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On the Representation of Natural Numbers in Positional Numeral Systems[1]

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Summary. In this paper we show that every natural number can be uniquely represented as a base-b numeral. The formalization is based on the proof presented in [11]. We also prove selected divisibility criteria in the base-10 numeral system.

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The notation and terminology used in this paper have been introduced in the following articles: [13], [15], [2], [1], [17], [12], [14], [6], [4], [5], [8], [9], [10], [16], [7], and [3].

1. Preliminaries

One can prove the following propositions:

(1) For all finite 0-sequences d, e of $\mathbb{N}$ holds $\sum(d \frown e) = \sum d + \sum e$.

(2) Let S be a sequence of real numbers, d be a finite 0-sequence of $\mathbb{N}$, and n be a natural number. If $d = S\restriction(n+1)$, then $\sum d = (\sum_{\alpha=0}^{\kappa} S(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(3) For all natural numbers k, l, m holds $(k\,(l^{\kappa})_{\kappa\in\mathbb{N}})\restriction m$ is a finite 0-sequence of $\mathbb{N}$.

(4) Let d, e be finite 0-sequences of $\mathbb{N}$. Suppose $\operatorname{len} d \geq 1$ and $\operatorname{len} d = \operatorname{len} e$ and for every natural number i such that $i \in \operatorname{dom} d$ holds $d(i) \leq e(i)$. Then $\sum d \leq \sum e$.

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(5) Let d be a finite 0-sequence of $\mathbb{N}$ and n be a natural number. If for every natural number i such that $i \in \operatorname{dom} d$ holds $n \mid d(i)$, then $n \mid \sum d$.

(6) Let d, e be finite 0-sequences of $\mathbb{N}$ and n be a natural number. Suppose $\operatorname{dom} d = \operatorname{dom} e$ and for every natural number i such that $i \in \operatorname{dom} d$ holds $e(i) = d(i) \bmod n$. Then $\sum d \bmod n = \sum e \bmod n$.

2. Representation of Numbers in the Base-b Numeral System

Let d be a finite 0-sequence of $\mathbb{N}$ and let b be a natural number. The functor $\mathrm{value}(d, b)$ yields a natural number and is defined by the condition (Def. 1).

(Def. 1) There exists a finite 0-sequence d' of $\mathbb{N}$ such that $\operatorname{dom} d' = \operatorname{dom} d$ and for every natural number i such that $i \in \operatorname{dom} d'$ holds $d'(i) = d(i) \cdot b^i$ and $\mathrm{value}(d, b) = \sum d'$.

Let n, b be natural numbers. Let us assume that $b > 1$. The functor $\mathrm{digits}(n, b)$ yields a finite 0-sequence of $\mathbb{N}$ and is defined as follows:

(Def. 2)(i) $\mathrm{value}(\mathrm{digits}(n, b), b) = n$ and $(\mathrm{digits}(n, b))(\operatorname{len} \mathrm{digits}(n, b) - 1) \neq 0$ and for every natural number i such that $i \in \operatorname{dom} \mathrm{digits}(n, b)$ holds $0 \leq (\mathrm{digits}(n, b))(i)$ and $(\mathrm{digits}(n, b))(i) < b$ if $n \neq 0$,

(ii) $\mathrm{digits}(n, b) = \langle 0 \rangle$, otherwise.

One can prove the following two propositions:

(7) For all natural numbers n, b such that $b > 1$ holds $\operatorname{len} \mathrm{digits}(n, b) \geq 1$.

(8) For all natural numbers n, b such that $b > 1$ holds $\mathrm{value}(\mathrm{digits}(n, b), b) = n$.

3. Selected Divisibility Criteria

One can prove the following propositions:

(9) For all natural numbers n, k such that $k = 10^n - 1$ holds $9 \mid k$.

(10) For all natural numbers n, b such that $b > 1$ holds $b \mid n$ iff $(\mathrm{digits}(n, b))(0) = 0$.

(11) For every natural number n holds $2 \mid n$ iff $2 \mid (\mathrm{digits}(n, 10))(0)$.

(12) For every natural number n holds $3 \mid n$ iff $3 \mid \sum \mathrm{digits}(n, 10)$.

(13) For every natural number n holds $5 \mid n$ iff $5 \mid (\mathrm{digits}(n, 10))(0)$.

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