrm{d}M = \int_B f\,\mathrm{d}M + \int_B g\,\mathrm{d}M$.

(104) If f is integrable on M and f is measurable on B, then $f{\restriction}B$ is integrable on M and $\int_B r\,f\,\mathrm{d}M = \overline{\mathbb{R}}(r) \cdot \int_B f\,\mathrm{d}M$.

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The Catalan Numbers. Part II[1]

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Summary. In this paper, we define sequence dominated by 0, in which every initial fragment contains more zeroes than ones. If $n \geq 2 \cdot m$ and $n > 0$, then the number of sequences dominated by 0 the length n including m of ones, is given by the formula

$$D(n, m) = \frac{n + 1 - 2 \cdot m}{n + 1 - m} \cdot \binom{n}{m}$$

and satisfies the recurrence relation

$$D(n, m) = D(n - 1, m) \; + \; \sum_{i=0}^{m-1} D(2 \cdot i, i) \cdot D(n - 2 \cdot (i + 1), m - (i + 1)).$$

Obviously, if $n = 2 \cdot m$, then we obtain the recurrence relation for the Catalan numbers (starting from 0)

$$C_{m+1} = \sum_{i=0}^{m-1} C_{i+1} \cdot C_{m-i}.$$

Using the above recurrence relation we can see that

$$\sum_{i=0}^{\infty} C_{i+1} \cdot x^i \; = \; 1 \; + \; \left(\sum_{i=0}^{\infty} C_{i+1} \cdot x^i \right)^2$$

where $(|x| < \frac{1}{4})$ and hence

$$\sum_{i=0}^{\infty} C_{i+1} \cdot x^i \; = \; \frac{1 - \sqrt{1 - 4 \cdot x}}{2 \cdot x}.$$

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The notation and terminology used here are introduced in the following papers: [2], [23], [7], [25], [19], [27], [5], [28], [9], [1], [26], [21], [6], [3], [14], [12], [16], [13], [20], [15], [8], [22], [11], [10], [18], [24], [17], and [4].

1. Preliminaries

For simplicity, we adopt the following convention: x, D denote sets, i, j, k, l, m, n denote elements of $\mathbb{N}$, p, q denote finite 0-sequences of $\mathbb{N}$, p', q' denote finite 0-sequences, and p_1, q_1 denote finite 0-sequences of D.

Next we state several propositions:

(1) $(p' \frown q') \restriction \operatorname{dom} p' = p'$.

(2) If $n \leq \operatorname{dom} p'$, then $(p' \frown q') \restriction n = p' \restriction n$.

(3) If $n = \operatorname{dom} p' + k$, then $(p' \frown q') \restriction n = p' \frown (q' \restriction k)$.

(4) There exists q' such that $p' = (p' \restriction n) \frown q'$.

(5) There exists q_1 such that $p_1 = (p_1 \restriction n) \frown q_1$.

Let us consider p. We say that p is dominated by 0 if and only if:

(Def. 1) $\operatorname{rng} p \subseteq \{0, 1\}$ and for every k such that $k \leq \operatorname{dom} p$ holds $2 \cdot \sum (p \restriction k) \leq k$.

The following propositions are true:

(6) If p is dominated by 0, then $2 \cdot \sum (p \restriction k) \leq k$.

(7) If p is dominated by 0, then $p(0) = 0$.

Let us consider k, m. Then $k \longmapsto m$ is a finite 0-sequence of $\mathbb{N}$.

One can check that there exists a finite 0-sequence of $\mathbb{N}$ which is empty and dominated by 0 and there exists a finite 0-sequence of $\mathbb{N}$ which is non empty and dominated by 0.

The following propositions are true:

(8) $n \longmapsto 0$ is dominated by 0.

(9) If $n \geq m$, then $(n \longmapsto 0) \frown (m \longmapsto 1)$ is dominated by 0.

(10) If p is dominated by 0, then $p \restriction n$ is dominated by 0.

(11) If p is dominated by 0 and q is dominated by 0, then $p \frown q$ is dominated by 0.

(12) If p is dominated by 0, then $2 \cdot \sum (p \restriction (2 \cdot n + 1)) < 2 \cdot n + 1$.

(13) If p is dominated by 0 and $n \leq \operatorname{len} p - 2 \cdot \sum p$, then $p \frown (n \longmapsto 1)$ is dominated by 0.

(14) If p is dominated by 0 and $n \leq (k + \operatorname{len} p) - 2 \cdot \sum p$, then $(k \longmapsto 0) \frown p \frown (n \longmapsto 1)$ is dominated by 0.

(15) If p is dominated by 0 and $2 \cdot \sum (p \restriction k) = k$, then $k \leq \operatorname{len} p$ and $\operatorname{len}(p \restriction k) = k$.

(16) If p is dominated by 0 and $2 \cdot \sum (p \restriction k) = k$ and $p = (p \restriction k) \frown q$, then q is dominated by 0.

(17) If p is dominated by 0 and $2 \cdot \sum(p \restriction k) = k$ and $k = n + 1$, then $p \restriction k = (p \restriction n) \mathbin{^\frown} (1 \longmapsto 1)$.

(18) Let given m, p. Suppose $m = \min^*\{n : 2 \cdot \sum(p \restriction n) = n \ \wedge \ n > 0\}$ and $m > 0$ and p is dominated by 0. Then there exists q such that $p \restriction m = (1 \longmapsto 0) \mathbin{^\frown} q \mathbin{^\frown} (1 \longmapsto 1)$ and q is dominated by 0.

(19) Let given p. Suppose $\operatorname{rng} p \subseteq \{0, 1\}$ and p is not dominated by 0. Then there exists k such that $2 \cdot k + 1 = \min^*\{n : 2 \cdot \sum(p \restriction n) > n\}$ and $2 \cdot k + 1 \leq \operatorname{dom} p$ and $k = \sum(p \restriction (2 \cdot k))$ and $p(2 \cdot k) = 1$.

(20) Let given p, q, k. Suppose $p \restriction (2 \cdot k + \operatorname{len} q) = (k \longmapsto 0) \mathbin{^\frown} q \mathbin{^\frown} (k \longmapsto 1)$ and q is dominated by 0 and $2 \cdot \sum q = \operatorname{len} q$ and $k > 0$. Then $\min^*\{n : 2 \cdot \sum(p \restriction n) = n \ \wedge \ n > 0\} = 2 \cdot k + \operatorname{len} q$.

(21) Let given p. Suppose p is dominated by 0 and $\{i : 2 \cdot \sum(p \restriction i) = i \ \wedge \ i > 0\} = \emptyset$ and $\operatorname{len} p > 0$. Then there exists q such that $p = \langle 0 \rangle \mathbin{^\frown} q$ and q is dominated by 0.

(22) If p is dominated by 0, then $\langle 0 \rangle \mathbin{^\frown} p$ is dominated by 0 and $\{i : 2 \cdot \sum((\langle 0 \rangle \mathbin{^\frown} p) \restriction i) = i \ \wedge \ i > 0\} = \emptyset$.

(23) If $\operatorname{rng} p \subseteq \{0, 1\}$ and p is not dominated by 0 and $2 \cdot k + 1 = \min^*\{n : 2 \cdot \sum(p \restriction n) > n\}$, then $p \restriction (2 \cdot k)$ is dominated by 0.

2. The Recurrence Relation for the Catalan Numbers

Let n, m be natural numbers. The functor $\mathrm{Domin}_0(n, m)$ yields a subset of $\{0, 1\}^\omega$ and is defined as follows:

(Def. 2) $x \in \mathrm{Domin}_0(n, m)$ iff there exists a finite 0-sequence p of $\mathbb{N}$ such that $p = x$ and p is dominated by 0 and $\operatorname{dom} p = n$ and $\sum p = m$.

Next we state two propositions:

(24) $p \in \mathrm{Domin}_0(n, m)$ iff p is dominated by 0 and $\operatorname{dom} p = n$ and $\sum p = m$.

(25) $\mathrm{Domin}_0(n, m) \subseteq \mathrm{Choose}(n, m, 1, 0)$.

Let us consider n, m. One can check that $\mathrm{Domin}_0(n, m)$ is finite.

One can prove the following propositions:

(26) $\mathrm{Domin}_0(n, m)$ is empty iff $2 \cdot m > n$.

(27) $\mathrm{Domin}_0(n, 0) = \{n \longmapsto 0\}$.

(28) $\operatorname{card} \mathrm{Domin}_0(n, 0) = 1$.

(29) Let given p, n. Suppose $\operatorname{rng} p \subseteq \{0, n\}$. Then there exists q such that $\operatorname{len} p = \operatorname{len} q$ and $\operatorname{rng} q \subseteq \{0, n\}$ and for every k such that $k \leq \operatorname{len} p$ holds $\sum(p \restriction k) + \sum(q \restriction k) = n \cdot k$ and for every k such that $k \in \operatorname{len} p$ holds $q(k) = n - p(k)$.

(30) If $m \leq n$, then $\binom{n}{m} > 0$.

(31) If $2\cdot(m+1) \leq n$, then $\mathrm{card}(\mathrm{Choose}(n, m+1, 1, 0) \setminus \mathrm{Domin}_0(n, m+1)) = \mathrm{card}\,\mathrm{Choose}(n, m, 1, 0)$.

(32) If $2\cdot(m+1) \leq n$, then $\mathrm{card}\,\mathrm{Domin}_0(n, m+1) = \binom{n}{m+1} - \binom{n}{m}$.

(33) If $2\cdot m \leq n$, then $\mathrm{card}\,\mathrm{Domin}_0(n, m) = \frac{(n+1)-2\cdot m}{(n+1)-m} \cdot \binom{n}{m}$.

(34) $\mathrm{card}\,\mathrm{Domin}_0(2+k, 1) = k+1$.

(35) $\mathrm{card}\,\mathrm{Domin}_0(4+k, 2) = \frac{(k+1)\cdot(k+4)}{2}$.

(36) $\mathrm{card}\,\mathrm{Domin}_0(6+k, 3) = \frac{(k+1)\cdot(k+5)\cdot(k+6)}{6}$.

(37) $\mathrm{card}\,\mathrm{Domin}_0(2\cdot n, n) = \frac{\binom{2\cdot n}{n}}{n+1}$.

(38) $\mathrm{card}\,\mathrm{Domin}_0(2\cdot n, n) = \mathrm{Catalan}(n+1)$.

Let us consider D. A functional non empty set is said to be a set of ω-sequences of D if:

(Def. 3) For every x such that $x \in$ it holds x is a finite 0-sequence of D.

Let us consider D. Then D^ω is a set of ω-sequences of D. Let F be a set of ω-sequences of D. We see that the element of F is a finite 0-sequence of D.

In the sequel p_2 denotes an element of $\mathbb{N}^\omega$.

We now state several propositions:

(39) $\overline{\overline{\{p_2 : \mathrm{dom}\, p_2 = 2\cdot n \ \wedge\ p_2 \text{ is dominated by } 0\}}} = \binom{2\cdot n}{n}$.

(40) Let given n, m, k, j, l. Suppose $j = n - 2\cdot(k+1)$ and $l = m - (k+1)$. Then $\overline{\overline{\{p_2 : p_2 \in \mathrm{Domin}_0(n, m) \ \wedge\ 2\cdot(k+1) = \min^*\{i : 2\cdot\sum(p_2{\restriction}i) = i \ \wedge\ i > 0\}\}}} = \mathrm{card}\,\mathrm{Domin}_0(2\cdot k, k)\cdot\mathrm{card}\,\mathrm{Domin}_0(j, l)$.

(41) Let given n, m. Suppose $2\cdot m \leq n$. Then there exists a finite 0-sequence C_1 of $\mathbb{N}$ such that $\overline{\overline{\{p_2 : p_2 \in \mathrm{Domin}_0(n, m) \ \wedge\ \{i : 2\cdot\sum(p_2{\restriction}i) = i \ \wedge\ i > 0\} \neq \emptyset\}}} = \sum C_1$ and $\mathrm{dom}\, C_1 = m$ and for every j such that $j < m$ holds $C_1(j) = \mathrm{card}\,\mathrm{Domin}_0(2\cdot j, j)\cdot\mathrm{card}\,\mathrm{Domin}_0(n -' 2\cdot(j+1), m -' (j+1))$.

(42) For every n such that $n > 0$ holds $\mathrm{Domin}_0(2\cdot n, n) = \{p_2 : p_2 \in \mathrm{Domin}_0(2\cdot n, n) \ \wedge\ \{i : 2\cdot\sum(p_2{\restriction}i) = i \ \wedge\ i > 0\} \neq \emptyset\}$.

(43) Let given n. Suppose $n > 0$. Then there exists a finite 0-sequence C_2 of $\mathbb{N}$ such that $\sum C_2 = \mathrm{Catalan}(n+1)$ and $\mathrm{dom}\, C_2 = n$ and for every j such that $j < n$ holds $C_2(j) = \mathrm{Catalan}(j+1)\cdot\mathrm{Catalan}(n -' j)$.

(44) $\overline{\overline{\{p_2 : p_2 \in \mathrm{Domin}_0(n+1, m) \ \wedge\ \{i : 2\cdot\sum(p_2{\restriction}i) = i \ \wedge\ i > 0\} = \emptyset\}}} = \mathrm{card}\,\mathrm{Domin}_0(n, m)$.

(45) Let given n, m. Suppose $2\cdot m \leq n$. Then there exists a finite 0-sequence C_1 of $\mathbb{N}$ such that $\mathrm{card}\,\mathrm{Domin}_0(n, m) = \sum C_1 + \mathrm{card}\,\mathrm{Domin}_0(n -' 1, m)$ and $\mathrm{dom}\, C_1 = m$ and for every j such that $j < m$ holds $C_1(j) = \mathrm{card}\,\mathrm{Domin}_0(2\cdot j, j)\cdot\mathrm{card}\,\mathrm{Domin}_0(n -' 2\cdot(j+1), m -' (j+1))$.

(46) For all n, k there exists p such that $\sum p = \mathrm{card}\,\mathrm{Domin}_0(2\cdot n + 2 + k, n+1)$ and $\mathrm{dom}\, p = k+1$ and for every i such that $i \leq k$ holds $p(i) =$

card $\mathrm{Domin}_0(2 \cdot n + 1 + i, n)$.

3. Cauchy Product

We use the following convention: s_1, s_2, s_3 denote sequences of real numbers, r denotes a real number, and F_1, F_2, F_3 denote finite 0-sequences of $\mathbb{R}$.

Let us consider F_1. The functor $\sum F_1$ yields a real number and is defined as follows:

(Def. 4) $\sum F_1 = +_{\mathbb{R}} \odot F_1$.

Let us consider F_1, x. Then $F_1(x)$ is a real number.

Let s_1, s_2 be sequences of real numbers. The functor $s_1 \, (\#) \, s_2$ yields a sequence of real numbers and is defined by the condition (Def. 5).

(Def. 5) Let k be a natural number. Then there exists a finite 0-sequence F_1 of $\mathbb{R}$ such that $\mathrm{dom}\, F_1 = k + 1$ and for every n such that $n \in k + 1$ holds $F_1(n) = s_1(n) \cdot s_2(k -' n)$ and $\sum F_1 = (s_1 \, (\#) \, s_2)(k)$.

Let us notice that the functor $s_1 \, (\#) \, s_2$ is commutative.

One can prove the following propositions:

(47) For all F_1, n such that $n \in \mathrm{dom}\, F_1$ holds $\sum(F_1{\restriction}n) + F_1(n) = \sum(F_1{\restriction}(n+1))$.

(48) For all F_2, F_3 such that $\mathrm{dom}\, F_2 = \mathrm{dom}\, F_3$ and for every n such that $n \in \mathrm{len}\, F_2$ holds $F_2(n) = F_3(\mathrm{len}\, F_2 -' (1 + n))$ holds $\sum F_2 = \sum F_3$.

(49) For all F_2, F_3 such that $\mathrm{dom}\, F_2 = \mathrm{dom}\, F_3$ and for every n such that $n \in \mathrm{len}\, F_2$ holds $F_2(n) = r \cdot F_3(n)$ holds $\sum F_2 = r \cdot \sum F_3$.

(50) $s_1 \, (\#) \, r \, s_2 = r \, (s_1 \, (\#) \, s_2)$.

(51) $s_1 \, (\#)(s_2 + s_3) = (s_1 \, (\#) \, s_2) + (s_1 \, (\#) \, s_3)$.

(52) $(s_1 \, (\#) \, s_2)(0) = s_1(0) \cdot s_2(0)$.

(53) For all s_1, s_2, n there exists F_1 such that $(\sum_{\alpha=0}^{\kappa}(s_1 \, (\#) \, s_2)(\alpha))_{\kappa \in \mathbb{N}}(n) = \sum F_1$ and $\mathrm{dom}\, F_1 = n + 1$ and for every i such that $i \in n + 1$ holds $F_1(i) = s_1(i) \cdot (\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa \in \mathbb{N}}(n -' i)$.

(54) Let given s_1, s_2, n. Suppose s_2 is summable. Then there exists F_1 such that $(\sum_{\alpha=0}^{\kappa}(s_1 \, (\#) \, s_2)(\alpha))_{\kappa \in \mathbb{N}}(n) = \sum s_2 \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}(n) - \sum F_1$ and $\mathrm{dom}\, F_1 = n + 1$ and for every i such that $i \in n + 1$ holds $F_1(i) = s_1(i) \cdot \sum(s_2 \uparrow ((n -' i) + 1))$.

(55) For every F_1 there exists a finite 0-sequence a_1 of $\mathbb{R}$ such that $\mathrm{dom}\, a_1 = \mathrm{dom}\, F_1$ and $|\sum F_1| \leq \sum a_1$ and for every i such that $i \in \mathrm{dom}\, a_1$ holds $a_1(i) = |F_1(i)|$.

(56) For every s_1 such that s_1 is summable there exists r such that $0 < r$ and for every k holds $|\sum(s_1 \uparrow k)| < r$.

(57) For all s_1, n, m such that $n \leq m$ and for every i holds $s_1(i) \geq 0$ holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m)$.

(58) For all s_1, s_2 such that s_1 is absolutely summable and s_2 is summable holds $s_1\,(\#)\,s_2$ is summable and $\sum(s_1\,(\#)\,s_2) = \sum s_1 \cdot \sum s_2$.

(59) If $p = F_1$, then $\sum p = \sum F_1$.

4. The Generating Function for the Catalan Numbers

Next we state the proposition

(60) Let given r. Then there exists a sequence C_2 of real numbers such that

(i) for every n holds $C_2(n) = \mathrm{Catalan}(n+1) \cdot r^n$, and

(ii) if $|r| < \frac{1}{4}$, then C_2 is absolutely summable and $\sum C_2 = 1 + r \cdot (\sum C_2)^2$ and $\sum C_2 = \frac{2}{1+\sqrt{1-4\cdot r}}$ and if $r \neq 0$, then $\sum C_2 = \frac{1-\sqrt{1-4\cdot r}}{2\cdot r}$.

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The Quaternion Numbers

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Summary. In this article, we define the set $\mathbb{H}$ of quaternion numbers as the set of all ordered sequences $q = \langle x, y, w, z \rangle$ where x,y,w and z are real numbers. The addition, difference and multiplication of the quaternion numbers are also defined. We define the real and imaginary parts of q and denote this by $x = \Re(q)$, $y = \Im_1(q)$, $w = \Im_2(q)$, $z = \Im_3(q)$. We define the addition, difference, multiplication again and denote this operation by real and three imaginary parts. We define the conjugate of q denoted by $q{*}'$ and the absolute value of q denoted by $|q|$. We also give some properties of quaternion numbers.

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The articles [14], [16], [2], [1], [12], [17], [4], [5], [6], [13], [3], [11], [7], [8], [15], [18], [9], and [10] provide the terminology and notation for this paper.

We use the following convention: a, b, c, d, x, y, w, z, x_1, x_2, x_3, x_4 denote sets and A denotes a non empty set.

The functor $\mathbb{H}$ is defined by:

(Def. 1) $\mathbb{H} = (\mathbb{R}^4 \setminus \{x; x \text{ ranges over elements of } \mathbb{R}^4\text{: } x(2) = 0 \ \wedge \ x(3) = 0\}) \cup \mathbb{C}$.

Let x be a number. We say that x is quaternion if and only if:

(Def. 2) $x \in \mathbb{H}$.

Let us observe that $\mathbb{H}$ is non empty.

Let us consider x, y, w, z, a, b, c, d. The functor $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ yields a set and is defined as follows:

(Def. 3) $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = [x \longmapsto a, y \longmapsto b] {+\cdot} [w \longmapsto c, z \longmapsto d]$.

Let us consider x, y, w, z, a, b, c, d. Note that $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ is function-like and relation-like.

Next we state several propositions:

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(1) $\mathrm{dom}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = \{x, y, w, z\}$.

(2) $\mathrm{rng}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] \subseteq \{a, b, c, d\}$.

(3) Suppose x, y, w, z are mutually different. Then $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](x) = a$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](y) = b$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](w) = c$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](z) = d$.

(4) If x, y, w, z are mutually different, then $\mathrm{rng}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = \{a, b, c, d\}$.

(5) $\{x_1, x_2, x_3, x_4\} \subseteq X$ iff $x_1 \in X$ and $x_2 \in X$ and $x_3 \in X$ and $x_4 \in X$.

Let us consider A, x, y, w, z and let a, b, c, d be elements of A. Then $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ is a function from $\{x, y, w, z\}$ into A.

The functor j is defined by:

(Def. 4) $j = [0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 1, 3 \mapsto 0]$.

The functor k is defined by:

(Def. 5) $k = [0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 0, 3 \mapsto 1]$.

One can check the following observations:

* i is quaternion,
* j is quaternion, and
* k is quaternion.

Let us observe that there exists a number which is quaternion.

Let us mention that every element of $\mathbb{H}$ is quaternion.

Let x, y, w, z be elements of $\mathbb{R}$. The functor $\langle x, y, w, z \rangle_{\mathbb{H}}$ yields an element of $\mathbb{H}$ and is defined as follows:

(Def. 6) $$\langle x, y, w, z \rangle_{\mathbb{H}} = \begin{cases} x + yi, & \text{if } w = 0 \text{ and } z = 0, \\ [0 \mapsto x, 1 \mapsto y, 2 \mapsto w, 3 \mapsto z], & \text{otherwise.} \end{cases}$$

Next we state three propositions:

(6) Let a, b, c, d, e, i, j, k be sets and g be a function. Suppose $a \neq b$ and $c \neq d$ and $\mathrm{dom}\, g = \{a, b, c, d\}$ and $g(a) = e$ and $g(b) = i$ and $g(c) = j$ and $g(d) = k$. Then $g = [a \mapsto e, b \mapsto i, c \mapsto j, d \mapsto k]$.

(7) For every element g of $\mathbb{H}$ there exist elements r, s, t, u of $\mathbb{R}$ such that $g = \langle r, s, t, u \rangle_{\mathbb{H}}$.

(8) If a, c, x, w are mutually different, then $[a \mapsto b, c \mapsto d, x \mapsto y, w \mapsto z] = \{\langle a, b \rangle, \langle c, d \rangle, \langle x, y \rangle, \langle w, z \rangle\}$.

We adopt the following convention: a, b, c, d are elements of $\mathbb{R}$ and r, s, t are elements of $\mathbb{Q}_+$.

One can prove the following four propositions:

(9) Let A be a subset of $\mathbb{Q}_+$. Suppose there exists t such that $t \in A$ and $t \neq \emptyset$ and for all r, s such that $r \in A$ and $s \leq r$ holds $s \in A$. Then there exist elements r_1, r_2, r_3, r_4, r_5 of $\mathbb{Q}_+$ such that

$r_1 \in A$ and $r_2 \in A$ and $r_3 \in A$ and $r_4 \in A$ and $r_5 \in A$ and $r_1 \neq r_2$ and $r_1 \neq r_3$ and $r_1 \neq r_4$ and $r_1 \neq r_5$ and $r_2 \neq r_3$ and $r_2 \neq r_4$ and $r_2 \neq r_5$ and $r_3 \neq r_4$ and $r_3 \neq r_5$ and $r_4 \neq r_5$.

(10) $[0 \mapsto a, 1 \mapsto b, 2 \mapsto c, 3 \mapsto d] \notin \mathbb{C}$.

(11) Let a, b, c, d, x, y, z, w, x', y', z', w' be sets. Suppose a, b, c, d are mutually different and $[a \mapsto x, b \mapsto y, c \mapsto z, d \mapsto w] = [a \mapsto x', b \mapsto y', c \mapsto z', d \mapsto w']$. Then $x = x'$ and $y = y'$ and $z = z'$ and $w = w'$.

(12) For all elements x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $\langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}} = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ holds $x_1 = y_1$ and $x_2 = y_2$ and $x_3 = y_3$ and $x_4 = y_4$.

Let x, y be quaternion numbers. The functor $x + y$ is defined by:

(Def. 7) There exist elements x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $x = \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}}$ and $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x + y = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 \rangle_{\mathbb{H}}$.

Let us observe that the functor $x + y$ is commutative.

Let z be a quaternion number. The functor $-z$ yields a quaternion number and is defined by:

(Def. 8) $z + -z = 0$.

Let us observe that the functor $-z$ is involutive.

Let x, y be quaternion numbers. The functor $x - y$ is defined as follows:

(Def. 9) $x - y = x + -y$.

Let x, y be quaternion numbers. The functor $x \cdot y$ is defined by the condition (Def. 10).

(Def. 10) There exist elements x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $x = \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}}$ and $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x \cdot y = \langle x_1 \cdot y_1 - x_2 \cdot y_2 - x_3 \cdot y_3 - x_4 \cdot y_4, (x_1 \cdot y_2 + x_2 \cdot y_1 + x_3 \cdot y_4) - x_4 \cdot y_3, (x_1 \cdot y_3 + y_1 \cdot x_3 + y_2 \cdot x_4) - y_4 \cdot x_2, (x_1 \cdot y_4 + x_4 \cdot y_1 + x_2 \cdot y_3) - x_3 \cdot y_2 \rangle_{\mathbb{H}}$.

Let z, z' be quaternion numbers. One can verify the following observations:

* $z + z'$ is quaternion,
* $z \cdot z'$ is quaternion, and
* $z - z'$ is quaternion.

j Is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 11) $j = \langle 0, 0, 1, 0 \rangle_{\mathbb{H}}$.

Then k is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 12) $k = \langle 0, 0, 0, 1 \rangle_{\mathbb{H}}$.

One can prove the following propositions:

(13) $i \cdot i = -1$.

(14) $j \cdot j = -1$.

(15) $k \cdot k = -1$.

(16) $i \cdot j = k$.

(17) $j \cdot k = i$.

(18) $k \cdot i = j$.

(19) $i \cdot j = -j \cdot i$.

(20) $j \cdot k = -k \cdot j$.

(21) $k \cdot i = -i \cdot k$.

Let z be a quaternion number. The functor $\Re(z)$ is defined as follows:

(Def. 13)(i) There exists a complex number z' such that $z = z'$ and $\Re(z) = \Re(z')$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Re(z) = f(0)$, otherwise.

The functor $\Im_1(z)$ is defined by:

(Def. 14)(i) There exists a complex number z' such that $z = z'$ and $\Im_1(z) = \Im(z')$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Im_1(z) = f(1)$, otherwise.

The functor $\Im_2(z)$ is defined as follows:

(Def. 15)(i) $\Im_2(z) = 0$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Im_2(z) = f(2)$, otherwise.

The functor $\Im_3(z)$ is defined by:

(Def. 16)(i) $\Im_3(z) = 0$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Im_3(z) = f(3)$, otherwise.

Let z be a quaternion number. One can check the following observations:

* $\Re(z)$ is real,
* $\Im_1(z)$ is real,
* $\Im_2(z)$ is real, and
* $\Im_3(z)$ is real.

Let z be a quaternion number. Then $\Re(z)$ is a real number. Then $\Im_1(z)$ is a real number. Then $\Im_2(z)$ is a real number. Then $\Im_3(z)$ is a real number.

One can prove the following two propositions:

(22) For every function f from 4 into $\mathbb{R}$ there exist a, b, c, d such that $f = [0 \mapsto a, 1 \mapsto b, 2 \mapsto c, 3 \mapsto d]$.

(23) $\Re(\langle a, b, c, d \rangle_{\mathbb{H}}) = a$ and $\Im_1(\langle a, b, c, d \rangle_{\mathbb{H}}) = b$ and $\Im_2(\langle a, b, c, d \rangle_{\mathbb{H}}) = c$ and $\Im_3(\langle a, b, c, d \rangle_{\mathbb{H}}) = d$.

In the sequel z, z_1, z_2, z_3, z_4 denote quaternion numbers.

Next we state two propositions:

(24) $z = \langle \Re(z), \Im_1(z), \Im_2(z), \Im_3(z) \rangle_{\mathbb{H}}$.

(25) If $\Re(z_1) = \Re(z_2)$ and $\Im_1(z_1) = \Im_1(z_2)$ and $\Im_2(z_1) = \Im_2(z_2)$ and $\Im_3(z_1) = \Im_3(z_2)$, then $z_1 = z_2$.

The quaternion number $0_{\mathbb{H}}$ is defined as follows:

(Def. 17) $0_{\mathbb{H}} = 0$.

The quaternion number $1_{\mathbb{H}}$ is defined as follows:

(Def. 18) $1_{\mathbb{H}} = 1$.

One can prove the following propositions:

(26) If $\Re(z) = 0$ and $\Im_1(z) = 0$ and $\Im_2(z) = 0$ and $\Im_3(z) = 0$, then $z = 0_{\mathbb{H}}$.

(27) If $z = 0$, then $(\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}} = 0$.

(28) If $(\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}} = 0$, then $z = 0_{\mathbb{H}}$.

(29) $\Re(1_{\mathbb{H}}) = 1$ and $\Im_1(1_{\mathbb{H}}) = 0$ and $\Im_2(1_{\mathbb{H}}) = 0$ and $\Im_3(1_{\mathbb{H}}) = 0$.

(30) $\Re(i) = 0$ and $\Im_1(i) = 1$ and $\Im_2(i) = 0$ and $\Im_3(i) = 0$.

(31) $\Re(j) = 0$ and $\Im_1(j) = 0$ and $\Im_2(j) = 1$ and $\Im_3(j) = 0$ and $\Re(k) = 0$ and $\Im_1(k) = 0$ and $\Im_2(k) = 0$ and $\Im_3(k) = 1$.

(32) $\Re(z_1 + z_2 + z_3 + z_4) = \Re(z_1) + \Re(z_2) + \Re(z_3) + \Re(z_4)$ and $\Im_1(z_1 + z_2 + z_3 + z_4) = \Im_1(z_1) + \Im_1(z_2) + \Im_1(z_3) + \Im_1(z_4)$ and $\Im_2(z_1 + z_2 + z_3 + z_4) = \Im_2(z_1) + \Im_2(z_2) + \Im_2(z_3) + \Im_2(z_4)$ and $\Im_3(z_1 + z_2 + z_3 + z_4) = \Im_3(z_1) + \Im_3(z_2) + \Im_3(z_3) + \Im_3(z_4)$.

In the sequel x denotes a real number.

We now state three propositions:

(33) If $z_1 = x$, then $\Re(z_1 \cdot i) = 0$ and $\Im_1(z_1 \cdot i) = x$ and $\Im_2(z_1 \cdot i) = 0$ and $\Im_3(z_1 \cdot i) = 0$.

(34) If $z_1 = x$, then $\Re(z_1 \cdot j) = 0$ and $\Im_1(z_1 \cdot j) = 0$ and $\Im_2(z_1 \cdot j) = x$ and $\Im_3(z_1 \cdot j) = 0$.

(35) If $z_1 = x$, then $\Re(z_1 \cdot k) = 0$ and $\Im_1(z_1 \cdot k) = 0$ and $\Im_2(z_1 \cdot k) = 0$ and $\Im_3(z_1 \cdot k) = x$.

Let x be a real number and let y be a quaternion number. The functor $x + y$ is defined as follows:

(Def. 19) There exist elements y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x + y = \langle x + y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$.

Let x be a real number and let y be a quaternion number. The functor $x - y$ is defined by:

(Def. 20) $x - y = x + -y$.

Let x be a real number and let y be a quaternion number. The functor $x \cdot y$ is defined as follows:

(Def. 21) There exist elements y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $y = \langle y_1, y_2, y_3, y_4\rangle_{\mathbb{H}}$ and $x \cdot y = \langle x \cdot y_1, x \cdot y_2, x \cdot y_3, x \cdot y_4\rangle_{\mathbb{H}}$.

Let x be a real number and let z' be a quaternion number. One can verify the following observations:

* $x + z'$ is quaternion,
* $x \cdot z'$ is quaternion, and
* $x - z'$ is quaternion.

Let z_1, z_2 be quaternion numbers. Then $z_1 + z_2$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 22) $z_1 + z_2 = \Re(z_1) + \Re(z_2) + (\Im_1(z_1) + \Im_1(z_2)) \cdot i + (\Im_2(z_1) + \Im_2(z_2)) \cdot j + (\Im_3(z_1) + \Im_3(z_2)) \cdot k$.

The following proposition is true

(36) $\Re(z_1 + z_2) = \Re(z_1) + \Re(z_2)$ and $\Im_1(z_1 + z_2) = \Im_1(z_1) + \Im_1(z_2)$ and $\Im_2(z_1 + z_2) = \Im_2(z_1) + \Im_2(z_2)$ and $\Im_3(z_1 + z_2) = \Im_3(z_1) + \Im_3(z_2)$.

Let z_1, z_2 be elements of $\mathbb{H}$. Then $z_1 \cdot z_2$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 23) $z_1 \cdot z_2 = (\Re(z_1) \cdot \Re(z_2) - \Im_1(z_1) \cdot \Im_1(z_2) - \Im_2(z_1) \cdot \Im_2(z_2) - \Im_3(z_1) \cdot \Im_3(z_2)) + ((\Re(z_1) \cdot \Im_1(z_2) + \Im_1(z_1) \cdot \Re(z_2) + \Im_2(z_1) \cdot \Im_3(z_2)) - \Im_3(z_1) \cdot \Im_2(z_2)) \cdot i + ((\Re(z_1) \cdot \Im_2(z_2) + \Im_2(z_1) \cdot \Re(z_2) + \Im_3(z_1) \cdot \Im_1(z_2)) - \Im_1(z_1) \cdot \Im_3(z_2)) \cdot j + ((\Re(z_1) \cdot \Im_3(z_2) + \Im_3(z_1) \cdot \Re(z_2) + \Im_1(z_1) \cdot \Im_2(z_2)) - \Im_2(z_1) \cdot \Im_1(z_2)) \cdot k$.

We now state four propositions:

(37) $z = \Re(z) + \Im_1(z) \cdot i + \Im_2(z) \cdot j + \Im_3(z) \cdot k$.

(38) Suppose $\Im_1(z_1) = 0$ and $\Im_1(z_2) = 0$ and $\Im_2(z_1) = 0$ and $\Im_2(z_2) = 0$ and $\Im_3(z_1) = 0$ and $\Im_3(z_2) = 0$. Then $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2)$ and $\Im_1(z_1 \cdot z_2) = \Im_2(z_1) \cdot \Im_3(z_2) - \Im_3(z_1) \cdot \Im_2(z_2)$ and $\Im_2(z_1 \cdot z_2) = \Im_3(z_1) \cdot \Im_1(z_2) - \Im_1(z_1) \cdot \Im_3(z_2)$ and $\Im_3(z_1 \cdot z_2) = \Im_1(z_1) \cdot \Im_2(z_2) - \Im_2(z_1) \cdot \Im_1(z_2)$.

(39) Suppose $\Re(z_1) = 0$ and $\Re(z_2) = 0$. Then $\Re(z_1 \cdot z_2) = -\Im_1(z_1) \cdot \Im_1(z_2) - \Im_2(z_1) \cdot \Im_2(z_2) - \Im_3(z_1) \cdot \Im_3(z_2)$ and $\Im_1(z_1 \cdot z_2) = \Im_2(z_1) \cdot \Im_3(z_2) - \Im_3(z_1) \cdot \Im_2(z_2)$ and $\Im_2(z_1 \cdot z_2) = \Im_3(z_1) \cdot \Im_1(z_2) - \Im_1(z_1) \cdot \Im_3(z_2)$ and $\Im_3(z_1 \cdot z_2) = \Im_1(z_1) \cdot \Im_2(z_2) - \Im_2(z_1) \cdot \Im_1(z_2)$.

(40) $\Re(z \cdot z) = (\Re(z))^{\mathbf{2}} - (\Im_1(z))^{\mathbf{2}} - (\Im_2(z))^{\mathbf{2}} - (\Im_3(z))^{\mathbf{2}}$ and $\Im_1(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_1(z))$ and $\Im_2(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_2(z))$ and $\Im_3(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_3(z))$.

Let z be a quaternion number. Then $-z$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 24) $-z = -\Re(z) + (-\Im_1(z)) \cdot i + (-\Im_2(z)) \cdot j + (-\Im_3(z)) \cdot k$.

The following proposition is true

(41) $\Re(-z) = -\Re(z)$ and $\Im_1(-z) = -\Im_1(z)$ and $\Im_2(-z) = -\Im_2(z)$ and $\Im_3(-z) = -\Im_3(z)$.

Let z_1, z_2 be quaternion numbers. Then $z_1 - z_2$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 25) $z_1 - z_2 = (\Re(z_1) - \Re(z_2)) + (\Im_1(z_1) - \Im_1(z_2)) \cdot i + (\Im_2(z_1) - \Im_2(z_2)) \cdot j + (\Im_3(z_1) - \Im_3(z_2)) \cdot k$.

One can prove the following proposition

(42) $\Re(z_1 - z_2) = \Re(z_1) - \Re(z_2)$ and $\Im_1(z_1 - z_2) = \Im_1(z_1) - \Im_1(z_2)$ and $\Im_2(z_1 - z_2) = \Im_2(z_1) - \Im_2(z_2)$ and $\Im_3(z_1 - z_2) = \Im_3(z_1) - \Im_3(z_2)$.

Let z be a quaternion number. The functor $\overline{z}$ yielding a quaternion number is defined by:

(Def. 26) $\overline{z} = \Re(z) + (-\Im_1(z)) \cdot i + (-\Im_2(z)) \cdot j + (-\Im_3(z)) \cdot k$.

Let z be a quaternion number. Then $\overline{z}$ is an element of $\mathbb{H}$.

We now state a number of propositions:

(43) $\overline{z} = \langle \Re(z), -\Im_1(z), -\Im_2(z), -\Im_3(z) \rangle_{\mathbb{H}}$.

(44) $\Re(\overline{z}) = \Re(z)$ and $\Im_1(\overline{z}) = -\Im_1(z)$ and $\Im_2(\overline{z}) = -\Im_2(z)$ and $\Im_3(\overline{z}) = -\Im_3(z)$.

(45) If $z = 0$, then $\overline{z} = 0$.

(46) If $\overline{z} = 0$, then $z = 0$.

(47) $\overline{1_{\mathbb{H}}} = 1_{\mathbb{H}}$.

(48) $\Re(\overline{i}) = 0$ and $\Im_1(\overline{i}) = -1$ and $\Im_2(\overline{i}) = 0$ and $\Im_3(\overline{i}) = 0$.

(49) $\Re(\overline{j}) = 0$ and $\Im_1(\overline{j}) = 0$ and $\Im_2(\overline{j}) = -1$ and $\Im_3(\overline{j}) = 0$.

(50) $\Re(\overline{k}) = 0$ and $\Im_1(\overline{k}) = 0$ and $\Im_2(\overline{k}) = 0$ and $\Im_3(\overline{k}) = -1$.

(51) $\overline{i} = -i$.

(52) $\overline{j} = -j$.

(53) $\overline{k} = -k$.

(54) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

(55) $\overline{-z} = -\overline{z}$.

(56) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

(57) If $\Im_2(z_1) \cdot \Im_3(z_2) \neq \Im_3(z_1) \cdot \Im_2(z_2)$, then $\overline{z_1 \cdot z_2} \neq \overline{z_1} \cdot \overline{z_2}$.

(58) If $\Im_1(z) = 0$ and $\Im_2(z) = 0$ and $\Im_3(z) = 0$, then $\overline{z} = z$.

(59) If $\Re(z) = 0$, then $\overline{z} = -z$.

(60) $\Re(z \cdot \overline{z}) = (\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}$ and $\Im_1(z \cdot \overline{z}) = 0$ and $\Im_2(z \cdot \overline{z}) = 0$ and $\Im_3(z \cdot \overline{z}) = 0$.

(61) $\Re(z+\overline{z}) = 2 \cdot \Re(z)$ and $\Im_1(z+\overline{z}) = 0$ and $\Im_2(z+\overline{z}) = 0$ and $\Im_3(z+\overline{z}) = 0$.

(62) $-z = \langle -\Re(z), -\Im_1(z), -\Im_2(z), -\Im_3(z) \rangle_{\mathbb{H}}$.

(63) $z_1 - z_2 = \langle \Re(z_1) - \Re(z_2), \Im_1(z_1) - \Im_1(z_2), \Im_2(z_1) - \Im_2(z_2), \Im_3(z_1) - \Im_3(z_2) \rangle_{\mathbb{H}}$.

(64) $\Re(z - \overline{z}) = 0$ and $\Im_1(z - \overline{z}) = 2 \cdot \Im_1(z)$ and $\Im_2(z - \overline{z}) = 2 \cdot \Im_2(z)$ and $\Im_3(z - \overline{z}) = 2 \cdot \Im_3(z)$.

Let us consider z. The functor $|z|$ yielding a real number is defined by:

(Def. 27) $|z| = \sqrt{(\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}}$.

We now state a number of propositions:

(65) $|0_{\mathbb{H}}| = 0$.

(66) If $|z| = 0$, then $z = 0$.

(67) $0 \leq |z|$.

(68) $|1_{\mathbb{H}}| = 1$.

(69) $|i| = 1$.

(70) $|j| = 1$.

(71) $|k| = 1$.

(72) $|-z| = |z|$.

(73) $|\overline{z}| = |z|$.

(74) $0 \leq (\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}$.

(75) $\Re(z) \leq |z|$.

(76) $\Im_1(z) \leq |z|$.

(77) $\Im_2(z) \leq |z|$.

(78) $\Im_3(z) \leq |z|$.

(79) $|z_1 + z_2| \leq |z_1| + |z_2|$.

(80) $|z_1 - z_2| \leq |z_1| + |z_2|$.

(81) $|z_1| - |z_2| \leq |z_1 + z_2|$.

(82) $|z_1| - |z_2| \leq |z_1 - z_2|$.

(83) $|z_1 - z_2| = |z_2 - z_1|$.

(84) $|z_1 - z_2| = 0$ iff $z_1 = z_2$.

(85) $|z_1 - z_2| \leq |z_1 - z| + |z - z_2|$.

(86) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

(87) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

(88) $|z \cdot z| = (\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}$.

(89) $|z \cdot z| = |z \cdot \overline{z}|$.

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Model Checking. Part I

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Summary. This text includes definitions of the Kripke structure, CTL (Computation Tree Logic), and verification of the basic algorithm for Model Checking based on CTL in [10].

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The articles [21], [20], [16], [9], [18], [14], [6], [7], [4], [3], [5], [11], [2], [8], [13], [12], [17], [15], [1], and [19] provide the notation and terminology for this paper.

Let x, S be sets and let a be an element of S. The functor $\mathrm{k.id}(x,S,a)$ yields an element of S and is defined by:

(Def. 1) $\mathrm{k.id}(x,S,a) = \begin{cases} x, & \text{if } x \in S, \\ a, & \text{otherwise.} \end{cases}$

Let x be a set. The functor $\mathrm{k.nat}\, x$ yields an element of $\mathbb{N}$ and is defined by:

(Def. 2) $\mathrm{k.nat}\, x = \begin{cases} x, & \text{if } x \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$

Let f be a function and let x, a be sets. The functor $\mathrm{UnivF}(x,f,a)$ yielding a set is defined by:

(Def. 3) $\mathrm{UnivF}(x,f,a) = \begin{cases} f(x), & \text{if } x \in \mathrm{dom}\, f, \\ a, & \text{otherwise.} \end{cases}$

Let a be a set. The functor $\mathrm{Castboolean}\, a$ yields a boolean set and is defined by:

(Def. 4) $\mathrm{Castboolean}\, a = \begin{cases} a, & \text{if } a \text{ is a boolean set,} \\ \mathit{false}, & \text{otherwise.} \end{cases}$

Let X, a be sets. The functor $\mathrm{CastBool}(a,X)$ yielding a subset of X is defined as follows:

(Def. 5) $\mathrm{CastBool}(a,X) = \begin{cases} a, & \text{if } a \subseteq X, \\ \emptyset, & \text{otherwise.} \end{cases}$

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For simplicity, we adopt the following rules: n denotes an element of $\mathbb{N}$, a denotes a set, D denotes a non empty set, and p, q denote finite sequences of elements of $\mathbb{N}$.

Let x be a variable. Then $\langle x \rangle$ is a finite sequence of elements of $\mathbb{N}$.

Let us consider n. The functor atom. n yields a finite sequence of elements of $\mathbb{N}$ and is defined by:

(Def. 6) $\text{atom.}\, n = \langle 5 + n \rangle$.

Let us consider p. The functor $\neg p$ yielding a finite sequence of elements of $\mathbb{N}$ is defined by:

(Def. 7) $\neg p = \langle 0 \rangle \frown p$.

Let us consider q. The functor $p \wedge q$ yielding a finite sequence of elements of $\mathbb{N}$ is defined by:

(Def. 8) $p \wedge q = \langle 1 \rangle \frown p \frown q$.

Let us consider p. The functor $\mathrm{EX}\, p$ yielding a finite sequence of elements of $\mathbb{N}$ is defined as follows:

(Def. 9) $\mathrm{EX}\, p = \langle 2 \rangle \frown p$.

The functor $\mathrm{EG}\, p$ yielding a finite sequence of elements of $\mathbb{N}$ is defined by:

(Def. 10) $\mathrm{EG}\, p = \langle 3 \rangle \frown p$.

Let us consider q. The functor $p\,\mathrm{EU}\,q$ yields a finite sequence of elements of $\mathbb{N}$ and is defined as follows:

(Def. 11) $p\,\mathrm{EU}\,q = \langle 4 \rangle \frown p \frown q$.

The non empty set CTL-WFF is defined by the conditions (Def. 12).

(Def. 12) For every a such that $a \in$ CTL-WFF holds a is a finite sequence of elements of $\mathbb{N}$ and for every n holds atom. $n \in$ CTL-WFF and for every p such that $p \in$ CTL-WFF holds $\neg p \in$ CTL-WFF and for all p, q such that $p \in$ CTL-WFF and $q \in$ CTL-WFF holds $p \wedge q \in$ CTL-WFF and for every p such that $p \in$ CTL-WFF holds $\mathrm{EX}\, p \in$ CTL-WFF and for every p such that $p \in$ CTL-WFF holds $\mathrm{EG}\, p \in$ CTL-WFF and for all p, q such that $p \in$ CTL-WFF and $q \in$ CTL-WFF holds $p\,\mathrm{EU}\,q \in$ CTL-WFF and for every D such that for every a such that $a \in D$ holds a is a finite sequence of elements of $\mathbb{N}$ and for every n holds atom. $n \in D$ and for every p such that $p \in D$ holds $\neg p \in D$ and for all p, q such that $p \in D$ and $q \in D$ holds $p \wedge q \in D$ and for every p such that $p \in D$ holds $\mathrm{EX}\, p \in D$ and for every p such that $p \in D$ holds $\mathrm{EG}\, p \in D$ and for all p, q such that $p \in D$ and $q \in D$ holds $p\,\mathrm{EU}\,q \in D$ holds CTL-WFF $\subseteq D$.

Let I_1 be a finite sequence of elements of $\mathbb{N}$. We say that I_1 is CTL-formula-like if and only if:

(Def. 13) I_1 is an element of CTL-WFF.

Let us mention that there exists a finite sequence of elements of $\mathbb{N}$ which is CTL-formula-like.

A CTL-formula is a CTL-formula-like finite sequence of elements of $\mathbb{N}$.

One can prove the following proposition

(1) a is a CTL-formula iff $a \in$ CTL-WFF.

In the sequel F, G, H, H_1, H_2 denote CTL-formulae.

Let us consider n. One can verify that atom. n is CTL-formula-like.

Let us consider H. One can verify the following observations:

* $\neg H$ is CTL-formula-like,
* $\mathrm{EX}\, H$ is CTL-formula-like, and
* $\mathrm{EG}\, H$ is CTL-formula-like.

Let us consider G. One can verify that $H \wedge G$ is CTL-formula-like and $H \,\mathrm{EU}\, G$ is CTL-formula-like.

Let us consider H. We say that H is atomic if and only if:

(Def. 14) There exists n such that $H =$ atom. n.

We say that H is negative if and only if:

(Def. 15) There exists H_1 such that $H = \neg H_1$.

We say that H is conjunctive if and only if:

(Def. 16) There exist F, G such that $H = F \wedge G$.

We say that H is exist-next-formula if and only if:

(Def. 17) There exists H_1 such that $H = \mathrm{EX}\, H_1$.

We say that H is exist-global-formula if and only if:

(Def. 18) There exists H_1 such that $H = \mathrm{EG}\, H_1$.

We say that H is exist-until-formula if and only if:

(Def. 19) There exist F, G such that $H = F \,\mathrm{EU}\, G$.

Let us consider F, G. The functor $F \vee G$ yielding a CTL-formula is defined by:

(Def. 20) $F \vee G = \neg(\neg F \wedge \neg G)$.

One can prove the following proposition

(2) H is atomic, or negative, or conjunctive, or exist-next-formula, or exist-global-formula, or exist-until-formula.

Let us consider H. Let us assume that H is negative, or exist-next-formula, or exist-global-formula. The functor $\mathrm{Arg}(H)$ yielding a CTL-formula is defined as follows:

(Def. 21)(i) $\neg\, \mathrm{Arg}(H) = H$ if H is negative,

(ii) $\mathrm{EX}\, \mathrm{Arg}(H) = H$ if H is exist-next-formula,

(iii) $\mathrm{EG}\, \mathrm{Arg}(H) = H$, otherwise.

Let us consider H. Let us assume that H is conjunctive or exist-until-formula. The functor $\mathrm{LeftArg}(H)$ yields a CTL-formula and is defined as follows:

(Def. 22)(i) There exists H_1 such that $\mathrm{LeftArg}(H) \wedge H_1 = H$ if H is conjunctive,

(ii) there exists H_1 such that $\mathrm{LeftArg}(H)\,\mathrm{EU}\,H_1 = H$, otherwise.

The functor $\mathrm{RightArg}(H)$ yielding a CTL-formula is defined by:

(Def. 23)(i) There exists H_1 such that $H_1 \wedge \mathrm{RightArg}(H) = H$ if H is conjunctive,

(ii) there exists H_1 such that $H_1\,\mathrm{EU}\,\mathrm{RightArg}(H) = H$, otherwise.

Let x be a set. The functor $\mathrm{CastCTLformula}\,x$ yields a CTL-formula and is defined by:

(Def. 24) $\mathrm{CastCTLformula}\,x = \begin{cases} x, & \text{if } x \in \text{CTL-WFF}, \\ \mathrm{atom}.\,0, & \text{otherwise.} \end{cases}$

Let P_1 be a set. We consider Kripke structures over P_1 as systems

$\langle$ worlds, starts, possibilities, a label $\rangle$,

where the worlds constitute a set, the starts constitute a subset of the worlds, the possibilities constitute a total relation between the worlds and the worlds, and the label is a function from the worlds into 2^{P_1}.

We introduce CTL model structures which are systems

$\langle$ assignations, basic assignations, a conjunction, a negation, a next-operation, a global-operation, an until-operation $\rangle$,

where the assignations constitute a non empty set, the basic assignations constitute a non empty subset of the assignations, the conjunction is a binary operation on the assignations, the negation is a unary operation on the assignations, the next-operation is a unary operation on the assignations, the global-operation is a unary operation on the assignations, and the until-operation is a binary operation on the assignations.

Let V be a CTL model structure. An assignation of V is an element of the assignations of V.

The subset the atomic WFF of CTL-WFF is defined by:

(Def. 25) The atomic WFF $= \{x; x$ ranges over CTL-formulae: x is atomic$\}$.

Let V be a CTL model structure, let K_1 be a function from the atomic WFF into the basic assignations of V, and let f be a function from CTL-WFF into the assignations of V. We say that f is an evaluation for K_1 if and only if the condition (Def. 26) is satisfied.

(Def. 26) Let H be a CTL-formula. Then

(i) if H is atomic, then $f(H) = K_1(H)$,

(ii) if H is negative, then $f(H) = (\text{the negation of } V)(f(\mathrm{Arg}(H)))$,

(iii) if H is conjunctive, then $f(H) = (\text{the conjunction of } V)(f(\mathrm{LeftArg}(H)), f(\mathrm{RightArg}(H)))$,

(iv) if H is exist-next-formula, then $f(H) = (\text{the next-operation of } V)(f(\mathrm{Arg}(H)))$,

(v) if H is exist-global-formula, then $f(H)$ = (the global-operation of V)($f(\mathrm{Arg}(H))$), and

(vi) if H is exist-until-formula, then $f(H)$ = (the until-operation of V)($f(\mathrm{LeftArg}(H))$, $f(\mathrm{RightArg}(H))$).

Let V be a CTL model structure, let K_1 be a function from the atomic WFF into the basic assignations of V, let f be a function from CTL-WFF into the assignations of V, and let n be an element of $\mathbb{N}$. We say that f is a n-pre-evaluation for K_1 if and only if the condition (Def. 27) is satisfied.

(Def. 27) Let H be a CTL-formula such that $\mathrm{len}\, H \leq n$. Then

(i) if H is atomic, then $f(H) = K_1(H)$,

(ii) if H is negative, then $f(H)$ = (the negation of V)($f(\mathrm{Arg}(H))$),

(iii) if H is conjunctive, then $f(H)$ = (the conjunction of V)($f(\mathrm{LeftArg}(H))$, $f(\mathrm{RightArg}(H))$),

(iv) if H is exist-next-formula, then $f(H)$ = (the next-operation of V)($f(\mathrm{Arg}(H))$),

(v) if H is exist-global-formula, then $f(H)$ = (the global-operation of V)($f(\mathrm{Arg}(H))$), and

(vi) if H is exist-until-formula, then $f(H)$ = (the until-operation of V)($f(\mathrm{LeftArg}(H))$, $f(\mathrm{RightArg}(H))$).

Let V be a CTL model structure, let K_1 be a function from the atomic WFF into the basic assignations of V, let f, h be functions from CTL-WFF into the assignations of V, let n be an element of $\mathbb{N}$, and let H be a CTL-formula. The functor $\mathrm{GraftEval}(V, K_1, f, h, n, H)$ yields a set and is defined as follows:

(Def. 28) $\mathrm{GraftEval}(V, K_1, f, h, n, H) =$

$$\begin{cases} f(H), & \text{if } \mathrm{len}\, H > n+1, \\ K_1(H), & \text{if } \mathrm{len}\, H = n+1 \text{ and } H \text{ is atomic}, \\ (\text{the negation of } V)(h(\mathrm{Arg}(H))), & \text{if } \mathrm{len}\, H = n+1 \text{ and } H \text{ is negative}, \\ (\text{the conjunction of } V)(h(\mathrm{LeftArg}(H)), h(\mathrm{RightArg}(H))), & \text{if } \mathrm{len}\, H = n+1 \text{ and } H \text{ is conjunctive}, \\ (\text{the next-operation of } V)(h(\mathrm{Arg}(H))), & \text{if } \mathrm{len}\, H = n+1 \text{ and } H \text{ is exist-next-formula}, \\ (\text{the global-operation of } V)(h(\mathrm{Arg}(H))), & \text{if } \mathrm{len}\, H = n+1 \text{ and } H \text{ is exist-global-formula}, \\ (\text{the until-operation of } V)(h(\mathrm{LeftArg}(H)), h(\mathrm{RightArg}(H))), & \text{if } \mathrm{len}\, H = n+1 \text{ and } H \text{ is exist-until-formula}, \\ h(H), & \text{if } \mathrm{len}\, H < n+1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We follow the rules: V is a CTL model structure, K_1 is a function from the atomic WFF into the basic assignations of V, and f, f_1, f_2 are functions from CTL-WFF into the assignations of V.

Let V be a CTL model structure, let K_1 be a function from the atomic

WFF into the basic assignations of V, and let n be an element of $\mathbb{N}$. The functor $\mathrm{EvalSet}(V, K_1, n)$ yields a non empty set and is defined by:

(Def. 29) $\mathrm{EvalSet}(V, K_1, n) = \{h; h$ ranges over functions from CTL-WFF into the assignations of V: h is a n-pre-evaluation for $K_1\}$.

Let V be a CTL model structure, let v_0 be an element of the assignations of V, and let x be a set. The functor $\mathrm{CastEval}(V, x, v_0)$ yielding a function from CTL-WFF into the assignations of V is defined by:

(Def. 30) $\mathrm{CastEval}(V, x, v_0) = \begin{cases} x, & \text{if } x \in (\text{the assignations of } V)^{\text{CTL-WFF}}, \\ \text{CTL-WFF} \longmapsto v_0, & \text{otherwise.} \end{cases}$

Let V be a CTL model structure and let K_1 be a function from the atomic WFF into the basic assignations of V. The functor $\mathrm{EvalFamily}(V, K_1)$ yielding a non empty set is defined by the condition (Def. 31).

(Def. 31) Let p be a set. Then $p \in \mathrm{EvalFamily}(V, K_1)$ if and only if the following conditions are satisfied:

 (i) $p \in 2^{(\text{the assignations of } V)^{\text{CTL-WFF}}}$, and

 (ii) there exists an element n of $\mathbb{N}$ such that $p = \mathrm{EvalSet}(V, K_1, n)$.

We now state two propositions:

(3) There exists f which is an evaluation for K_1.

(4) If f_1 is an evaluation for K_1 and f_2 is an evaluation for K_1, then $f_1 = f_2$.

Let V be a CTL model structure, let K_1 be a function from the atomic WFF into the basic assignations of V, and let H be a CTL-formula. The functor $\mathrm{Evaluate}(H, K_1)$ yields an assignation of V and is defined by:

(Def. 32) There exists a function f from CTL-WFF into the assignations of V such that f is an evaluation for K_1 and $\mathrm{Evaluate}(H, K_1) = f(H)$.

Let V be a CTL model structure and let f be an assignation of V. The functor $\neg f$ yields an assignation of V and is defined as follows:

(Def. 33) $\neg f = (\text{the negation of } V)(f)$.

Let V be a CTL model structure and let f, g be assignations of V. The functor $f \wedge g$ yielding an assignation of V is defined by:

(Def. 34) $f \wedge g = (\text{the conjunction of } V)(f, g)$.

Let V be a CTL model structure and let f be an assignation of V. The functor $\mathrm{EX}\, f$ yields an assignation of V and is defined by:

(Def. 35) $\mathrm{EX}\, f = (\text{the next-operation of } V)(f)$.

The functor $\mathrm{EG}\, f$ yielding an assignation of V is defined as follows:

(Def. 36) $\mathrm{EG}\, f = (\text{the global-operation of } V)(f)$.

Let V be a CTL model structure and let f, g be assignations of V. The functor $f\, \mathrm{EU}\, g$ yields an assignation of V and is defined as follows:

(Def. 37) $f\, \mathrm{EU}\, g = (\text{the until-operation of } V)(f, g)$.

The functor $f \vee g$ yielding an assignation of V is defined as follows:

(Def. 38) $f \vee g = \neg(\neg f \wedge \neg g)$.

Next we state several propositions:

(5) $\mathrm{Evaluate}(\neg H, K_1) = \neg\, \mathrm{Evaluate}(H, K_1)$.

(6) $\mathrm{Evaluate}(H_1 \wedge H_2, K_1) = \mathrm{Evaluate}(H_1, K_1) \wedge \mathrm{Evaluate}(H_2, K_1)$.

(7) $\mathrm{Evaluate}(\mathrm{EX}\, H, K_1) = \mathrm{EX}\, \mathrm{Evaluate}(H, K_1)$.

(8) $\mathrm{Evaluate}(\mathrm{EG}\, H, K_1) = \mathrm{EG}\, \mathrm{Evaluate}(H, K_1)$.

(9) $\mathrm{Evaluate}(H_1\, \mathrm{EU}\, H_2, K_1) = \mathrm{Evaluate}(H_1, K_1)\, \mathrm{EU}\, \mathrm{Evaluate}(H_2, K_1)$.

(10) $\mathrm{Evaluate}(H_1 \vee H_2, K_1) = \mathrm{Evaluate}(H_1, K_1) \vee \mathrm{Evaluate}(H_2, K_1)$.

Let f be a function and let n be an element of $\mathbb{N}$. We introduce f^n as a synonym of f^n.

Let S be a set, let f be a function from S into S, and let n be an element of $\mathbb{N}$. Then f^n is a function from S into S.

We use the following convention: S is a non empty set, R is a total relation between S and S, and s, s_0, s_1 are elements of S.

The scheme *ExistPath* deals with a non empty set $\mathcal{A}$, a total relation $\mathcal{B}$ between $\mathcal{A}$ and $\mathcal{A}$, an element $\mathcal{C}$ of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{C}$ and for every element n of $\mathbb{N}$ holds $\langle f(n), f(n+1)\rangle \in \mathcal{B}$ and $f(n+1) \in \mathcal{F}(f(n))$

provided the following requirement is met:

- For every element s of $\mathcal{A}$ holds $\mathcal{B}^\circ\{s\} \cap \mathcal{F}(s)$ is a non empty subset of $\mathcal{A}$.

Let S be a non empty set and let R be a total relation between S and S. A function from $\mathbb{N}$ into S is said to be an infinity path of R if:

(Def. 39) For every element n of $\mathbb{N}$ holds $\langle \mathrm{it}(n), \mathrm{it}(n+1)\rangle \in R$.

Let S be a non empty set. The functor $\mathrm{ModelSP}\, S$ yields a non empty set and is defined by:

(Def. 40) $\mathrm{ModelSP}\, S = \mathit{Boolean}^S$.

Let S be a non empty set. Observe that $\mathrm{ModelSP}\, S$ is non empty.

Let S be a non empty set and let f be a set. The functor $\mathrm{Fid}(f, S)$ yielding a function from S into *Boolean* is defined by:

(Def. 41) $\mathrm{Fid}(f, S) = \begin{cases} f, & \text{if } f \in \mathrm{ModelSP}\, S, \\ S \longmapsto \mathit{false}, & \text{otherwise.} \end{cases}$

Now we present several schemes. The scheme *Func1EX* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$ from $\mathcal{A}$ into *Boolean*, and a binary functor $\mathcal{F}$ yielding a boolean set, and states that:

> There exists a set g such that $g \in \mathrm{ModelSP}\, \mathcal{A}$ and for every set s such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}) = \mathit{true}$ iff $(\mathrm{Fid}(g, \mathcal{A}))(s) = \mathit{true}$

for all values of the parameters.

The scheme *Func1Unique* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$ from $\mathcal{A}$ into *Boolean*, and a binary functor $\mathcal{F}$ yielding a boolean set, and states that:

Let g_1, g_2 be sets. Suppose that
(i) $g_1 \in \mathrm{ModelSP}\,\mathcal{A}$,
(ii) for every set s such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}) = \mathit{true}$ iff $(\mathrm{Fid}(g_1, \mathcal{A}))(s) = \mathit{true}$,
(iii) $g_2 \in \mathrm{ModelSP}\,\mathcal{A}$, and
(iv) for every set s such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}) = \mathit{true}$ iff $(\mathrm{Fid}(g_2, \mathcal{A}))(s) = \mathit{true}$.
Then $g_1 = g_2$

for all values of the parameters.

The scheme *UnOpEX* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a unary operation o on $\mathcal{A}$ such that for every set f such that $f \in \mathcal{A}$ holds $o(f) = \mathcal{F}(f)$

for all values of the parameters.

The scheme *UnOpUnique* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

Let o_1, o_2 be unary operations on $\mathcal{B}$. Suppose for every set f such that $f \in \mathcal{B}$ holds $o_1(f) = \mathcal{F}(f)$ and for every set f such that $f \in \mathcal{B}$ holds $o_2(f) = \mathcal{F}(f)$. Then $o_1 = o_2$

for all values of the parameters.

The scheme *Func2EX* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$ from $\mathcal{A}$ into *Boolean*, a function $\mathcal{C}$ from $\mathcal{A}$ into *Boolean*, and a ternary functor $\mathcal{F}$ yielding a boolean set, and states that:

There exists a set h such that $h \in \mathrm{ModelSP}\,\mathcal{A}$ and for every set s such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}, \mathcal{C}) = \mathit{true}$ iff $(\mathrm{Fid}(h, \mathcal{A}))(s) = \mathit{true}$

for all values of the parameters.

The scheme *Func2Unique* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$ from $\mathcal{A}$ into *Boolean*, a function $\mathcal{C}$ from $\mathcal{A}$ into *Boolean*, and a ternary functor $\mathcal{F}$ yielding a boolean set, and states that:

Let h_1, h_2 be sets. Suppose that
(i) $h_1 \in \mathrm{ModelSP}\,\mathcal{A}$,
(ii) for every set s such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}, \mathcal{C}) = \mathit{true}$ iff $(\mathrm{Fid}(h_1, \mathcal{A}))(s) = \mathit{true}$,
(iii) $h_2 \in \mathrm{ModelSP}\,\mathcal{A}$, and
(iv) for every set s such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}, \mathcal{C}) = \mathit{true}$ iff $(\mathrm{Fid}(h_2, \mathcal{A}))(s) = \mathit{true}$.
Then $h_1 = h_2$

for all values of the parameters.

Let S be a non empty set and let f be a set. The functor $\mathrm{Not}_0(f, S)$ yielding an element of $\mathrm{ModelSP}\,S$ is defined as follows:

(Def. 42) For every set s such that $s \in S$ holds $\neg$ Castboolean(Fid$(f, S))(s) = true$ iff (Fid(Not$_0(f, S), S))(s) = true$.

Let S be a non empty set. The functor Not S yields a unary operation on ModelSP S and is defined by:

(Def. 43) For every set f such that $f \in$ ModelSP S holds (Not $S)(f) = \mathrm{Not}_0(f, S)$.

Let S be a non empty set, let R be a total relation between S and S, let f be a function from S into *Boolean*, and let x be a set. The functor $\mathrm{EneXt}_{\mathrm{univ}}(x, f, R)$ yielding an element of *Boolean* is defined by:

(Def. 44) $$\mathrm{EneXt}_{\mathrm{univ}}(x, f, R) = \begin{cases} \mathit{true}, \\ \quad \text{if } x \in S \text{ and there exists an infinity path } p_1 \\ \quad \text{of } R \text{ such that } p_1(0) = x \text{ and } f(p_1(1)) = \mathit{true}, \\ \mathit{false}, \text{ otherwise.} \end{cases}$$

Let S be a non empty set, let R be a total relation between S and S, and let f be a set. The functor $\mathrm{EneXt}_0(f, R)$ yielding an element of ModelSP S is defined as follows:

(Def. 45) For every set s such that $s \in S$ holds $\mathrm{EneXt}_{\mathrm{univ}}(s, \mathrm{Fid}(f, S), R) = true$ iff (Fid(EneXt$_0(f, R), S))(s) = true$.

Let S be a non empty set and let R be a total relation between S and S. The functor EneXt R yields a unary operation on ModelSP S and is defined by:

(Def. 46) For every set f such that $f \in$ ModelSP S holds (EneXt $R)(f) = \mathrm{EneXt}_0(f, R)$.

Let S be a non empty set, let R be a total relation between S and S, let f be a function from S into *Boolean*, and let x be a set. The functor $\mathrm{EGlobal}_{\mathrm{univ}}(x, f, R)$ yielding an element of *Boolean* is defined by:

(Def. 47) $$\mathrm{EGlobal}_{\mathrm{univ}}(x, f, R) = \begin{cases} \mathit{true}, \\ \quad \text{if } x \in S \text{ and there exists an infinity path} \\ \quad p_1 \text{ of } R \text{ such that } p_1(0) = x \text{ and for every} \\ \quad \text{element } n \text{ of } \mathbb{N} \text{ holds } f(p_1(n)) = \mathit{true}, \\ \mathit{false}, \text{ otherwise.} \end{cases}$$

Let S be a non empty set, let R be a total relation between S and S, and let f be a set. The functor $\mathrm{EGlobal}_0(f, R)$ yielding an element of ModelSP S is defined as follows:

(Def. 48) For every set s such that $s \in S$ holds $\mathrm{EGlobal}_{\mathrm{univ}}(s, \mathrm{Fid}(f, S), R) = true$ iff (Fid(EGlobal$_0(f, R), S))(s) = true$.

Let S be a non empty set and let R be a total relation between S and S. The functor EGlobal R yields a unary operation on ModelSP S and is defined as follows:

(Def. 49) For every set f such that $f \in$ ModelSP S holds (EGlobal $R)(f) = \mathrm{EGlobal}_0(f, R)$.

Let S be a non empty set and let f, g be sets. The functor $\mathrm{And}_0(f,g,S)$ yields an element of ModelSP S and is defined as follows:

(Def. 50) For every set s such that $s \in S$ holds $\mathrm{Castboolean}(\mathrm{Fid}(f,S))(s) \wedge \mathrm{Castboolean}(\mathrm{Fid}(g,S))(s) = \mathit{true}$ iff $(\mathrm{Fid}(\mathrm{And}_0(f,g,S),S))(s) = \mathit{true}$.

Let S be a non empty set. The and S yielding a binary operation on ModelSP S is defined by:

(Def. 51) For all sets f, g such that $f \in \mathrm{ModelSP}\,S$ and $g \in \mathrm{ModelSP}\,S$ holds (the and $S)(f,\,g) = \mathrm{And}_0(f,g,S)$.

Let S be a non empty set, let R be a total relation between S and S, let f, g be functions from S into *Boolean*, and let x be a set. The functor $\mathrm{EUntill}_{\mathrm{univ}}(x,f,g,R)$ yielding an element of *Boolean* is defined as follows:

(Def. 52) $$\mathrm{EUntill}_{\mathrm{univ}}(x,f,g,R) = \begin{cases} \mathit{true}, \text{if } x \in S \text{ and there exists an infinity path } p_1 \text{ of } R \text{ such that } p_1(0) = x \text{ and there exists an element } m \text{ of } \mathbb{N} \text{ such that for every element } j \text{ of } \mathbb{N} \text{ such that } j < m \text{ holds } f(p_1(j)) = \mathit{true} \text{ and } g(p_1(m)) = \mathit{true}, \\ \mathit{false}, \text{otherwise.} \end{cases}$$

Let S be a non empty set, let R be a total relation between S and S, and let f, g be sets. The functor $\mathrm{EUntill}_0(f,g,R)$ yields an element of ModelSP S and is defined by:

(Def. 53) For every set s such that $s \in S$ holds $\mathrm{EUntill}_{\mathrm{univ}}(s,\mathrm{Fid}(f,S),\mathrm{Fid}(g,S),R) = \mathit{true}$ iff $(\mathrm{Fid}(\mathrm{EUntill}_0(f,g,R),S))(s) = \mathit{true}$.

Let S be a non empty set and let R be a total relation between S and S. The functor EUntill R yields a binary operation on ModelSP S and is defined as follows:

(Def. 54) For all sets f, g such that $f \in \mathrm{ModelSP}\,S$ and $g \in \mathrm{ModelSP}\,S$ holds $(\mathrm{EUntill}\,R)(f,\,g) = \mathrm{EUntill}_0(f,g,R)$.

Let S be a non empty set, let X be a non empty subset of ModelSP S, and let s be a set. The functor $\mathrm{F\text{-}LABEL}(s,X)$ yields a subset of X and is defined as follows:

(Def. 55) For every set x holds $x \in \mathrm{F\text{-}LABEL}(s,X)$ iff $x \in X$ and there exists a function f from S into *Boolean* such that $f = x$ and $f(s) = \mathit{true}$.

Let S be a non empty set and let X be a non empty subset of ModelSP S. The functor Label X yields a function from S into 2^X and is defined by:

(Def. 56) For every set x such that $x \in S$ holds $(\mathrm{Label}\,X)(x) = \mathrm{F\text{-}LABEL}(x,X)$.

Let S be a non empty set, let S_0 be a subset of S, let R be a total relation between S and S, and let P_1 be a non empty subset of ModelSP S. The functor $\mathrm{KModel}(R,S_0,P_1)$ yields a Kripke structure over P_1 and is defined as follows:

(Def. 57) $\mathrm{KModel}(R,S_0,P_1) = \langle S,S_0,R,\mathrm{Label}\,P_1\rangle$.

Let S be a non empty set, let S_0 be a subset of S, let R be a total relation between S and S, and let P_1 be a non empty subset of ModelSP S. One can check that the worlds of KModel(R, S_0, P_1) is non empty.

Let S be a non empty set, let S_0 be a subset of S, let R be a total relation between S and S, and let P_1 be a non empty subset of ModelSP S. One can verify that ModelSP (the worlds of KModel(R, S_0, P_1)) is non empty.

Let S be a non empty set, let R be a total relation between S and S, and let B_1 be a non empty subset of ModelSP S. The functor CTLModel(R, B_1) yielding a CTL model structure is defined as follows:

(Def. 58) $\mathrm{CTLModel}(R, B_1) = \langle \mathrm{ModelSP}\, S, B_1,$ the and S, $\mathrm{Not}\, S, \mathrm{EneXt}\, R,$ $\mathrm{EGlobal}\, R, \mathrm{EUntill}\, R\rangle$.

In the sequel B_1 is a non empty subset of ModelSP S and k_1 is a function from the atomic WFF into the basic assignations of CTLModel(R, B_1).

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of ModelSP S, let s be an element of S, and let f be an assignation of CTLModel(R, B_1). The predicate $s \models f$ is defined by:

(Def. 59) $(\mathrm{Fid}(f, S))(s) = \mathit{true}$.

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of ModelSP S, let s be an element of S, and let f be an assignation of CTLModel(R, B_1). We introduce $s \not\models f$ as an antonym of $s \models f$.

Next we state several propositions:

(11) For every assignation a of CTLModel(R, B_1) such that $a \in B_1$ holds $s \models a$ iff $a \in (\mathrm{Label}\, B_1)(s)$.

(12) For every assignation f of CTLModel(R, B_1) holds $s \models \neg f$ iff $s \not\models f$.

(13) For all assignations f, g of CTLModel(R, B_1) holds $s \models f \wedge g$ iff $s \models f$ and $s \models g$.

(14) For every assignation f of CTLModel(R, B_1) holds $s \models \mathrm{EX}\, f$ iff there exists an infinity path p_1 of R such that $p_1(0) = s$ and $p_1(1) \models f$.

(15) Let f be an assignation of CTLModel(R, B_1). Then $s \models \mathrm{EG}\, f$ if and only if there exists an infinity path p_1 of R such that $p_1(0) = s$ and for every element n of $\mathbb{N}$ holds $p_1(n) \models f$.

(16) Let f, g be assignations of CTLModel(R, B_1). Then $s \models f\, \mathrm{EU}\, g$ if and only if there exists an infinity path p_1 of R such that $p_1(0) = s$ and there exists an element m of $\mathbb{N}$ such that for every element j of $\mathbb{N}$ such that $j < m$ holds $p_1(j) \models f$ and $p_1(m) \models g$.

(17) For all assignations f, g of CTLModel(R, B_1) holds $s \models f \vee g$ iff $s \models f$ or $s \models g$.

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of ModelSP S, let k_1 be a function from the atomic

WFF into the basic assignations of CTLModel(R, B_1), let s be an element of S, and let H be a CTL-formula. The predicate $s \models_{k_1} H$ is defined by:

(Def. 60) $s \models \text{Evaluate}(H, k_1)$.

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of ModelSP S, let k_1 be a function from the atomic WFF into the basic assignations of CTLModel(R, B_1), let s be an element of S, and let H be a CTL-formula. We introduce $s \not\models_{k_1} H$ as an antonym of $s \models_{k_1} H$.

The following propositions are true:

(18) If H is atomic, then $s \models_{k_1} H$ iff $k_1(H) \in (\text{Label}\, B_1)(s)$.

(19) $s \models_{k_1} \neg H$ iff $s \not\models_{k_1} H$.

(20) $s \models_{k_1} H_1 \wedge H_2$ iff $s \models_{k_1} H_1$ and $s \models_{k_1} H_2$.

(21) $s \models_{k_1} H_1 \vee H_2$ iff $s \models_{k_1} H_1$ or $s \models_{k_1} H_2$.

(22) $s \models_{k_1} \text{EX}\, H$ iff there exists an infinity path p_1 of R such that $p_1(0) = s$ and $p_1(1) \models_{k_1} H$.

(23) $s \models_{k_1} \text{EG}\, H$ iff there exists an infinity path p_1 of R such that $p_1(0) = s$ and for every element n of $\mathbb{N}$ holds $p_1(n) \models_{k_1} H$.

(24) $s \models_{k_1} H_1 \,\text{EU}\, H_2$ if and only if there exists an infinity path p_1 of R such that $p_1(0) = s$ and there exists an element m of $\mathbb{N}$ such that for every element j of $\mathbb{N}$ such that $j < m$ holds $p_1(j) \models_{k_1} H_1$ and $p_1(m) \models_{k_1} H_2$.

(25) For every s_0 there exists an infinity path p_1 of R such that $p_1(0) = s_0$.

(26) Let R be a relation between S and S. Then R is total if and only if for every set x such that $x \in S$ there exists a set y such that $y \in S$ and $\langle x, y \rangle \in R$.

Let S be a non empty set, let R be a total relation between S and S, let s_0 be an element of S, let p_1 be an infinity path of R, and let n be a set. The functor PrePath(n, s_0, p_1) yielding an element of S is defined as follows:

(Def. 61) $\text{PrePath}(n, s_0, p_1) = \begin{cases} s_0, & \text{if } n = 0, \\ p_1(\text{k.nat}(\text{k.nat}\, n - 1)), & \text{otherwise.} \end{cases}$

The following propositions are true:

(27) If $\langle s_0, s_1 \rangle \in R$, then there exists an infinity path p_1 of R such that $p_1(0) = s_0$ and $p_1(1) = s_1$.

(28) For every assignation f of CTLModel(R, B_1) holds $s \models \text{EX}\, f$ iff there exists an element s_1 of S such that $\langle s, s_1 \rangle \in R$ and $s_1 \models f$.

Let S be a non empty set, let R be a total relation between S and S, and let H be a subset of S. The functor Pred(H, R) yields a subset of S and is defined by:

(Def. 62) $\text{Pred}(H, R) = \{s; s \text{ ranges over elements of } S\colon \bigvee_{t\,:\,\text{element of } S} (t \in H \wedge \langle s, t \rangle \in R)\}$.

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of ModelSP S, and let f be an assignation of CTLModel(R, B_1). The functor SIGMA f yields a subset of S and is defined as follows:

(Def. 63) SIGMA $f = \{s; s$ ranges over elements of S: $s \models f\}$.

One can prove the following proposition

(29) For all assignations f, g of CTLModel(R, B_1) such that SIGMA f = SIGMA g holds $f = g$.

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of ModelSP S, and let T be a subset of S. The functor Tau(T, R, B_1) yielding an assignation of CTLModel(R, B_1) is defined as follows:

(Def. 64) For every set s such that $s \in S$ holds (Fid(Tau$(T, R, B_1), S$))$(s) = \chi_{T,S}(s)$.

The following propositions are true:

(30) For all subsets T_1, T_2 of S such that Tau(T_1, R, B_1) = Tau(T_2, R, B_1) holds $T_1 = T_2$.

(31) For every assignation f of CTLModel(R, B_1) holds Tau(SIGMA $f, R, B_1) = f$.

(32) For every subset T of S holds SIGMA Tau$(T, R, B_1) = T$.

(33) For all assignations f, g of CTLModel(R, B_1) holds SIGMA $\neg f = S \setminus$ SIGMA f and SIGMA$(f \wedge g)$ = SIGMA $f \cap$ SIGMA g and SIGMA$(f \vee g)$ = SIGMA $f \cup$ SIGMA g.

(34) For all subsets G_1, G_2 of S such that $G_1 \subseteq G_2$ and for every element s of S such that $s \models$ Tau(G_1, R, B_1) holds $s \models$ Tau(G_2, R, B_1).

(35) For all assignations f_1, f_2 of CTLModel(R, B_1) such that for every element s of S such that $s \models f_1$ holds $s \models f_2$ holds SIGMA $f_1 \subseteq$ SIGMA f_2.

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of ModelSP S, and let f, g be assignations of CTLModel(R, B_1). The functor Fax(f, g) yielding an assignation of CTLModel(R, B_1) is defined by:

(Def. 65) Fax$(f, g) = f \wedge$ EX g.

Next we state the proposition

(36) Let f, g_1, g_2 be assignations of CTLModel(R, B_1). Suppose that for every element s of S such that $s \models g_1$ holds $s \models g_2$. Let s be an element of S. If $s \models$ Fax(f, g_1), then $s \models$ Fax(f, g_2).

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of ModelSP S, let f be an assignation of CTLModel(R, B_1), and let G be a subset of S. The functor SigFaxTau(f, G, R, B_1) yielding a subset of S is defined by:

(Def. 66) $\mathrm{SigFaxTau}(f, G, R, B_1) = \mathrm{SIGMA}\,\mathrm{Fax}(f, \mathrm{Tau}(G, R, B_1))$.

One can prove the following proposition

(37) For every assignation f of $\mathrm{CTLModel}(R, B_1)$ and for all subsets G_1, G_2 of S such that $G_1 \subseteq G_2$ holds $\mathrm{SigFaxTau}(f, G_1, R, B_1) \subseteq \mathrm{SigFaxTau}(f, G_2, R, B_1)$.

Let S be a non empty set, let R be a total relation between S and S, let p_1 be an infinity path of R, and let k be an element of $\mathbb{N}$. The functor $\mathrm{PathShift}(p_1, k)$ yielding an infinity path of R is defined as follows:

(Def. 67) For every element n of $\mathbb{N}$ holds $(\mathrm{PathShift}(p_1, k))(n) = p_1(n + k)$.

Let S be a non empty set, let R be a total relation between S and S, let p_2, p_3 be infinity paths of R, and let n, k be elements of $\mathbb{N}$. The functor $\mathrm{PathChange}(p_2, p_3, k, n)$ yielding a set is defined by:

(Def. 68) $\mathrm{PathChange}(p_2, p_3, k, n) = \begin{cases} p_2(n), & \text{if } n < k, \\ p_3(n - k), & \text{otherwise.} \end{cases}$

Let S be a non empty set, let R be a total relation between S and S, let p_2, p_3 be infinity paths of R, and let k be an element of $\mathbb{N}$. The functor $\mathrm{PathConc}(p_2, p_3, k)$ yielding a function from $\mathbb{N}$ into S is defined as follows:

(Def. 69) For every element n of $\mathbb{N}$ holds $(\mathrm{PathConc}(p_2, p_3, k))(n) = \mathrm{PathChange}(p_2, p_3, k, n)$.

We now state four propositions:

(38) Let p_2, p_3 be infinity paths of R and k be an element of $\mathbb{N}$. If $p_2(k) = p_3(0)$, then $\mathrm{PathConc}(p_2, p_3, k)$ is an infinity path of R.

(39) For every assignation f of $\mathrm{CTLModel}(R, B_1)$ and for every element s of S holds $s \models \mathrm{EG}\,f$ iff $s \models \mathrm{Fax}(f, \mathrm{EG}\,f)$.

(40) Let g be an assignation of $\mathrm{CTLModel}(R, B_1)$ and s_0 be an element of S. Suppose $s_0 \models g$. Suppose that for every element s of S such that $s \models g$ holds $s \models \mathrm{EX}\,g$. Then there exists an infinity path p_1 of R such that $p_1(0) = s_0$ and for every element n of $\mathbb{N}$ holds $p_1(n) \models g$.

(41) Let f, g be assignations of $\mathrm{CTLModel}(R, B_1)$. Suppose that for every element s of S holds $s \models g$ iff $s \models \mathrm{Fax}(f, g)$. Let s be an element of S. If $s \models g$, then $s \models \mathrm{EG}\,f$.

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of $\mathrm{ModelSP}\,S$, and let f be an assignation of $\mathrm{CTLModel}(R, B_1)$. The functor $\mathrm{TransEG}\,f$ yielding a $\subseteq$-monotone function from 2^S into 2^S is defined as follows:

(Def. 70) For every subset G of S holds $(\mathrm{TransEG}\,f)(G) = \mathrm{SigFaxTau}(f, G, R, B_1)$.

One can prove the following two propositions:

(42) Let f, g be assignations of $\mathrm{CTLModel}(R, B_1)$. Then for every element s of S holds $s \models g$ iff $s \models \mathrm{Fax}(f, g)$ if and only if $\mathrm{SIGMA}\,g$ is a fixpoint of

TransEG f.

(43) For every assignation f of CTLModel(R, B_1) holds SIGMA EG f = gfp$(S,$ TransEG $f)$.

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of ModelSP S, and let f, g, h be assignations of CTLModel(R, B_1). The functor Foax(g, f, h) yields an assignation of CTLModel(R, B_1) and is defined as follows:

(Def. 71) Foax$(g, f, h) = g \vee$ Fax(f, h).

We now state the proposition

(44) Let f, g, h_1, h_2 be assignations of CTLModel(R, B_1). Suppose that for every element s of S such that $s \models h_1$ holds $s \models h_2$. Let s be an element of S. If $s \models$ Foax(g, f, h_1), then $s \models$ Foax(g, f, h_2).

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of ModelSP S, let f, g be assignations of CTLModel(R, B_1), and let H be a subset of S. The functor SigFoaxTau(g, f, H, R, B_1) yields a subset of S and is defined as follows:

(Def. 72) SigFoaxTau(g, f, H, R, B_1) = SIGMA Foax$(g, f,$ Tau$(H, R, B_1))$.

Next we state three propositions:

(45) For all assignations f, g of CTLModel(R, B_1) and for all subsets H_1, H_2 of S such that $H_1 \subseteq H_2$ holds SigFoaxTau$(g, f, H_1, R, B_1) \subseteq$ SigFoaxTau(g, f, H_2, R, B_1).

(46) For all assignations f, g of CTLModel(R, B_1) and for every element s of S holds $s \models f$ EU g iff $s \models$ Foax$(g, f, f$ EU $g)$.

(47) Let f, g, h be assignations of CTLModel(R, B_1). Suppose that for every element s of S holds $s \models h$ iff $s \models$ Foax(g, f, h). Let s be an element of S. If $s \models f$ EU g, then $s \models h$.

Let S be a non empty set, let R be a total relation between S and S, let B_1 be a non empty subset of ModelSP S, and let f, g be assignations of CTLModel(R, B_1). The functor TransEU(f, g) yields a $\subseteq$-monotone function from 2^S into 2^S and is defined by:

(Def. 73) For every subset H of S holds (TransEU$(f, g))(H)$ = SigFoaxTau(g, f, H, R, B_1).

One can prove the following propositions:

(48) Let f, g, h be assignations of CTLModel(R, B_1). Then for every element s of S holds $s \models h$ iff $s \models$ Foax(g, f, h) if and only if SIGMA h is a fixpoint of TransEU(f, g).

(49) For all assignations f, g of CTLModel(R, B_1) holds SIGMA$(f$ EU $g)$ = lfp$(S,$ TransEU$(f, g))$.

(50) For every assignation f of CTLModel(R, B_1) holds SIGMA EX f = Pred(SIGMA f, R).

(51) For every assignation f of CTLModel(R, B_1) and for every subset X of S holds (TransEG f)(X) = SIGMA $f \cap$ Pred(X, R).

(52) For all assignations f, g of CTLModel(R, B_1) and for every subset X of S holds (TransEU(f, g))(X) = SIGMA $g \cup$ SIGMA $f \cap$ Pred(X, R).

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Recognizing Chordal Graphs: Lex BFS and MCS[1]

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Summary. We are formalizing the algorithm for recognizing chordal graphs by lexicographic breadth-first search as presented in [13, Section 3 of Chapter 4, pp. 81–84]. Then we follow with a formalization of another algorithm serving the same end but based on maximum cardinality search as presented by Tarjan and Yannakakis [25].

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The notation and terminology used in this paper are introduced in the following articles: [28], [11], [26], [32], [33], [35], [30], [10], [7], [8], [20], [29], [4], [2], [14], [23], [12], [3], [6], [9], [18], [15], [19], [16], [17], [24], [21], [1], [5], [31], [27], [22], and [34].

1. Preliminaries

The following propositions are true:

(1) Let A, B be elements of $\mathbb{N}$, X be a non empty set, and F be a function from $\mathbb{N}$ into X. If F is one-to-one, then $\overline{\overline{\{F(w); w \text{ ranges over elements of } \mathbb{N}\colon A \leq w \wedge w \leq A+B\}}} = B+1$.

(2) For all natural numbers n, m, k such that $m \leq k$ and $n < m$ holds $k -' m < k -' n$.

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(3) For all natural numbers n, k such that $n < k$ holds $(k -' (n+1)) + 1 = k -' n$.

(4) For all natural numbers n, m, k such that $k \neq 0$ holds $(n + m \cdot k) \div k = (n \div k) + m$.

Let S be a set. We say that S has finite elements if and only if:

(Def. 1) Every element of S is finite.

Let us note that there exists a set which is non empty and has finite elements and there exists a subset of $2^{\mathbb{N}}$ which is non empty and finite and has finite elements.

Let S be a set with finite elements. One can check that every element of S is finite.

Let f, g be functions. The functor $f[\cup]g$ yielding a function is defined by:

(Def. 2) $\mathrm{dom}(f[\cup]g) = \mathrm{dom}\, f \cup \mathrm{dom}\, g$ and for every set x such that $x \in \mathrm{dom}\, f \cup \mathrm{dom}\, g$ holds $(f[\cup]g)(x) = f(x) \cup g(x)$.

The following three propositions are true:

(5) For all natural numbers m, n, k holds $m \in \mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' n)$ iff $k -' n < m$ and $m \leq k$.

(6) For all natural numbers n, k, m such that $n \leq m$ holds $\mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' n) \subseteq \mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' m)$.

(7) For all natural numbers n, k such that $n < k$ holds $(\mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' n)) \cup \{k -' n\} = \mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' (n+1))$.

Let f be a binary relation. We say that f is natsubset yielding if and only if:

(Def. 3) $\mathrm{rng}\, f \subseteq 2^{\mathbb{N}}$.

Let us mention that there exists a function which is finite-yielding and natsubset yielding.

Let f be a finite-yielding natsubset yielding function and let x be a set. Then $f(x)$ is a finite subset of $\mathbb{N}$.

One can prove the following proposition

(8) For every ordinal number X and for all finite subsets a, b of X such that $a \neq b$ holds $(a, 1)$-bag $\neq (b, 1)$-bag.

Let F be a natural-yielding function, let S be a set, and let k be a natural number. The functor F.incSubset(S, k) yielding a natural-yielding function is defined by the conditions (Def. 4).

(Def. 4)(i) $\mathrm{dom}(F.\mathrm{incSubset}(S,k)) = \mathrm{dom}\, F$, and

(ii) for every set y holds if $y \in S$ and $y \in \mathrm{dom}\, F$, then $(F.\mathrm{incSubset}(S,k))(y) = F(y) + k$ and if $y \notin S$, then $(F.\mathrm{incSubset}(S,k))(y) = F(y)$.

Let n be an ordinal number, let T be a connected term order of n, and let B be a non empty finite subset of Bags n. The functor $\max(B,T)$ yields a bag of n and is defined as follows:

(Def. 5) $\max(B,T) \in B$ and for every bag x of n such that $x \in B$ holds $x \leq_T \max(B,T)$.

Let O be an ordinal number. Observe that InvLexOrder O is connected.

2. Miscellany on Graphs

Let G be a graph. Note that there exists a vertex sequence of G which is non empty and one-to-one.

Let G be a graph and let V be a non empty vertex sequence of G. A walk of G is called a walk of V if:

(Def. 6) It.vertexSeq() $= V$.

Let G be a graph and let V be a non empty one-to-one vertex sequence of G. One can check that every walk of V is path-like.

We now state two propositions:

(9) For every graph G and for all walks W_1, W_2 of G such that W_1 is trivial and W_1.last() $= W_2$.first() holds W_1.append(W_2) $= W_2$.

(10) Let G, H be graphs, A, B, C be sets, G_1 be a subgraph of G induced by A, H_1 be a subgraph of H induced by B, G_2 be a subgraph of G_1 induced by C, and H_2 be a subgraph of H_1 induced by C. Suppose $G =_G H$ and $A \subseteq B$ and $C \subseteq A$ and C is a non empty subset of the vertices of G. Then $G_2 =_G H_2$.

Let G be a v-graph. We say that G is natural v-labeled if and only if:

(Def. 7) The vlabel of G is natural-yielding.

3. Graphs with Two Vertex Labels

The natural number V2-LabelSelector is defined by:

(Def. 8) V2-LabelSelector $= 8$.

Let G be a graph structure. We say that G is v2-labeled if and only if:

(Def. 9) V2-LabelSelector $\in \operatorname{dom} G$ and there exists a function f such that G(V2-LabelSelector) $= f$ and $\operatorname{dom} f \subseteq$ the vertices of G.

Let us note that there exists a graph structure which is graph-like, weighted, elabeled, vlabeled, and v2-labeled.

A v2-graph is a v2-labeled graph. A vv-graph is a vlabeled v2-labeled graph.

Let G be a v2-graph. The v2-label of G yields a function and is defined as follows:

(Def. 10) The v2-label of $G = G(\text{V2-LabelSelector})$.

Next we state the proposition

(11) For every v2-graph G holds dom (the v2-label of G) $\subseteq$ the vertices of G.

Let G be a graph and let X be a set. Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is graph-like.

We now state the proposition

(12) For every graph G and for every set X holds $G.\text{set}(\text{V2-LabelSelector}, X) =_G G$.

Let G be a finite graph and let X be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is finite.

Let G be a loopless graph and let X be a set.

Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is loopless.

Let G be a trivial graph and let X be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is trivial.

Let G be a non trivial graph and let X be a set. One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non trivial.

Let G be a non-multi graph and let X be a set. One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non-multi.

Let G be a non-directed-multi graph and let X be a set. One can verify that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non-directed-multi.

Let G be a connected graph and let X be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is connected.

Let G be an acyclic graph and let X be a set.

One can verify that $G.\text{set}(\text{V2-LabelSelector}, X)$ is acyclic.

Let G be a v-graph and let X be a set.

One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is vlabeled.

Let G be a e-graph and let X be a set. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is elabeled.

Let G be a w-graph and let X be a set. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is weighted.

Let G be a v2-graph and let X be a set.

One can verify that $G.\text{set}(\text{VLabelSelector}, X)$ is v2-labeled.

Let G be a graph, let Y be a set, and let X be a partial function from the vertices of G to Y. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is v2-labeled.

Let G be a graph and let X be a many sorted set indexed by the vertices of G. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is v2-labeled.

Let G be a graph. One can verify that $G.\text{set}(\text{V2-LabelSelector}, \emptyset)$ is v2-labeled.

Let G be a v2-graph. We say that G is natural v2-labeled if and only if:

(Def. 11) The v2-label of G is natural-yielding.

We say that G is finite v2-labeled if and only if:

(Def. 12) The v2-label of G is finite-yielding.

We say that G is natsubset v2-labeled if and only if:

(Def. 13) The v2-label of G is natsubset yielding.

One can check that there exists a weighted elabeled vlabeled v2-labeled graph which is finite, natural v-labeled, finite v2-labeled, natsubset v2-labeled, and chordal and there exists a weighted elabeled vlabeled v2-labeled graph which is finite, natural v-labeled, natural v2-labeled, and chordal.

Let G be a natural v-labeled v-graph. Observe that the vlabel of G is natural-yielding.

Let G be a natural v2-labeled v2-graph. Observe that the v2-label of G is natural-yielding.

Let G be a finite v2-labeled v2-graph. Observe that the v2-label of G is finite-yielding.

Let G be a natsubset v2-labeled v2-graph. One can verify that the v2-label of G is natsubset yielding.

Let G be a vv-graph and let v, x be sets. One can check that G.labelVertex(v, x) is v2-labeled.

Next we state the proposition

(13) For every vv-graph G and for all sets v, x holds the v2-label of $G =$ the v2-label of G.labelVertex(v, x).

Let G be a natural v-labeled vv-graph, let v be a set, and let x be a natural number. Observe that G.labelVertex(v, x) is natural v-labeled.

Let G be a natural v2-labeled vv-graph, let v be a set, and let x be a natural number. Observe that G.labelVertex(v, x) is natural v2-labeled.

Let G be a finite v2-labeled vv-graph, let v be a set, and let x be a natural number. Note that G.labelVertex(v, x) is finite v2-labeled.

Let G be a natsubset v2-labeled vv-graph, let v be a set, and let x be a natural number. One can check that G.labelVertex(v, x) is natsubset v2-labeled.

Let G be a graph. Note that there exists a subgraph of G which is vlabeled and v2-labeled.

Let G be a v2-graph and let G_2 be a v2-labeled subgraph of G. We say that G_2 inherits v2-label if and only if:

(Def. 14) The v2-label of $G_2 =$ (the v2-label of G)$\restriction$(the vertices of G_2).

Let G be a v2-graph. Note that there exists a v2-labeled subgraph of G which inherits v2-label.

Let G be a v2-graph. A v2-subgraph of G is a v2-labeled subgraph of G inheriting v2-label.

Let G be a vv-graph. Note that there exists a vlabeled v2-labeled subgraph of G which inherits vlabel and v2-label.

Let G be a vv-graph. A vv-subgraph of G is a vlabeled v2-labeled subgraph of G inheriting vlabel and v2-label.

Let G be a natural v-labeled v-graph. Note that every v-subgraph of G is natural v-labeled.

Let G be a graph and let V, E be sets. Observe that there exists a subgraph of G induced by V and E which is weighted, elabeled, vlabeled, and v2-labeled.

Let G be a vv-graph and let V, E be sets. Observe that there exists a vlabeled v2-labeled subgraph of G induced by V and E which inherits vlabel and v2-label.

Let G be a vv-graph and let V, E be sets. A (V, E)-induced vv-subgraph of G is a vlabeled v2-labeled subgraph of G induced by V and E inheriting vlabel and v2-label.

Let G be a vv-graph and let V be a set. A V-induced vv-subgraph of G is a $(V, G.\mathrm{edgesBetween}(V))$-induced vv-subgraph of G.

4. More on Graph Sequences

Let s be a many sorted set indexed by $\mathbb{N}$. We say that s is iterative if and only if:

(Def. 15) For all natural numbers k, n such that $s(k) = s(n)$ holds $s(k+1) = s(n+1)$.

Let G_3 be a many sorted set indexed by $\mathbb{N}$. We say that G_3 is eventually constant if and only if:

(Def. 16) There exists a natural number n such that for every natural number m such that $n \leq m$ holds $G_3(n) = G_3(m)$.

Let us observe that there exists a many sorted set indexed by $\mathbb{N}$ which is halting, iterative, and eventually constant.

The following proposition is true

(14) For every many sorted set G_4 indexed by $\mathbb{N}$ such that G_4 is halting and iterative holds G_4 is eventually constant.

One can check that every many sorted set indexed by $\mathbb{N}$ which is halting and iterative is also eventually constant.

The following proposition is true

(15) For every many sorted set G_4 indexed by $\mathbb{N}$ such that G_4 is eventually constant holds G_4 is halting.

Let us mention that every many sorted set indexed by $\mathbb{N}$ which is eventually constant is also halting.

One can prove the following two propositions:

(16) Let G_4 be an iterative eventually constant many sorted set indexed by $\mathbb{N}$ and n be a natural number. If $G_4.\mathrm{Lifespan}() \leq n$, then $G_4(G_4.\mathrm{Lifespan}()) = G_4(n)$.

(17) Let G_4 be an iterative eventually constant many sorted set indexed by $\mathbb{N}$ and n, m be natural numbers. If G_4.Lifespan() $\leq n$ and $n \leq m$, then $G_4(m) = G_4(n)$.

Let G_3 be a v-graph sequence. We say that G_3 is natural v-labeled if and only if:

(Def. 17) For every natural number x holds $G_3(x)$ is natural v-labeled.

Let G_3 be a graph sequence. We say that G_3 is chordal if and only if:

(Def. 18) For every natural number x holds $G_3(x)$ is chordal.

We say that G_3 has fixed vertices if and only if:

(Def. 19) For all natural numbers n, m holds the vertices of $G_3(n) =$ the vertices of $G_3(m)$.

We say that G_3 is v2-labeled if and only if:

(Def. 20) For every natural number x holds $G_3(x)$ is v2-labeled.

Let us observe that there exists a graph sequence which is weighted, elabeled, vlabeled, and v2-labeled.

A v2-graph sequence is a v2-labeled graph sequence. A vv-graph sequence is a vlabeled v2-labeled graph sequence.

Let G_5 be a v2-graph sequence and let x be a natural number. Note that $G_5(x)$ is v2-labeled.

Let G_5 be a v2-graph sequence. We say that G_5 is natural v2-labeled if and only if:

(Def. 21) For every natural number x holds $G_5(x)$ is natural v2-labeled.

We say that G_5 is finite v2-labeled if and only if:

(Def. 22) For every natural number x holds $G_5(x)$ is finite v2-labeled.

We say that G_5 is natsubset v2-labeled if and only if:

(Def. 23) For every natural number x holds $G_5(x)$ is natsubset v2-labeled.

Let us mention that there exists a weighted elabeled vlabeled v2-labeled graph sequence which is finite, natural v-labeled, finite v2-labeled, natsubset v2-labeled, and chordal and there exists a weighted elabeled vlabeled v2-labeled graph sequence which is finite, natural v-labeled, natural v2-labeled, and chordal.

Let G_4 be a v-graph sequence and let x be a natural number. Then $G_4(x)$ is a v-graph.

Let G_5 be a natural v-labeled v-graph sequence and let x be a natural number. Observe that $G_5(x)$ is natural v-labeled.

Let G_5 be a natural v2-labeled v2-graph sequence and let x be a natural number. One can check that $G_5(x)$ is natural v2-labeled.

Let G_5 be a finite v2-labeled v2-graph sequence and let x be a natural number. One can verify that $G_5(x)$ is finite v2-labeled.

Let G_5 be a natsubset v2-labeled v2-graph sequence and let x be a natural number. Note that $G_5(x)$ is natsubset v2-labeled.

Let G_5 be a chordal graph sequence and let x be a natural number. One can check that $G_5(x)$ is chordal.

Let G_4 be a v-graph sequence and let n be a natural number. Then $G_4(n)$ is a v-graph.

Let G_4 be a finite v-graph sequence and let n be a natural number. One can check that $G_4(n)$ is finite.

Let G_4 be a vv-graph sequence and let n be a natural number. Then $G_4(n)$ is a vv-graph.

Let G_4 be a finite vv-graph sequence and let n be a natural number. One can verify that $G_4(n)$ is finite.

Let G_4 be a chordal vv-graph sequence and let n be a natural number. Note that $G_4(n)$ is chordal.

Let G_4 be a natural v-labeled vv-graph sequence and let n be a natural number. One can check that $G_4(n)$ is natural v-labeled.

Let G_4 be a finite v2-labeled vv-graph sequence and let n be a natural number. Note that $G_4(n)$ is finite v2-labeled.

Let G_4 be a natsubset v2-labeled vv-graph sequence and let n be a natural number. One can check that $G_4(n)$ is natsubset v2-labeled.

Let G_4 be a natural v2-labeled vv-graph sequence and let n be a natural number. Observe that $G_4(n)$ is natural v2-labeled.

5. Vertices Numbering Sequences

Let G_3 be a v-graph sequence. We say that G_3 has initially empty v-label if and only if:

(Def. 24) The vlabel of $G_3(0) = \emptyset$.

We say that G_3 is adding one at a step if and only if the condition (Def. 25) is satisfied.

(Def. 25) Let n be a natural number. Suppose $n < G_3$.Lifespan(). Then there exists a set w such that $w \notin \operatorname{dom}$ (the vlabel of $G_3(n)$) and the vlabel of $G_3(n+1) =$ (the vlabel of $G_3(n)$)$+\cdot(w \longmapsto (G_3.\text{Lifespan}() -' n))$.

Let G_3 be a v-graph sequence. We say that G_3 is v-label numbering if and only if the condition (Def. 26) is satisfied.

(Def. 26) G_3 is iterative, eventually constant, finite, natural v-labeled, and adding one at a step and has fixed vertices and initially empty v-label.

One can check that there exists a v-graph sequence which is iterative, eventually constant, finite, natural v-labeled, and adding one at a step and has fixed vertices and initially empty v-label.

Let us observe that there exists a v-graph sequence which is v-label numbering.

One can check the following observations:

* every v-graph sequence which is v-label numbering is also iterative,
* every v-graph sequence which is v-label numbering is also eventually constant,
* every v-graph sequence which is v-label numbering is also finite,
* every v-graph sequence which is v-label numbering has also fixed vertices,
* every v-graph sequence which is v-label numbering is also natural v-labeled,
* every v-graph sequence which is v-label numbering has also initially empty v-label, and
* every v-graph sequence which is v-label numbering is also adding one at a step.

A v-label numbering sequence is a v-label numbering v-graph sequence.

Let G_3 be a v-label numbering sequence and let n be a natural number. The functor $G_3\,.\mathrm{PickedAt}\,n$ yields a set and is defined by:

(Def. 27)(i) $G_3\,.\mathrm{PickedAt}\,n = \mathrm{choose}(\text{the vertices of } G_3(0))$ if $n \geq G_3.\mathrm{Lifespan}()$,

(ii) $G_3\,.\mathrm{PickedAt}\,n \notin \mathrm{dom}\,(\text{the vlabel of } G_3(n))$ and the vlabel of $G_3(n+1) = (\text{the vlabel of } G_3(n)) {+\cdot} ((G_3\,.\mathrm{PickedAt}\,n) \dot{\longmapsto} (G_3.\mathrm{Lifespan}() -' n))$, otherwise.

The following propositions are true:

(18) Let G_3 be a v-label numbering sequence and n be a natural number. If $n < G_3.\mathrm{Lifespan}()$, then $G_3\,.\mathrm{PickedAt}\,n \in G_3(n+1).\mathrm{labeledV}()$ and $G_3(n+1).\mathrm{labeledV}() = G_3(n).\mathrm{labeledV}() \cup \{G_3\,.\mathrm{PickedAt}\,n\}$.

(19) Let G_3 be a v-label numbering sequence and n be a natural number. If $n < G_3.\mathrm{Lifespan}()$, then $(\text{the vlabel of } G_3(n+1))(G_3\,.\mathrm{PickedAt}\,n) = G_3.\mathrm{Lifespan}() -' n$.

(20) For every v-label numbering sequence G_3 and for every natural number n such that $n \leq G_3.\mathrm{Lifespan}()$ holds $\mathrm{card}(G_3(n).\mathrm{labeledV}()) = n$.

(21) For every v-label numbering sequence G_3 and for every natural number n holds $\mathrm{rng}\,(\text{the vlabel of } G_3(n)) = \mathrm{Seg}(G_3.\mathrm{Lifespan}()) \setminus \mathrm{Seg}(G_3.\mathrm{Lifespan}() -' n)$.

(22) Let G_3 be a v-label numbering sequence, n be a natural number, and x be a set. Then $(\text{the vlabel of } G_3(n))(x) \leq G_3.\mathrm{Lifespan}()$ and if $x \in G_3(n).\mathrm{labeledV}()$, then $1 \leq (\text{the vlabel of } G_3(n))(x)$.

(23) Let G_3 be a v-label numbering sequence and n, m be natural numbers. Suppose $G_3.\mathrm{Lifespan}() -' n < m$ and $m \leq G_3.\mathrm{Lifespan}()$. Then there exists a vertex v of $G_3(n)$ such that $v \in G_3(n).\mathrm{labeledV}()$ and (the vlabel

of $G_3(n))(v) = m$.

(24) Let G_3 be a v-label numbering sequence and m, n be natural numbers. If $m \leq n$, then the vlabel of $G_3(m) \subseteq$ the vlabel of $G_3(n)$.

(25) For every v-label numbering sequence G_3 and for every natural number n holds the vlabel of $G_3(n)$ is one-to-one.

(26) Let G_3 be a v-label numbering sequence, m, n be natural numbers, and v be a set. Suppose $v \in G_3(m)$.labeledV() and $v \in G_3(n)$.labeledV(). Then (the vlabel of $G_3(m))(v) =$ (the vlabel of $G_3(n))(v)$.

(27) Let G_3 be a v-label numbering sequence, v be a set, and m, n be natural numbers. If $v \in G_3(m)$.labeledV() and (the vlabel of $G_3(m))(v) = n$, then G_3 .PickedAt(G_3.Lifespan() $-'$ n) $= v$.

(28) Let G_3 be a v-label numbering sequence and m, n be natural numbers. If $n < G_3$.Lifespan() and $n < m$, then G_3 .PickedAt $n \in G_3(m)$.labeledV() and (the vlabel of $G_3(m))(G_3$ .PickedAt $n) = G_3$.Lifespan() $-'$ n.

(29) Let G_3 be a v-label numbering sequence, m be a natural number, and v be a set. Suppose $v \in G_3(m)$.labeledV(). Then G_3.Lifespan() $-'$ (the vlabel of $G_3(m))(v) < m$ and G_3.Lifespan() $-'$ $m <$ (the vlabel of $G_3(m))(v)$.

(30) Let G_3 be a v-label numbering sequence, i be a natural number, and a, b be sets. Suppose $a \in G_3(i)$.labeledV() and $b \in G_3(i)$.labeledV() and (the vlabel of $G_3(i))(a) <$ (the vlabel of $G_3(i))(b)$. Then $b \in G_3(G_3$.Lifespan() $-'$ (the vlabel of $G_3(i))(a))$.labeledV().

(31) Let G_3 be a v-label numbering sequence, i be a natural number, and a, b be sets. Suppose $a \in G_3(i)$.labeledV() and $b \in G_3(i)$.labeledV() and (the vlabel of $G_3(i))(a) <$ (the vlabel of $G_3(i))(b)$. Then $a \notin G_3(G_3$.Lifespan() $-'$ (the vlabel of $G_3(i))(b))$.labeledV().

6. Lexicographical Breadth-First Search

Let G be a graph. The functor LexBFS:Init G yields a natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph and is defined as follows:

(Def. 28) LexBFS:Init G = G.set(VLabelSelector, $\emptyset$).set(V2-LabelSelector, (the vertices of G) $\longmapsto \emptyset$).

Let G be a finite graph. Then LexBFS:Init G is a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph.

Let G be a finite finite v2-labeled natsubset v2-labeled vv-graph. Let us assume that dom (the v2-label of G) = the vertices of G. The functor LexBFS:PickUnnumbered G yields a vertex of G and is defined by:

(Def. 29)(i) LexBFS:PickUnnumbered G = choose(the vertices of G) if dom (the vlabel of G) = the vertices of G,

(ii) there exists a non empty finite subset S of $2^{\mathbb{N}}$ and there exists a non empty finite subset B of Bags $\mathbb{N}$ and there exists a function F such that $S = \operatorname{rng} F$ and $F =$ (the v2-label of G)$\restriction$((the vertices of G) $\setminus$ dom (the vlabel of G)) and for every finite subset x of $\mathbb{N}$ such that $x \in S$ holds $(x, 1)$-bag $\in B$ and for every set x such that $x \in B$ there exists a finite subset y of $\mathbb{N}$ such that $y \in S$ and $x = (y, 1)$-bag and LexBFS:PickUnnumbered G = choose(F^{-1}({support max(B, InvLexOrder $\mathbb{N}$)})), otherwise.

Let G be a vv-graph, let v be a set, and let k be a natural number. The functor LexBFS:LabelAdjacent(G, v, k) yielding a vv-graph is defined as follows:

(Def. 30) LexBFS:LabelAdjacent$(G, v, k) = G$.set(V2-LabelSelector, (the v2-label of G)[$\cup$]((G.adjacentSet($\{v\}$)) $\setminus$ dom (the vlabel of G) $\longmapsto \{k\}$)).

Next we state four propositions:

(32) Let G be a vv-graph, v, x be sets, and k be a natural number. If $x \notin G$.adjacentSet($\{v\}$), then (the v2-label of G)(x) = (the v2-label of LexBFS:LabelAdjacent(G, v, k))(x).

(33) Let G be a vv-graph, v, x be sets, and k be a natural number. Suppose $x \in$ dom (the vlabel of G). Then (the v2-label of G)(x) = (the v2-label of LexBFS:LabelAdjacent(G, v, k))(x).

(34) Let G be a vv-graph, v, x be sets, and k be a natural number. Suppose $x \in G$.adjacentSet($\{v\}$) and $x \notin$ dom (the vlabel of G). Then (the v2-label of LexBFS:LabelAdjacent(G, v, k))(x) = (the v2-label of G)(x) $\cup \{k\}$.

(35) Let G be a vv-graph, v be a set, and k be a natural number. Suppose dom (the v2-label of G) = the vertices of G. Then dom (the v2-label of LexBFS:LabelAdjacent(G, v, k)) = the vertices of G.

Let G be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph, let v be a vertex of G, and let k be a natural number. Then LexBFS:LabelAdjacent(G, v, k) is a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph.

Let G be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph, let v be a vertex of G, and let n be a natural number. The functor LexBFS:Update(G, v, n) yielding a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph is defined by:

(Def. 31) LexBFS:Update(G, v, n) =
LexBFS:LabelAdjacent(G.labelVertex(v, G.order() $-'$ n), v, G.order() $-'$ n).

Let G be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph. The functor LexBFS:Step G yields a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph and is defined as follows:

(Def. 32) $$\text{LexBFS:Step}\, G = \begin{cases} G, & \text{if } G.\text{order}() \leq \text{card dom (the vlabel of } G), \\ \text{LexBFS:Update}(G, \text{LexBFS:PickUnnumbered}\, G, & \\ \quad \text{card dom (the vlabel of } G)), & \text{otherwise.} \end{cases}$$

Let G be a finite graph. The functor LexBFS:CSeq G yields a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph sequence and is defined by:

(Def. 33) (LexBFS:CSeq G)(0) = LexBFS:Init G and for every natural number n holds (LexBFS:CSeq G)($n + 1$) = LexBFS:Step(LexBFS:CSeq G)(n).

We now state the proposition

(36) For every finite graph G holds LexBFS:CSeq G is iterative.

Let G be a finite graph. Observe that LexBFS:CSeq G is iterative.

Next we state a number of propositions:

(37) For every graph G holds the vlabel of LexBFS:Init $G = \emptyset$.

(38) Let G be a graph and v be a set. Then dom (the v2-label of LexBFS:Init G) = the vertices of G and (the v2-label of LexBFS:Init G)(v) = $\emptyset$.

(39) For every graph G holds $G =_G$ LexBFS:Init G.

(40) Let G be a finite finite v2-labeled natsubset v2-labeled vv-graph and x be a set. Suppose that

(i) $x \notin$ dom (the vlabel of G),

(ii) dom (the v2-label of G) = the vertices of G, and

(iii) dom (the vlabel of G) $\neq$ the vertices of G.

Then ((the v2-label of G)(x), 1) -bag $\leq_{\mathrm{InvLexOrder}\,\mathbb{N}}$ ((the v2-label of G)(LexBFS:PickUnnumbered G), 1) -bag .

(41) Let G be a finite finite v2-labeled natsubset v2-labeled vv-graph. Suppose dom (the v2-label of G) = the vertices of G and dom (the vlabel of G) $\neq$ the vertices of G. Then LexBFS:PickUnnumbered $G \notin$ dom (the vlabel of G).

(42) For every finite graph G and for every natural number n holds (LexBFS:CSeq G)(n) $=_G G$.

(43) For every finite graph G and for all natural numbers m, n holds (LexBFS:CSeq G)(m) $=_G$ (LexBFS:CSeq G)(n).

(44) Let G be a finite graph and n be a natural number. Suppose card dom (the vlabel of (LexBFS:CSeq G)(n)) $<$ G.order(). Then the vlabel of (LexBFS:CSeq G)(n + 1) = (the vlabel of (LexBFS:CSeq G)(n))+·(LexBFS:PickUnnumbered(LexBFS:CSeq G)(n) $\dot{\longmapsto}$(G.order() $-'$ card dom (the vlabel of (LexBFS:CSeq G)(n)))).

(45) For every finite graph G and for every natural number n holds dom (the v2-label of (LexBFS:CSeq G)(n)) = the vertices of (LexBFS:CSeq G)(n).

(46) For every finite graph G and for every natural number n such that $n \leq$ G.order() holds card dom (the vlabel of (LexBFS:CSeq G)(n)) = n.

(47) For every finite graph G and for every natural number n such that G.order() $\leq$ n holds (LexBFS:CSeq G)(G.order()) =

(LexBFS:CSeq G)(n).

(48) For every finite graph G and for all natural numbers m, n such that G.order() $\leq m$ and $m \leq n$ holds (LexBFS:CSeq G)(m) $=$ (LexBFS:CSeq G)(n).

(49) For every finite graph G holds LexBFS:CSeq G is eventually constant.

Let G be a finite graph. Note that LexBFS:CSeq G is eventually constant.

We now state two propositions:

(50) Let G be a finite graph and n be a natural number. Then dom (the vlabel of (LexBFS:CSeq G)(n)) = the vertices of (LexBFS:CSeq G)(n) if and only if G.order() $\leq n$.

(51) For every finite graph G holds (LexBFS:CSeq G).Lifespan() = G.order().

Let G be a finite chordal graph and let i be a natural number. One can check that (LexBFS:CSeq G)(i) is chordal.

Let G be a finite chordal graph. One can check that LexBFS:CSeq G is chordal.

One can prove the following proposition

(52) For every finite graph G holds LexBFS:CSeq G is v-label numbering.

Let G be a finite graph. Note that LexBFS:CSeq G is v-label numbering.

We now state several propositions:

(53) For every finite graph G and for every natural number n such that $n <$ G.order() holds LexBFS:CSeq G .PickedAt n $=$ LexBFS:PickUnnumbered(LexBFS:CSeq G)(n).

(54) Let G be a finite graph and n be a natural number. Suppose $n <$ G.order(). Then there exists a vertex w of (LexBFS:CSeq G)(n) such that
  (i) w = LexBFS:PickUnnumbered(LexBFS:CSeq G)(n), and
  (ii) for every set v holds if $v \in G$.adjacentSet($\{w\}$) and $v \notin$ dom (the vlabel of (LexBFS:CSeq G)(n)), then (the v2-label of (LexBFS:CSeq G)($n+1$))(v) = (the v2-label of (LexBFS:CSeq G)(n))(v)$\cup\{G$.order()$-'n\}$ and if $v \notin G$.adjacentSet($\{w\}$) or $v \in$ dom (the vlabel of (LexBFS:CSeq G)(n)), then (the v2-label of (LexBFS:CSeq G)($n+1$))(v) $=$ (the v2-label of (LexBFS:CSeq G)(n))(v).

(55) Let G be a finite graph, i be a natural number, and v be a set. Then (the v2-label of (LexBFS:CSeq G)(i))(v) $\subseteq$ Seg(G.order())$\setminus$Seg(G.order()$-'i$).

(56) Let G be a finite graph, x be a set, and i, j be natural numbers. If $i \leq j$, then (the v2-label of (LexBFS:CSeq G)(i))(x) $\subseteq$ (the v2-label of (LexBFS:CSeq G)(j))(x).

(57) Let G be a finite graph, m, n be natural numbers, and x, y be sets. Suppose $n <$ G.order() and $n < m$ and $y =$ LexBFS:PickUnnumbered(LexBFS:CSeq G)(n) and $x \notin$ dom (the vlabel of

(LexBFS:CSeq G)(n)) and $x \in G$.adjacentSet($\{y\}$). Then G.order() $-'$ $n \in$ (the v2-label of (LexBFS:CSeq G)(m))(x).

(58) Let G be a finite graph and m, n be natural numbers. Suppose $m < n$. Let x be a set. Suppose G.order() $-'$ $m \notin$ (the v2-label of (LexBFS:CSeq G)($m+1$))(x). Then G.order() $-'$ $m \notin$ (the v2-label of (LexBFS:CSeq G)(n))(x).

(59) Let G be a finite graph and m, n, k be natural numbers. Suppose $k < n$ and $n \leq m$. Let x be a set. Suppose G.order() $-'$ $k \notin$ (the v2-label of (LexBFS:CSeq G)(n))(x). Then G.order() $-'$ $k \notin$ (the v2-label of (LexBFS:CSeq G)(m))(x).

(60) Let G be a finite graph, m, n be natural numbers, and x be a vertex of (LexBFS:CSeq G)(m). Suppose $n \in$ (the v2-label of (LexBFS:CSeq G)(m))(x). Then there exists a vertex y of (LexBFS:CSeq G)(m) such that LexBFS:PickUnnumbered(LexBFS:CSeq G) (G.order() $-'$ n) $= y$ and $y \notin$ dom (the vlabel of (LexBFS:CSeq G)(G.order() $-'$ n)) and $x \in G$.adjacentSet($\{y\}$).

Let G_4 be a finite natural v-labeled vv-graph sequence. Then G_4.Result() is a finite natural v-labeled vv-graph.

The following four propositions are true:

(61) For every finite graph G holds (LexBFS:CSeq G).Result().labeledV() = the vertices of G.

(62) For every finite graph G holds (the vlabel of (LexBFS:CSeq G).Result())$^{-1}$ is a vertex scheme of G.

(63) Let G be a finite graph, i, j be natural numbers, and a, b be vertices of (LexBFS:CSeq G)(i). Suppose that

(i) $a \in$ dom (the vlabel of (LexBFS:CSeq G)(i)),

(ii) $b \in$ dom (the vlabel of (LexBFS:CSeq G)(i)),

(iii) (the vlabel of (LexBFS:CSeq G)(i))(a) $<$ (the vlabel of (LexBFS:CSeq G)(i))(b), and

(iv) $j = G$.order() $-'$ (the vlabel of (LexBFS:CSeq G)(i))(b).

Then ((the v2-label of (LexBFS:CSeq G)(j))(a), 1)-bag $\leq_{\mathrm{InvLexOrder}\,\mathbb{N}}$ ((the v2-label of (LexBFS:CSeq G)(j))(b), 1)-bag.

(64) Let G be a finite graph, i, j be natural numbers, and v be a vertex of (LexBFS:CSeq G)(i). Suppose $j \in$ (the v2-label of (LexBFS:CSeq G)(i))(v). Then there exists a vertex w of (LexBFS:CSeq G)(i) such that $w \in$ dom (the vlabel of (LexBFS:CSeq G)(i)) and (the vlabel of (LexBFS:CSeq G)(i))(w) $= j$ and $v \in G$.adjacentSet($\{w\}$).

Let G be a natural v-labeled v-graph. We say that G has property *L3* if and only if the condition (Def. 34) is satisfied.

(Def. 34) Let a, b, c be vertices of G. Suppose that $a \in \mathrm{dom}$ (the vlabel of G) and $b \in \mathrm{dom}$ (the vlabel of G) and $c \in \mathrm{dom}$ (the vlabel of G) and (the vlabel of G)(a) $<$ (the vlabel of G)(b) and (the vlabel of G)(b) $<$ (the vlabel of G)(c) and a and c are adjacent and b and c are not adjacent. Then there exists a vertex d of G such that

(i) $d \in \mathrm{dom}$ (the vlabel of G),

(ii) (the vlabel of G)(c) $<$ (the vlabel of G)(d),

(iii) b and d are adjacent,

(iv) a and d are not adjacent, and

(v) for every vertex e of G such that $e \neq d$ and e and b are adjacent and e and a are not adjacent holds (the vlabel of G)(e) $<$ (the vlabel of G)(d).

One can prove the following three propositions:

(65) For every finite graph G and for every natural number n holds (LexBFS:CSeq G)(n) has property *L3*.

(66) Let G be a finite chordal natural v-labeled v-graph. Suppose G has property *L3* and dom (the vlabel of G) = the vertices of G. Let V be a vertex scheme of G. If V^{-1} = the vlabel of G, then V is perfect.

(67) For every finite chordal vv-graph G holds (the vlabel of (LexBFS:CSeq G).Result())$^{-1}$ is a perfect vertex scheme of G.

7. The Maximum Cardinality Search Algorithm

Let G be a finite graph. The functor MCS:Init G yields a finite natural v-labeled natural v2-labeled vv-graph and is defined by:

(Def. 35) MCS:Init G = G.set(VLabelSelector, $\emptyset$).set(V2-LabelSelector, (the vertices of G) $\longmapsto 0$).

Let G be a finite natural v2-labeled vv-graph. Let us assume that dom (the v2-label of G) = the vertices of G. The functor MCS:PickUnnumbered G yields a vertex of G and is defined by:

(Def. 36)(i) MCS:PickUnnumbered G = choose(the vertices of G) if dom (the vlabel of G) = the vertices of G,

(ii) there exists a finite non empty natural-membered set S and there exists a function F such that $S = \mathrm{rng}\, F$ and F = (the v2-label of G)$\restriction$((the vertices of G) $\setminus$ dom (the vlabel of G)) and MCS:PickUnnumbered G = choose($F^{-1}(\{\max S\})$), otherwise.

Let G be a finite natural v2-labeled vv-graph and let v be a set. The functor MCS:LabelAdjacent(G, v) yields a finite natural v2-labeled vv-graph and is defined by:

(Def. 37) MCS:LabelAdjacent(G, v) $=$ G.set(V2-LabelSelector, (the v2-label of G) .incSubset((G.adjacentSet($\{v\}$)) $\setminus$ dom (the vlabel of G), 1)).

Let G be a finite natural v-labeled natural v2-labeled vv-graph and let v be a vertex of G. Then MCS:LabelAdjacent(G, v) is a finite natural v-labeled natural v2-labeled vv-graph.

Let G be a finite natural v-labeled natural v2-labeled vv-graph, let v be a vertex of G, and let n be a natural number. The functor MCS:Update(G, v, n) yielding a finite natural v-labeled natural v2-labeled vv-graph is defined as follows:

(Def. 38) MCS:Update(G, v, n) = MCS:LabelAdjacent(G.labelVertex$(v, G$.order$()-' n), v)$.

Let G be a finite natural v-labeled natural v2-labeled vv-graph. The functor MCS:Step G yielding a finite natural v-labeled natural v2-labeled vv-graph is defined by:

(Def. 39) MCS:Step G = $\begin{cases} G, \text{ if } G\text{.order}() \leq \text{card dom (the vlabel of } G), \\ \text{MCS:Update}(G, \text{MCS:PickUnnumbered } G, \text{card dom} \\ \quad \text{(the vlabel of } G)), \text{ otherwise.} \end{cases}$

Let G be a finite graph. The functor MCS:CSeq G yields a finite natural v-labeled natural v2-labeled vv-graph sequence and is defined by:

(Def. 40) (MCS:CSeq G)(0) = MCS:Init G and for every natural number n holds (MCS:CSeq G)$(n + 1)$ = MCS:Step(MCS:CSeq G)(n).

The following proposition is true

(68) For every finite graph G holds MCS:CSeq G is iterative.

Let G be a finite graph. Observe that MCS:CSeq G is iterative.

We now state a number of propositions:

(69) For every finite graph G holds the vlabel of MCS:Init $G = \emptyset$.

(70) Let G be a finite graph and v be a set. Then dom (the v2-label of MCS:Init G) = the vertices of G and (the v2-label of MCS:Init G)$(v) = 0$.

(71) For every finite graph G holds $G =_G$ MCS:Init G.

(72) Let G be a finite natural v2-labeled vv-graph and x be a set. Suppose that

(i) $x \notin$ dom (the vlabel of G),

(ii) dom (the v2-label of G) = the vertices of G, and

(iii) dom (the vlabel of G) $\neq$ the vertices of G.

Then (the v2-label of G)$(x) \leq$ (the v2-label of G)(MCS:PickUnnumbered G).

(73) Let G be a finite natural v2-labeled vv-graph. Suppose dom (the v2-label of G) = the vertices of G and dom (the vlabel of G) $\neq$ the vertices of G. Then MCS:PickUnnumbered $G \notin$ dom (the vlabel of G).

(74) Let G be a finite natural v2-labeled vv-graph and v, x be sets. If $x \notin G$.adjacentSet($\{v\}$), then (the v2-label of G)(x) = (the v2-label of

MCS:LabelAdjacent$(G, v))(x)$.

(75) Let G be a finite natural v2-labeled vv-graph and v, x be sets. Suppose $x \in$ dom (the vlabel of G). Then (the v2-label of $G)(x) =$ (the v2-label of MCS:LabelAdjacent$(G, v))(x)$.

(76) Let G be a finite natural v2-labeled vv-graph and v, x be sets. Suppose $x \in$ dom (the v2-label of G) and $x \in G$.adjacentSet($\{v\}$) and $x \notin$ dom (the vlabel of G). Then (the v2-label of MCS:LabelAdjacent$(G, v))(x) =$ (the v2-label of $G)(x) + 1$.

(77) Let G be a finite natural v2-labeled vv-graph and v be a set. Suppose dom (the v2-label of G) = the vertices of G. Then dom (the v2-label of MCS:LabelAdjacent(G, v)) = the vertices of G.

(78) For every finite graph G and for every natural number n holds (MCS:CSeq $G)(n) =_G G$.

(79) For every finite graph G and for all natural numbers m, n holds (MCS:CSeq $G)(m) =_G$ (MCS:CSeq $G)(n)$.

Let G be a finite chordal graph and let n be a natural number. Observe that (MCS:CSeq $G)(n)$ is chordal.

Let G be a finite chordal graph. Observe that MCS:CSeq G is chordal.

One can prove the following propositions:

(80) For every finite graph G and for every natural number n holds dom (the v2-label of (MCS:CSeq $G)(n)$) = the vertices of (MCS:CSeq $G)(n)$.

(81) Let G be a finite graph and n be a natural number. Suppose card dom (the vlabel of (MCS:CSeq $G)(n)$) $< G$.order(). Then the vlabel of (MCS:CSeq $G)(n+1) =$ (the vlabel of (MCS:CSeq $G)(n)$) $+\cdot$(MCS:PickUnnumbered(MCS:CSeq $G)(n) \dot{\longmapsto} (G$.order()$-'$card dom (the vlabel of (MCS:CSeq $G)(n)$))).

(82) For every finite graph G and for every natural number n such that $n \leq G$.order() holds card dom (the vlabel of (MCS:CSeq $G)(n)$) $= n$.

(83) For every finite graph G and for every natural number n such that G.order() $\leq n$ holds (MCS:CSeq $G)(G$.order()) = (MCS:CSeq $G)(n)$.

(84) For every finite graph G and for all natural numbers m, n such that G.order() $\leq m$ and $m \leq n$ holds (MCS:CSeq $G)(m) =$ (MCS:CSeq $G)(n)$.

(85) For every finite graph G holds MCS:CSeq G is eventually constant.

Let G be a finite graph. Observe that MCS:CSeq G is eventually constant.

The following propositions are true:

(86) Let G be a finite graph and n be a natural number. Then dom (the vlabel of (MCS:CSeq $G)(n)$) = the vertices of (MCS:CSeq $G)(n)$ if and only if G.order() $\leq n$.

(87) For every finite graph G holds (MCS:CSeq G).Lifespan() $= G$.order().

(88) For every finite graph G holds MCS:CSeq G is v-label numbering.

Let G be a finite graph. Note that MCS:CSeq G is v-label numbering.

Next we state three propositions:

(89) For every finite graph G and for every natural number n such that $n <$ G.order() holds MCS:CSeq G .PickedAt $n =$ MCS:PickUnnumbered(MCS:CSeq G)(n).

(90) Let G be a finite graph and n be a natural number. Suppose $n <$ G.order(). Then there exists a vertex w of (MCS:CSeq G)(n) such that

(i) $w =$ MCS:PickUnnumbered(MCS:CSeq G)(n), and

(ii) for every set v holds if $v \in G$.adjacentSet($\{w\}$) and $v \notin$ dom (the vlabel of (MCS:CSeq G)(n)), then (the v2-label of (MCS:CSeq G)($n+1$))(v) = (the v2-label of (MCS:CSeq G)(n))(v) + 1 and if $v \notin G$.adjacentSet($\{w\}$) or $v \in$ dom (the vlabel of (MCS:CSeq G)(n)), then (the v2-label of (MCS:CSeq G)($n+1$))(v) = (the v2-label of (MCS:CSeq G)(n))(v).

(91) Let G be a finite graph, n be a natural number, and x be a set. Suppose $x \notin$ dom (the vlabel of (MCS:CSeq G)(n)). Then (the v2-label of (MCS:CSeq G)(n))(x) = card((G.adjacentSet($\{x\}$)) $\cap$ dom (the vlabel of (MCS:CSeq G)(n))).

Let G be a natural v-labeled v-graph. We say that G has property T if and only if the condition (Def. 41) is satisfied.

(Def. 41) Let a, b, c be vertices of G. Suppose that $a \in$ dom (the vlabel of G) and $b \in$ dom (the vlabel of G) and $c \in$ dom (the vlabel of G) and (the vlabel of G)(a) < (the vlabel of G)(b) and (the vlabel of G)(b) < (the vlabel of G)(c) and a and c are adjacent and b and c are not adjacent. Then there exists a vertex d of G such that

(i) $d \in$ dom (the vlabel of G),

(ii) (the vlabel of G)(b) < (the vlabel of G)(d),

(iii) b and d are adjacent, and

(iv) a and d are not adjacent.

We now state three propositions:

(92) For every finite graph G and for every natural number n holds (MCS:CSeq G)(n) has property T.

(93) For every finite graph G holds (LexBFS:CSeq G).Result() has property T.

(94) Let G be a finite chordal natural v-labeled v-graph. Suppose G has property T and dom (the vlabel of G) = the vertices of G. Let V be a vertex scheme of G. If V^{-1} = the vlabel of G, then V is perfect.

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Integrability and the Integral of Partial Functions from $\mathbb{R}$ into $\mathbb{R}$[1]

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Summary. In this paper, we showed the linearity of the indefinite integral $\int_a^b f dx$, the form of which was introduced in [11]. In addition, we proved some theorems about the integral calculus on the subinterval of $[a, b]$. As a result, we described the fundamental theorem of calculus, that we developed in [11], by a more general expression.

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The articles [23], [25], [26], [2], [22], [4], [14], [1], [24], [5], [27], [7], [6], [21], [9], [3], [17], [16], [15], [18], [20], [8], [10], [13], [19], [12], and [11] provide the notation and terminology for this paper.

1. Preliminaries

We use the following convention: a, b, c, d, e, x are real numbers, A is a closed-interval subset of $\mathbb{R}$, and f, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

We now state several propositions:

(1) If $a \leq b$ and $c \leq d$ and $a + c = b + d$, then $a = b$ and $c = d$.

(2) If $a \leq b$, then $]x - a, x + a[\subseteq]x - b, x + b[$.

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(3) For every binary relation R and for all sets A, B, C such that $A \subseteq B$ and $A \subseteq C$ holds $R{\restriction}B{\restriction}A = R{\restriction}C{\restriction}A$.

(4) For all sets A, B, C such that $A \subseteq B$ and $A \subseteq C$ holds $\chi_{B,B}{\restriction}A = \chi_{C,C}{\restriction}A$.

(5) If $a \leq b$, then $\mathrm{vol}([\prime a, b\prime]) = b - a$.

(6) $\mathrm{vol}([\prime \min(a, b), \max(a, b)\prime]) = |b - a|$.

2. Integrability and the Integral of Partial Functions

The following propositions are true:

(7) If $A \subseteq \mathrm{dom}\, f$ and f is integrable on A and f is bounded on A, then $|f|$ is integrable on A and $|\int\limits_{A} f(x)dx| \leq \int\limits_{A} |f|(x)dx$.

(8) If $a \leq b$ and $[\prime a, b\prime] \subseteq \mathrm{dom}\, f$ and f is integrable on $[\prime a, b\prime]$ and f is bounded on $[\prime a, b\prime]$, then $|\int\limits_{a}^{b} f(x)dx| \leq \int\limits_{a}^{b} |f|(x)dx$.

(9) Let r be a real number. Suppose $A \subseteq \mathrm{dom}\, f$ and f is integrable on A and f is bounded on A. Then $r\, f$ is integrable on A and $\int\limits_{A} (r\, f)(x)dx = r \cdot \int\limits_{A} f(x)dx$.

(10) If $a \leq b$ and $[\prime a, b\prime] \subseteq \mathrm{dom}\, f$ and f is integrable on $[\prime a, b\prime]$ and f is bounded on $[\prime a, b\prime]$, then $\int\limits_{a}^{b} (c\, f)(x)dx = c \cdot \int\limits_{a}^{b} f(x)dx$.

(11) Suppose $A \subseteq \mathrm{dom}\, f$ and $A \subseteq \mathrm{dom}\, g$ and f is integrable on A and f is bounded on A and g is integrable on A and g is bounded on A. Then $f + g$ is integrable on A and $f - g$ is integrable on A and $\int\limits_{A} (f + g)(x)dx = \int\limits_{A} f(x)dx + \int\limits_{A} g(x)dx$ and $\int\limits_{A} (f - g)(x)dx = \int\limits_{A} f(x)dx - \int\limits_{A} g(x)dx$.

(12) Suppose that $a \leq b$ and $[\prime a, b\prime] \subseteq \mathrm{dom}\, f$ and $[\prime a, b\prime] \subseteq \mathrm{dom}\, g$ and f is integrable on $[\prime a, b\prime]$ and g is integrable on $[\prime a, b\prime]$ and f is bounded on $[\prime a, b\prime]$ and g is bounded on $[\prime a, b\prime]$. Then $\int\limits_{a}^{b} (f + g)(x)dx = \int\limits_{a}^{b} f(x)dx + \int\limits_{a}^{b} g(x)dx$

and $\int\limits_a^b (f-g)(x)dx = \int\limits_a^b f(x)dx - \int\limits_a^b g(x)dx$.

(13) If f is bounded on A and g is bounded on A, then $f\,g$ is bounded on A.

(14) Suppose $A \subseteq \operatorname{dom} f$ and $A \subseteq \operatorname{dom} g$ and f is integrable on A and f is bounded on A and g is integrable on A and g is bounded on A. Then $f\,g$ is integrable on A.

(15) Let n be an element of $\mathbb{N}$. Suppose $n > 0$ and $\operatorname{vol}(A) > 0$. Then there exists an element D of $\operatorname{divs} A$ such that $\operatorname{len} D = n$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} D$ holds $D(i) = \inf A + \frac{\operatorname{vol}(A)}{n} \cdot i$.

3. Integrability on a Subinterval

The following propositions are true:

(16) Suppose $\operatorname{vol}(A) > 0$. Then there exists a DivSequence T of A such that

(i) δ_T is convergent,

(ii) $\lim(\delta_T) = 0$, and

(iii) for every element n of $\mathbb{N}$ there exists an element T_1 of $\operatorname{divs} A$ such that T_1 divides into equal $n+1$ and $T(n) = T_1$.

(17) Suppose $a \leq b$ and f is integrable on $['a, b']$ and f is bounded on $['a, b']$ and $['a, b'] \subseteq \operatorname{dom} f$ and $c \in ['a, b']$. Then f is integrable on $['a, c']$ and f is integrable on $['c, b']$ and $\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx$.

(18) Suppose $a \leq c$ and $c \leq d$ and $d \leq b$ and f is integrable on $['a, b']$ and f is bounded on $['a, b']$ and $['a, b'] \subseteq \operatorname{dom} f$. Then f is integrable on $['c, d']$ and f is bounded on $['c, d']$ and $['c, d'] \subseteq \operatorname{dom} f$.

(19) Suppose that $a \leq c$ and $c \leq d$ and $d \leq b$ and f is integrable on $['a, b']$ and g is integrable on $['a, b']$ and f is bounded on $['a, b']$ and g is bounded on $['a, b']$ and $['a, b'] \subseteq \operatorname{dom} f$ and $['a, b'] \subseteq \operatorname{dom} g$. Then $f + g$ is integrable on $['c, d']$ and $f + g$ is bounded on $['c, d']$.

(20) Suppose $a \leq b$ and f is integrable on $['a, b']$ and f is bounded on $['a, b']$ and $['a, b'] \subseteq \operatorname{dom} f$ and $c \in ['a, b']$ and $d \in ['a, b']$. Then $\int\limits_a^d f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^d f(x)dx$.

(21) Suppose $a \leq b$ and f is integrable on $['a, b']$ and f is bounded on $['a, b']$ and $['a, b'] \subseteq \operatorname{dom} f$ and $c \in ['a, b']$ and $d \in ['a, b']$. Then $['\min(c, d), \max(c, d)'] \subseteq \operatorname{dom}|f|$ and $|f|$ is integrable on

$['\min(c,d),\max(c,d)']$ and $|f|$ is bounded on $['\min(c,d),\max(c,d)']$ and $|\int_{c}^{d} f(x)dx| \leq \int_{\min(c,d)}^{\max(c,d)} |f|(x)dx$.

(22) Suppose $a \leq b$ and $c \leq d$ and f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $c \in ['a,b']$ and $d \in ['a,b']$. Then $['c,d'] \subseteq \operatorname{dom}|f|$ and $|f|$ is integrable on $['c,d']$ and $|f|$ is bounded on $['c,d']$ and $|\int_{c}^{d} f(x)dx| \leq \int_{c}^{d} |f|(x)dx$ and $|\int_{d}^{c} f(x)dx| \leq \int_{c}^{d} |f|(x)dx$.

(23) Suppose that $a \leq b$ and $c \leq d$ and f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $c \in ['a,b']$ and $d \in ['a,b']$ and for every real number x such that $x \in ['c,d']$ holds $|f(x)| \leq e$. Then $|\int_{c}^{d} f(x)dx| \leq e \cdot (d-c)$ and $|\int_{d}^{c} f(x)dx| \leq e \cdot (d-c)$.

(24) Suppose that $a \leq b$ and f is integrable on $['a,b']$ and g is integrable on $['a,b']$ and f is bounded on $['a,b']$ and g is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $['a,b'] \subseteq \operatorname{dom} g$ and $c \in ['a,b']$ and $d \in ['a,b']$. Then $\int_{c}^{d} (f+g)(x)dx = \int_{c}^{d} f(x)dx + \int_{c}^{d} g(x)dx$ and $\int_{c}^{d} (f-g)(x)dx = \int_{c}^{d} f(x)dx - \int_{c}^{d} g(x)dx$.

(25) Suppose $a \leq b$ and f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $c \in ['a,b']$ and $d \in ['a,b']$. Then $\int_{c}^{d} (e\, f)(x)dx = e \cdot \int_{c}^{d} f(x)dx$.

(26) Suppose $a \leq b$ and $['a,b'] \subseteq \operatorname{dom} f$ and for every real number x such that $x \in ['a,b']$ holds $f(x) = e$. Then f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $\int_{a}^{b} f(x)dx = e \cdot (b-a)$.

(27) Suppose $a \leq b$ and for every real number x such that $x \in ['a,b']$ holds $f(x) = e$ and $['a,b'] \subseteq \operatorname{dom} f$ and $c \in ['a,b']$ and $d \in ['a,b']$. Then $\int_{c}^{d} f(x)dx = e \cdot (d-c)$.

4. Fundamental Theorem of Calculus

Next we state two propositions:

(28) Let x_0 be a real number and F be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $a \leq b$ and f is integrable on $['a, b']$ and f is bounded on $['a, b']$ and $['a, b'] \subseteq \operatorname{dom} f$ and $]a, b[\subseteq \operatorname{dom} F$ and for every real number x such that $x \in]a, b[$ holds $F(x) = \int\limits_{a}^{x} f(x)dx$ and $x_0 \in]a, b[$ and f is continuous in x_0. Then F is differentiable in x_0 and $F'(x_0) = f(x_0)$.

(29) Let x_0 be a real number. Suppose $a \leq b$ and f is integrable on $['a, b']$ and f is bounded on $['a, b']$ and $['a, b'] \subseteq \operatorname{dom} f$ and $x_0 \in]a, b[$ and f is continuous in x_0. Then there exists a partial function F from $\mathbb{R}$ to $\mathbb{R}$ such that $]a, b[\subseteq \operatorname{dom} F$ and for every real number x such that $x \in]a, b[$ holds $F(x) = \int\limits_{a}^{x} f(x)dx$ and F is differentiable in x_0 and $F'(x_0) = f(x_0)$.

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Baire's Category Theorem and Some Spaces Generated from Real Normed Space[1]

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Summary. As application of complete metric space, we proved a Baire's category theorem. Then we defined some spaces generated from real normed space and discussed each of them. In the second section, we showed the equivalence of convergence and the continuity of a function. In other sections, we showed some topological properties of two spaces, which are topological space and linear topological space generated from real normed space.

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The papers [23], [7], [26], [4], [1], [21], [15], [27], [6], [5], [17], [19], [20], [24], [22], [2], [25], [9], [10], [13], [16], [12], [11], [3], [18], [8], and [14] provide the notation and terminology for this paper.

1. Baire's Category Theorem

The following proposition is true

(1) Let X be a non empty metric space and Y be a sequence of subsets of X. Suppose X is complete and $\bigcup \operatorname{rng} Y =$ the carrier of X and for every element n of $\mathbb{N}$ holds $Y(n)^{\mathrm{c}} \in$ the open set family of X. Then there exists an element n_0 of $\mathbb{N}$ and there exists a real number r and there exists a point x_0 of X such that $0 < r$ and $\mathrm{Ball}(x_0, r) \subseteq Y(n_0)$.

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2. Metric Space Generated from Real Normed Space

Let X be a real normed space. The distance by norm of X yields a function from $[\!:$ the carrier of X, the carrier of $X :\!]$ into $\mathbb{R}$ and is defined as follows:

(Def. 1) For all points x, y of X holds (the distance by norm of X)$(x, y) = \|x - y\|$.

Let X be a real normed space. The functor MetricSpaceNorm X yields a non empty metric space and is defined by:

(Def. 2) MetricSpaceNorm $X = \langle$the carrier of X, the distance by norm of $X\rangle$.

Next we state several propositions:

(2) Let X be a real normed space, z be an element of MetricSpaceNorm X, and r be a real number. Then there exists a point x of X such that $x = z$ and $\mathrm{Ball}(z, r) = \{y; y$ ranges over points of X: $\|x - y\| < r\}$.

(3) Let X be a real normed space, z be an element of MetricSpaceNorm X, and r be a real number. Then there exists a point x of X such that $x = z$ and $\overline{\mathrm{Ball}}(z, r) = \{y; y$ ranges over points of X: $\|x - y\| \leq r\}$.

(4) Let X be a real normed space, S be a sequence of X, S_1 be a sequence of MetricSpaceNorm X, x be a point of X, and x_1 be a point of MetricSpaceNorm X. Suppose $S = S_1$ and $x = x_1$. Then S_1 is convergent to x_1 if and only if for every real number r such that $0 < r$ there exists an element m of $\mathbb{N}$ such that for every element n of $\mathbb{N}$ such that $m \leq n$ holds $\|S(n) - x\| < r$.

(5) Let X be a real normed space, S be a sequence of X, and S_1 be a sequence of MetricSpaceNorm X. If $S = S_1$, then S_1 is convergent iff S is convergent.

(6) Let X be a real normed space, S be a sequence of X, and S_1 be a sequence of MetricSpaceNorm X. If $S = S_1$ and S_1 is convergent, then $\lim S_1 = \lim S$.

3. Topological Space Generated from Real Normed Space

Let X be a real normed space. The functor TopSpaceNorm X yields a non empty topological space and is defined by:

(Def. 3) TopSpaceNorm $X = (\text{MetricSpaceNorm}\, X)_{\mathrm{top}}$.

The following propositions are true:

(7) Let X be a real normed space and V be a subset of TopSpaceNorm X. Then V is open if and only if for every point x of X such that $x \in V$ there exists a real number r such that $r > 0$ and $\{y; y$ ranges over points of X: $\|x - y\| < r\} \subseteq V$.

(8) Let X be a real normed space, x be a point of X, and r be a real number. Then $\{y; y$ ranges over points of X: $\|x - y\| < r\}$ is an open subset of TopSpaceNorm X.

(9) Let X be a real normed space, x be a point of X, and r be a real number. Then $\{y; y$ ranges over points of X: $\|x - y\| \leq r\}$ is a closed subset of TopSpaceNorm X.

(10) For every Hausdorff non empty topological space X such that X is locally-compact holds X is Baire.

(11) For every real normed space X holds TopSpaceNorm X is sequential.

Let X be a real normed space. Observe that TopSpaceNorm X is sequential.

One can prove the following propositions:

(12) Let X be a real normed space, S be a sequence of X, S_1 be a sequence of TopSpaceNorm X, x be a point of X, and x_1 be a point of TopSpaceNorm X. Suppose $S = S_1$ and $x = x_1$. Then S_1 is convergent to x_1 if and only if for every real number r such that $0 < r$ there exists an element m of $\mathbb{N}$ such that for every element n of $\mathbb{N}$ such that $m \leq n$ holds $\|S(n) - x\| < r$.

(13) Let X be a real normed space, S be a sequence of X, and S_1 be a sequence of TopSpaceNorm X. If $S = S_1$, then S_1 is convergent iff S is convergent.

(14) Let X be a real normed space, S be a sequence of X, and S_1 be a sequence of TopSpaceNorm X. If $S = S_1$ and S_1 is convergent, then $\operatorname{Lim} S_1 = \{\lim S\}$ and $\lim S_1 = \lim S$.

(15) Let X be a real normed space, V be a subset of X, and V_1 be a subset of TopSpaceNorm X. If $V = V_1$, then V is closed iff V_1 is closed.

(16) Let X be a real normed space, V be a subset of X, and V_1 be a subset of TopSpaceNorm X. If $V = V_1$, then V is open iff V_1 is open.

(17) Let X be a real normed space, U be a subset of X, U_1 be a subset of TopSpaceNorm X, x be a point of X, and x_1 be a point of TopSpaceNorm X. Suppose $U = U_1$ and $x = x_1$. Then U is a neighbourhood of x if and only if U_1 is a neighbourhood of x_1.

(18) Let X, Y be real normed spaces, f be a partial function from X to Y, f_1 be a function from TopSpaceNorm X into TopSpaceNorm Y, x be a point of X, and x_1 be a point of TopSpaceNorm X. Suppose $f = f_1$ and $x = x_1$. Then f is continuous in x if and only if f_1 is continuous at x_1.

(19) Let X, Y be real normed spaces, f be a partial function from X to Y, and f_1 be a function from TopSpaceNorm X into TopSpaceNorm Y. Suppose $f = f_1$. Then f is continuous on the carrier of X if and only if f_1 is continuous.

4. Linear Topological Space Generated from Real Normed Space

Let X be a real normed space. The functor LinearTopSpaceNorm X yields a strict non empty real linear topological structure and is defined by the conditions (Def. 4).

(Def. 4)(i) The carrier of LinearTopSpaceNorm X = the carrier of X,

(ii) the zero of LinearTopSpaceNorm X = the zero of X,

(iii) the addition of LinearTopSpaceNorm X = the addition of X,

(iv) the external multiplication of LinearTopSpaceNorm X = the external multiplication of X, and

(v) the topology of LinearTopSpaceNorm X = the topology of TopSpaceNorm X.

Let X be a real normed space. Note that LinearTopSpaceNorm X is add-continuous, mult-continuous, topological space-like, Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

We now state several propositions:

(20) Let X be a real normed space, V be a subset of TopSpaceNorm X, and V_1 be a subset of LinearTopSpaceNorm X. If $V = V_1$, then V is open iff V_1 is open.

(21) Let X be a real normed space, V be a subset of TopSpaceNorm X, and V_1 be a subset of LinearTopSpaceNorm X. If $V = V_1$, then V is closed iff V_1 is closed.

(22) Let X be a real normed space and V be a subset of LinearTopSpaceNorm X. Then V is open if and only if for every point x of X such that $x \in V$ there exists a real number r such that $r > 0$ and $\{y; y$ ranges over points of X: $\|x - y\| < r\} \subseteq V$.

(23) Let X be a real normed space, x be a point of X, r be a real number, and V be a subset of LinearTopSpaceNorm X. If $V = \{y; y$ ranges over points of X: $\|x - y\| < r\}$, then V is open.

(24) Let X be a real normed space, x be a point of X, r be a real number, and V be a subset of TopSpaceNorm X. If $V = \{y; y$ ranges over points of X: $\|x - y\| \leq r\}$, then V is closed.

Let X be a real normed space. Observe that LinearTopSpaceNorm X is T_2 and LinearTopSpaceNorm X is sober.

One can prove the following proposition

(25) Let X be a real normed space, S be a family of subsets of TopSpaceNorm X, S_1 be a family of subsets of LinearTopSpaceNorm X, x be a point of TopSpaceNorm X, and x_1 be a point of LinearTopSpaceNorm X. Suppose $S = S_1$ and $x = x_1$. Then S_1 is a basis of x_1 if and only if S is a basis of x.

Let X be a real normed space. One can verify the following observations:

* LinearTopSpaceNorm X is first-countable,
* LinearTopSpaceNorm X is Frechet, and
* LinearTopSpaceNorm X is sequential.

Next we state a number of propositions:

(26) Let X be a real normed space, S be a sequence of TopSpaceNorm X, S_1 be a sequence of LinearTopSpaceNorm X, x be a point of TopSpaceNorm X, and x_1 be a point of LinearTopSpaceNorm X. Suppose $S = S_1$ and $x = x_1$. Then S_1 is convergent to x_1 if and only if S is convergent to x.

(27) Let X be a real normed space, S be a sequence of TopSpaceNorm X, and S_1 be a sequence of LinearTopSpaceNorm X. If $S = S_1$, then S_1 is convergent iff S is convergent.

(28) Let X be a real normed space, S be a sequence of TopSpaceNorm X, and S_1 be a sequence of LinearTopSpaceNorm X. If $S = S_1$ and S_1 is convergent, then $\mathrm{Lim}\, S = \mathrm{Lim}\, S_1$ and $\lim S = \lim S_1$.

(29) Let X be a real normed space, S be a sequence of X, S_1 be a sequence of LinearTopSpaceNorm X, x be a point of X, and x_1 be a point of LinearTopSpaceNorm X. Suppose $S = S_1$ and $x = x_1$. Then S_1 is convergent to x_1 if and only if for every real number r such that $0 < r$ there exists an element m of $\mathbb{N}$ such that for every element n of $\mathbb{N}$ such that $m \leq n$ holds $\|S(n) - x\| < r$.

(30) Let X be a real normed space, S be a sequence of X, and S_1 be a sequence of LinearTopSpaceNorm X. If $S = S_1$, then S_1 is convergent iff S is convergent.

(31) Let X be a real normed space, S be a sequence of X, and S_1 be a sequence of LinearTopSpaceNorm X. If $S = S_1$ and S_1 is convergent, then $\mathrm{Lim}\, S_1 = \{\lim S\}$ and $\lim S_1 = \lim S$.

(32) Let X be a real normed space, V be a subset of X, and V_1 be a subset of LinearTopSpaceNorm X. If $V = V_1$, then V is closed iff V_1 is closed.

(33) Let X be a real normed space, V be a subset of X, and V_1 be a subset of LinearTopSpaceNorm X. If $V = V_1$, then V is open iff V_1 is open.

(34) Let X be a real normed space, U be a subset of TopSpaceNorm X, U_1 be a subset of LinearTopSpaceNorm X, x be a point of TopSpaceNorm X, and x_1 be a point of LinearTopSpaceNorm X. Suppose $U = U_1$ and $x = x_1$. Then U is a neighbourhood of x if and only if U_1 is a neighbourhood of x_1.

(35) Let X, Y be real normed spaces, f be a function from TopSpaceNorm X into TopSpaceNorm Y, f_1 be a function from LinearTopSpaceNorm X into LinearTopSpaceNorm Y, x be a point of TopSpaceNorm X, and x_1 be a

point of LinearTopSpaceNorm X. Suppose $f = f_1$ and $x = x_1$. Then f is continuous at x if and only if f_1 is continuous at x_1.

(36) Let X, Y be real normed spaces, f be a function from TopSpaceNorm X into TopSpaceNorm Y, and f_1 be a function from LinearTopSpaceNorm X into LinearTopSpaceNorm Y. If $f = f_1$, then f is continuous iff f_1 is continuous.

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On the Representation of Natural Numbers in Positional Numeral Systems[1]

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Summary. In this paper we show that every natural number can be uniquely represented as a base-b numeral. The formalization is based on the proof presented in [11]. We also prove selected divisibility criteria in the base-10 numeral system.

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The notation and terminology used in this paper have been introduced in the following articles: [13], [15], [2], [1], [17], [12], [14], [6], [4], [5], [8], [9], [10], [16], [7], and [3].

1. Preliminaries

One can prove the following propositions:

(1) For all finite 0-sequences d, e of $\mathbb{N}$ holds $\sum(d \frown e) = \sum d + \sum e$.

(2) Let S be a sequence of real numbers, d be a finite 0-sequence of $\mathbb{N}$, and n be a natural number. If $d = S{\restriction}(n+1)$, then $\sum d = (\sum_{\alpha=0}^{\kappa} S(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(3) For all natural numbers k, l, m holds $(k\,(l^{\kappa})_{\kappa\in\mathbb{N}}){\restriction}m$ is a finite 0-sequence of $\mathbb{N}$.

(4) Let d, e be finite 0-sequences of $\mathbb{N}$. Suppose $\operatorname{len} d \geq 1$ and $\operatorname{len} d = \operatorname{len} e$ and for every natural number i such that $i \in \operatorname{dom} d$ holds $d(i) \leq e(i)$. Then $\sum d \leq \sum e$.

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(5) Let d be a finite 0-sequence of $\mathbb{N}$ and n be a natural number. If for every natural number i such that $i \in \operatorname{dom} d$ holds $n \mid d(i)$, then $n \mid \sum d$.

(6) Let d, e be finite 0-sequences of $\mathbb{N}$ and n be a natural number. Suppose $\operatorname{dom} d = \operatorname{dom} e$ and for every natural number i such that $i \in \operatorname{dom} d$ holds $e(i) = d(i) \bmod n$. Then $\sum d \bmod n = \sum e \bmod n$.

2. Representation of Numbers in the Base-b Numeral System

Let d be a finite 0-sequence of $\mathbb{N}$ and let b be a natural number. The functor $\operatorname{value}(d, b)$ yields a natural number and is defined by the condition (Def. 1).

(Def. 1) There exists a finite 0-sequence d' of $\mathbb{N}$ such that $\operatorname{dom} d' = \operatorname{dom} d$ and for every natural number i such that $i \in \operatorname{dom} d'$ holds $d'(i) = d(i) \cdot b^i$ and $\operatorname{value}(d, b) = \sum d'$.

Let n, b be natural numbers. Let us assume that $b > 1$. The functor $\operatorname{digits}(n, b)$ yields a finite 0-sequence of $\mathbb{N}$ and is defined as follows:

(Def. 2)(i) $\operatorname{value}(\operatorname{digits}(n, b), b) = n$ and $(\operatorname{digits}(n, b))(\operatorname{len} \operatorname{digits}(n, b) - 1) \neq 0$ and for every natural number i such that $i \in \operatorname{dom} \operatorname{digits}(n, b)$ holds $0 \leq (\operatorname{digits}(n, b))(i)$ and $(\operatorname{digits}(n, b))(i) < b$ if $n \neq 0$,

(ii) $\operatorname{digits}(n, b) = \langle 0 \rangle$, otherwise.

One can prove the following two propositions:

(7) For all natural numbers n, b such that $b > 1$ holds $\operatorname{len} \operatorname{digits}(n, b) \geq 1$.

(8) For all natural numbers n, b such that $b > 1$ holds $\operatorname{value}(\operatorname{digits}(n, b), b) = n$.

3. Selected Divisibility Criteria

One can prove the following propositions:

(9) For all natural numbers n, k such that $k = 10^n - 1$ holds $9 \mid k$.

(10) For all natural numbers n, b such that $b > 1$ holds $b \mid n$ iff $(\operatorname{digits}(n, b))(0) = 0$.

(11) For every natural number n holds $2 \mid n$ iff $2 \mid (\operatorname{digits}(n, 10))(0)$.

(12) For every natural number n holds $3 \mid n$ iff $3 \mid \sum \operatorname{digits}(n, 10)$.

(13) For every natural number n holds $5 \mid n$ iff $5 \mid (\operatorname{digits}(n, 10))(0)$.

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The Relevance of Measure and Probability, and Definition of Completeness of Probability

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Summary. In this article, we first discuss the relation between measure defined using extended real numbers and probability defined using real numbers. Further, we define completeness of probability, and its completion method, and also show that they coincide with those of measure.

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The articles [18], [20], [2], [3], [5], [1], [12], [15], [21], [8], [19], [17], [4], [9], [14], [23], [6], [11], [16], [22], [10], [7], and [13] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: n denotes a natural number, X denotes a set, A_1 denotes a sequence of subsets of X, S_1 denotes a σ-field of subsets of X, X_1 denotes a sequence of subsets of S_1, O_1 denotes a non empty set, S_2 denotes a σ-field of subsets of O_1, A_2 denotes a sequence of subsets of S_2, and P denotes a probability on S_2.

Let us consider X, S_1, X_1, n. Then $X_1(n)$ is an element of S_1.

Next we state two propositions:

(1) $\operatorname{rng} X_1 \subseteq S_1$.

(2) For every function f holds f is a sequence of subsets of S_1 iff f is a function from $\mathbb{N}$ into S_1.

The scheme *LambdaSigmaSSeq* deals with a set $\mathcal{A}$, a σ-field $\mathcal{B}$ of subsets of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

> There exists a sequence f of subsets of $\mathcal{B}$ such that for every n holds $f(n) = \mathcal{F}(n)$

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for all values of the parameters.

Let us consider X. Note that there exists a sequence of subsets of X which is disjoint valued.

Let us consider X, S_1. Note that there exists a sequence of subsets of S_1 which is disjoint valued.

One can prove the following propositions:

(3) For all subsets A, B of X there exists A_1 such that $A_1(0) = A$ and $A_1(1) = B$ and for every n such that $n > 1$ holds $A_1(n) = \emptyset$.

(4) Let A, B be subsets of X. Suppose A misses B and $A_1(0) = A$ and $A_1(1) = B$ and for every n such that $n > 1$ holds $A_1(n) = \emptyset$. Then A_1 is disjoint valued and $\bigcup A_1 = A \cup B$.

(5) Let S be a non empty set. Then S is a σ-field of subsets of X if and only if the following conditions are satisfied:

(i) $S \subseteq 2^X$,

(ii) for every sequence A_1 of subsets of X such that for every n holds $A_1(n) \in S$ holds $\bigcup A_1 \in S$, and

(iii) for every subset A of X such that $A \in S$ holds $A^{\rm c} \in S$.

(6) For all events A, B of S_2 holds $P(A \setminus B) = P(A \cup B) - P(B)$.

(7) For all events A, B of S_2 such that $A \subseteq B$ and $P(B) = 0$ holds $P(A) = 0$.

(8) For every n holds $P(A_2(n)) = 0$ iff $P(\bigcup A_2) = 0$.

(9) For every set A such that $A \in \operatorname{rng} A_2$ holds $P(A) = 0$ iff $P(\bigcup \operatorname{rng} A_2) = 0$.

(10) For every function s_1 from $\mathbb{N}$ into $\mathbb{R}$ and for every function E_1 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $s_1 = E_1$ holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}} = \operatorname{Ser} E_1$.

(11) Let s_1 be a function from $\mathbb{N}$ into $\mathbb{R}$ and E_1 be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. If $s_1 = E_1$ and s_1 is upper bounded, then $\sup s_1 = \sup \operatorname{rng} E_1$.

(12) Let s_1 be a function from $\mathbb{N}$ into $\mathbb{R}$ and E_1 be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. If $s_1 = E_1$ and s_1 is lower bounded, then $\inf s_1 = \inf \operatorname{rng} E_1$.

(13) Let s_1 be a function from $\mathbb{N}$ into $\mathbb{R}$ and E_1 be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. If $s_1 = E_1$ and s_1 is non-negative and summable, then $\sum s_1 = \sum E_1$.

(14) P is a σ-measure on S_2.

Let us consider O_1, S_2, P. The functor P2M P yields a σ-measure on S_2 and is defined as follows:

(Def. 1) P2M $P = P$.

One can prove the following proposition

(15) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. If $M(X) = \overline{\mathbb{R}}(1)$, then M is a probability on S.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. Let us assume that $M(X) = \overline{\mathbb{R}}(1)$. The functor M2P M yielding a probability on S is defined as follows:

(Def. 2) M2P $M = M$.

One can prove the following propositions:

(16) If A_1 is non-decreasing, then the partial unions of $A_1 = A_1$.

(17) Suppose A_1 is non-decreasing. Then (the partial diff-unions of A_1)$(0) = A_1(0)$ and for every n holds (the partial diff-unions of A_1)$(n+1) = A_1(n+1) \setminus A_1(n)$.

(18) If A_1 is non-decreasing, then for every n holds $A_1(n+1) =$ (the partial diff-unions of A_1)$(n+1) \cup A_1(n)$.

(19) If A_1 is non-decreasing, then for every n holds (the partial diff-unions of A_1)$(n+1)$ misses $A_1(n)$.

(20) If X_1 is non-decreasing, then the partial unions of $X_1 = X_1$.

(21) Suppose X_1 is non-decreasing. Then (the partial diff-unions of X_1)$(0) = X_1(0)$ and for every n holds (the partial diff-unions of X_1)$(n+1) = X_1(n+1) \setminus X_1(n)$.

(22) If X_1 is non-decreasing, then for every n holds (the partial diff-unions of X_1)$(n+1)$ misses $X_1(n)$.

Let us consider O_1, S_2, P. We say that P is complete on S_2 if and only if:

(Def. 3) For every subset A of O_1 and for every set B such that $B \in S_2$ holds if $A \subseteq B$ and $P(B) = 0$, then $A \in S_2$.

Next we state the proposition

(23) P is complete on S_2 iff P2M P is complete on S_2.

Let us consider O_1, S_2, P. A subset of O_1 is called a set with measure zero w.r.t. P if:

(Def. 4) There exists a set A such that $A \in S_2$ and it $\subseteq A$ and $P(A) = 0$.

We now state three propositions:

(24) Let Y be a subset of O_1. Then Y is a set with measure zero w.r.t. P if and only if Y is a set with measure zero w.r.t. P2M P.

(25) $\emptyset$ is a set with measure zero w.r.t. P.

(26) Let B_1, B_2 be sets. Suppose $B_1 \in S_2$ and $B_2 \in S_2$. Let C_1, C_2 be sets with measure zero w.r.t. P. If $B_1 \cup C_1 = B_2 \cup C_2$, then $P(B_1) = P(B_2)$.

Let us consider O_1, S_2, P. The functor $\mathrm{COM}(S_2, P)$ yields a non empty family of subsets of O_1 and is defined by the condition (Def. 5).

(Def. 5) Let A be a set. Then $A \in \mathrm{COM}(S_2, P)$ if and only if there exists a set B such that $B \in S_2$ and there exists a set C with measure zero w.r.t. P such that $A = B \cup C$.

Next we state two propositions:

(27) For every set C with measure zero w.r.t. P holds $C \in \mathrm{COM}(S_2, P)$.

(28) $\mathrm{COM}(S_2, P) = \mathrm{COM}(S_2, \mathrm{P2M}\, P)$.

Let us consider O_1, S_2, P and let A be an element of $\mathrm{COM}(S_2, P)$. The functor $\mathrm{P_{COM}2M_{COM}}\, A$ yields an element of $\mathrm{COM}(S_2, \mathrm{P2M}\, P)$ and is defined by:

(Def. 6) $\mathrm{P_{COM}2M_{COM}}\, A = A$.

Next we state the proposition

(29) $S_2 \subseteq \mathrm{COM}(S_2, P)$.

Let us consider O_1, S_2, P and let A be an element of $\mathrm{COM}(S_2, P)$. The functor $\mathrm{ProbPart}\, A$ yielding a non empty family of subsets of O_1 is defined by:

(Def. 7) For every set B holds $B \in \mathrm{ProbPart}\, A$ iff $B \in S_2$ and $B \subseteq A$ and $A \setminus B$ is a set with measure zero w.r.t. P.

We now state several propositions:

(30) For every element A of $\mathrm{COM}(S_2, P)$ holds $\mathrm{ProbPart}\, A = \mathrm{MeasPart}\, \mathrm{P_{COM}2M_{COM}}\, A$.

(31) For every element A of $\mathrm{COM}(S_2, P)$ and for all sets A_1, A_3 such that $A_1 \in \mathrm{ProbPart}\, A$ and $A_3 \in \mathrm{ProbPart}\, A$ holds $P(A_1) = P(A_3)$.

(32) For every function F from $\mathbb{N}$ into $\mathrm{COM}(S_2, P)$ there exists a sequence B_3 of subsets of S_2 such that for every n holds $B_3(n) \in \mathrm{ProbPart}\, F(n)$.

(33) Let F be a function from $\mathbb{N}$ into $\mathrm{COM}(S_2, P)$ and B_3 be a sequence of subsets of S_2. Then there exists a sequence C_3 of subsets of O_1 such that for every n holds $C_3(n) = F(n) \setminus B_3(n)$.

(34) Let B_3 be a sequence of subsets of O_1. Suppose that for every n holds $B_3(n)$ is a set with measure zero w.r.t. P. Then there exists a sequence C_3 of subsets of S_2 such that for every n holds $B_3(n) \subseteq C_3(n)$ and $P(C_3(n)) = 0$.

(35) Let D be a non empty family of subsets of O_1. Suppose that for every set A holds $A \in D$ iff there exists a set B such that $B \in S_2$ and there exists a set C with measure zero w.r.t. P such that $A = B \cup C$. Then D is a σ-field of subsets of O_1.

Let us consider O_1, S_2, P. Then $\mathrm{COM}(S_2, P)$ is a σ-field of subsets of O_1.

Let us consider O_1, S_2, P. We see that the set with measure zero w.r.t. P is an event of $\mathrm{COM}(S_2, P)$.

Next we state two propositions:

(36) For every set A holds $A \in \mathrm{COM}(S_2, P)$ iff there exist sets A_1, A_3 such that $A_1 \in S_2$ and $A_3 \in S_2$ and $A_1 \subseteq A$ and $A \subseteq A_3$ and $P(A_3 \setminus A_1) = 0$.

(37) Let C be a non empty family of subsets of O_1. Suppose that for every set A holds $A \in C$ iff there exist sets A_1, A_3 such that $A_1 \in S_2$ and $A_3 \in S_2$ and $A_1 \subseteq A$ and $A \subseteq A_3$ and $P(A_3 \setminus A_1) = 0$. Then $C = \mathrm{COM}(S_2, P)$.

Let us consider O_1, S_2, P. The functor $\mathrm{COM}(P)$ yields a probability on $\mathrm{COM}(S_2, P)$ and is defined as follows:

(Def. 8) For every set B such that $B \in S_2$ and for every set C with measure zero w.r.t. P holds $(\mathrm{COM}(P))(B \cup C) = P(B)$.

One can prove the following propositions:

(38) $\mathrm{COM}(P) = \mathrm{COM}(\mathrm{P2M}\,P)$.

(39) $\mathrm{COM}(P)$ is complete on $\mathrm{COM}(S_2, P)$.

(40) For every event A of S_2 holds $P(A) = (\mathrm{COM}(P))(A)$.

(41) For every set C with measure zero w.r.t. P holds $(\mathrm{COM}(P))(C) = 0$.

(42) For every element A of $\mathrm{COM}(S_2, P)$ and for every set B such that $B \in \mathrm{ProbPart}\,A$ holds $P(B) = (\mathrm{COM}(P))(A)$.

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