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Laplace Expansion

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Summary. In the article the formula for Laplace expansion is proved.

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The notation and terminology used in this paper are introduced in the following articles: [23], [11], [29], [20], [12], [30], [31], [6], [9], [7], [3], [4], [21], [28], [26], [15], [22], [10], [5], [13], [24], [14], [33], [25], [18], [34], [1], [8], [2], [16], [17], [27], [19], and [32].

1. Preliminaries

For simplicity, we follow the rules: x, y are sets, N is an element of $\mathbb{N}$, c, i, j, k, m, n are natural numbers, D is a non empty set, s is an element of $2\mathrm{Set}\,\mathrm{Seg}(n+2)$, p is an element of the permutations of n-element set, p_1, q_1 are elements of the permutations of $(n+1)$-element set, p_2 is an element of the permutations of $(n+2)$-element set, K is a field, a, b are elements of K, f is a finite sequence of elements of K, A is a matrix over K, A_1 is a matrix over D of dimension $n \times m$, p_3 is a finite sequence of elements of D, and M is a matrix over K of dimension n.

The following propositions are true:

(1) For every finite sequence f and for every natural number i such that $i \in \mathrm{dom}\, f$ holds $\mathrm{len}(f_{\restriction i}) = \mathrm{len}\, f -' 1$.

(2) Let i, j, n be natural numbers and M be a matrix over K of dimension n. If $i \in \mathrm{dom}\, M$, then len (the deleting of i-row and j-column in M) $= n -' 1$.

(3) If $j \in \mathrm{Seg}\,\mathrm{width}\, A$, then width (the deleting of j-column in A) $= \mathrm{width}\, A -' 1$.

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(4) For every natural number i such that $\operatorname{len} A > 1$ holds $\operatorname{width} A = \operatorname{width}$ (the deleting of i-row in A).

(5) For every natural number i such that $j \in \operatorname{Seg} \operatorname{width} M$ holds width (the deleting of i-row and j-column in M) $= n -' 1$.

Let G be a non empty groupoid, let B be a function from $[\!:$ the carrier of G, $\mathbb{N}\,:\!]$ into the carrier of G, let g be an element of G, and let i be a natural number. Then $B(g, i)$ is an element of G.

One can prove the following propositions:

(6) $\overline{\overline{\text{the permutations of } n\text{-element set}}} = n!$.

(7) For all i, j such that $i \in \operatorname{Seg}(n+1)$ and $j \in \operatorname{Seg}(n+1)$ holds $\overline{\overline{\{p_1 : p_1(i) = j\}}} = n!$.

(8) Let K be a Fanoian field, given p_2, and X, Y be elements of Fin 2Set Seg$(n+2)$. Suppose $Y = \{s : s \in X \wedge (\text{Part-sgn}(p_2, K))(s) = -\mathbf{1}_K\}$. Then (the multiplication of K)-$\sum_X$ Part-sgn$(p_2, K) = \operatorname{power}_K(-\mathbf{1}_K, \operatorname{card} Y)$.

(9) Let K be a Fanoian field and given p_2, i, j. Suppose $i \in \operatorname{Seg}(n+2)$ and $p_2(i) = j$. Then there exists an element X of Fin 2Set Seg$(n+2)$ such that $X = \{\{N, i\} : \{N, i\} \in \text{2Set Seg}(n+2)\}$ and (the multiplication of K)-$\sum_X$ Part-sgn$(p_2, K) = \operatorname{power}_K(-\mathbf{1}_K, i+j)$.

(10) Let given i, j. Suppose $i \in \operatorname{Seg}(n+1)$ and $j \in \operatorname{Seg}(n+1)$ and $n \geq 2$. Then there exists a function P_1 from 2Set Seg n into 2Set Seg$(n+1)$ such that

(i) $\operatorname{rng} P_1 = \text{2Set Seg}(n+1) \setminus \{\{N, i\} : \{N, i\} \in \text{2Set Seg}(n+1)\}$,

(ii) P_1 is one-to-one, and

(iii) for all k, m such that $k < m$ and $\{k, m\} \in$ 2Set Seg n holds if $m < i$ and $k < i$, then $P_1(\{k, m\}) = \{k, m\}$ and if $m \geq i$ and $k < i$, then $P_1(\{k, m\}) = \{k, m+1\}$ and if $m \geq i$ and $k \geq i$, then $P_1(\{k, m\}) = \{k+1, m+1\}$.

(11) If $n < 2$, then for every element p of the permutations of n-element set holds p is even and $p = \operatorname{idseq}(n)$.

(12) Let X, Y, D be non empty sets, f be a function from X into Fin Y, g be a function from Fin Y into D, and F be a binary operation on D. Suppose that

(i) for all elements A, B of Fin Y such that A misses B holds $F(g(A), g(B)) = g(A \cup B)$,

(ii) F is commutative and associative and has a unity, and

(iii) $g(\emptyset) = \mathbf{1}_F$.

Let I be an element of Fin X. Suppose that for all x, y such that $x \in I$ and $y \in I$ and $f(x)$ meets $f(y)$ holds $x = y$. Then F-$\sum_I g \cdot f = F$-$\sum_{f^\circ I} g$ and F-$\sum_{f^\circ I} g = g(\bigcup(f^\circ I))$ and $\bigcup(f^\circ I)$ is an element of Fin Y.

2. Auxiliary Notions

Let i, j, n be natural numbers, let us consider K, and let M be a matrix over K of dimension n. Let us assume that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$. The functor $\operatorname{Delete}(M, i, j)$ yielding a matrix over K of dimension $n -' 1$ is defined as follows:

(Def. 1) $\operatorname{Delete}(M, i, j) =$ the deleting of i-row and j-column in M.

The following propositions are true:

(13) Let given i, j. Suppose $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$. Let given k, m such that $k \in \operatorname{Seg}(n -' 1)$ and $m \in \operatorname{Seg}(n -' 1)$. Then

(i) if $k < i$ and $m < j$, then $(\operatorname{Delete}(M, i, j))_{k,m} = M_{k,m}$,

(ii) if $k < i$ and $m \geq j$, then $(\operatorname{Delete}(M, i, j))_{k,m} = M_{k,m+1}$,

(iii) if $k \geq i$ and $m < j$, then $(\operatorname{Delete}(M, i, j))_{k,m} = M_{k+1,m}$, and

(iv) if $k \geq i$ and $m \geq j$, then $(\operatorname{Delete}(M, i, j))_{k,m} = M_{k+1,m+1}$.

(14) For all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ holds $(\operatorname{Delete}(M, i, j))^{\mathrm{T}} = \operatorname{Delete}(M^{\mathrm{T}}, j, i)$.

(15) For every finite sequence f of elements of K and for all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ holds $\operatorname{Delete}(M, i, j) = \operatorname{Delete}(\operatorname{RLine}(M, i, f), i, j)$.

Let us consider c, n, m, D, let M be a matrix over D of dimension $n \times m$, and let p_3 be a finite sequence of elements of D. The functor $\operatorname{ReplaceCol}(M, c, p_3)$ yielding a matrix over D of dimension $n \times m$ is defined by:

(Def. 2)(i) $\operatorname{len} \operatorname{ReplaceCol}(M, c, p_3) = \operatorname{len} M$ and $\operatorname{width} \operatorname{ReplaceCol}(M, c, p_3) = \operatorname{width} M$ and for all i, j such that $\langle i, j \rangle \in$ the indices of M holds if $j \neq c$, then $(\operatorname{ReplaceCol}(M, c, p_3))_{i,j} = M_{i,j}$ and if $j = c$, then $(\operatorname{ReplaceCol}(M, c, p_3))_{i,c} = p_3(i)$ if $\operatorname{len} p_3 = \operatorname{len} M$,

(ii) $\operatorname{ReplaceCol}(M, c, p_3) = M$, otherwise.

Let us consider c, n, m, D, let M be a matrix over D of dimension $n \times m$, and let p_3 be a finite sequence of elements of D. We introduce $\operatorname{RCol}(M, c, p_3)$ as a synonym of $\operatorname{ReplaceCol}(M, c, p_3)$.

We now state four propositions:

(16) For every i such that $i \in \operatorname{Seg} \operatorname{width} A_1$ holds if $i = c$ and $\operatorname{len} p_3 = \operatorname{len} A_1$, then $(\operatorname{RCol}(A_1, c, p_3))_{\Box,i} = p_3$ and if $i \neq c$, then $(\operatorname{RCol}(A_1, c, p_3))_{\Box,i} = (A_1)_{\Box,i}$.

(17) If $c \notin \operatorname{Seg} \operatorname{width} A_1$, then $\operatorname{RCol}(A_1, c, p_3) = A_1$.

(18) $\operatorname{RCol}(A_1, c, (A_1)_{\Box,c}) = A_1$.

(19) Let A be a matrix over D of dimension $n \times m$ and A' be a matrix over D of dimension $m \times n$. If $A' = A^{\mathrm{T}}$ and if $m = 0$, then $n = 0$, then $\operatorname{ReplaceCol}(A, c, p_3) = (\operatorname{ReplaceLine}(A', c, p_3))^{\mathrm{T}}$.

3. Permutations

Let us consider i, n and let p_4 be an element of the permutations of $(n+1)$-element set. Let us assume that $i \in \operatorname{Seg}(n+1)$. The functor $\operatorname{Rem}(p_4, i)$ yielding an element of the permutations of n-element set is defined by the condition (Def. 3).

(Def. 3) Let given k such that $k \in \operatorname{Seg} n$. Then

(i) if $k < i$, then if $p_4(k) < p_4(i)$, then $(\operatorname{Rem}(p_4, i))(k) = p_4(k)$ and if $p_4(k) \geq p_4(i)$, then $(\operatorname{Rem}(p_4, i))(k) = p_4(k) - 1$, and

(ii) if $k \geq i$, then if $p_4(k+1) < p_4(i)$, then $(\operatorname{Rem}(p_4, i))(k) = p_4(k+1)$ and if $p_4(k+1) \geq p_4(i)$, then $(\operatorname{Rem}(p_4, i))(k) = p_4(k+1) - 1$.

One can prove the following three propositions:

(20) Let given i, j. Suppose $i \in \operatorname{Seg}(n+1)$ and $j \in \operatorname{Seg}(n+1)$. Let P be a set. Suppose $P = \{p_1 : p_1(i) = j\}$. Then there exists a function P_1 from P into the permutations of n-element set such that P_1 is bijective and for every q_1 such that $q_1(i) = j$ holds $P_1(q_1) = \operatorname{Rem}(q_1, i)$.

(21) For all i, j such that $i \in \operatorname{Seg}(n+1)$ and $p_1(i) = j$ holds $(-1)^{\operatorname{sgn}(p_1)}a = \operatorname{power}_K(-\mathbf{1}_K, i+j) \cdot (-1)^{\operatorname{sgn}(\operatorname{Rem}(p_1,i))}a$.

(22) Let given i, j. Suppose $i \in \operatorname{Seg}(n+1)$ and $p_1(i) = j$. Let M be a matrix over K of dimension $n+1$ and D_1 be a matrix over K of dimension n. Suppose $D_1 = \operatorname{Delete}(M, i, j)$. Then (the product on paths of M)$(p_1) = \operatorname{power}_K(-\mathbf{1}_K, i+j) \cdot M_{i,j} \cdot$ (the product on paths of D_1)$(\operatorname{Rem}(p_1, i))$.

4. Minors and Cofactors

Let i, j, n be natural numbers, let us consider K, and let M be a matrix over K of dimension n. The functor $\operatorname{Minor}(M, i, j)$ yielding an element of K is defined by:

(Def. 4) $\operatorname{Minor}(M, i, j) = \operatorname{Det} \operatorname{Delete}(M, i, j)$.

Let i, j, n be natural numbers, let us consider K, and let M be a matrix over K of dimension n. The functor $\operatorname{Cofactor}(M, i, j)$ yielding an element of K is defined as follows:

(Def. 5) $\operatorname{Cofactor}(M, i, j) = \operatorname{power}_K(-\mathbf{1}_K, i+j) \cdot \operatorname{Minor}(M, i, j)$.

The following propositions are true:

(23) Let given i, j. Suppose $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$. Let P be an element of Fin (the permutations of n-element set). Suppose $P = \{p : p(i) = j\}$. Let M be a matrix over K of dimension n. Then (the addition of K)-$\sum_P$ (the product on paths of M) $= M_{i,j} \cdot \operatorname{Cofactor}(M, i, j)$.

(24) For all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ holds $\operatorname{Minor}(M, i, j) = \operatorname{Minor}(M^{\mathrm{T}}, j, i)$.

Let us consider n, K and let M be a matrix over K of dimension n. The matrix of cofactor M yielding a matrix over K of dimension n is defined by the condition (Def. 6).

(Def. 6) Let i, j be natural numbers. Suppose $\langle i, j\rangle \in$ the indices of the matrix of cofactor M. Then (the matrix of cofactor $M)_{i,j} = \mathrm{Cofactor}(M, i, j)$.

5. Laplace Expansion

Let us consider n, i, K and let M be a matrix over K of dimension n. The functor $\mathrm{LaplaceExpL}(M, i)$ yields a finite sequence of elements of K and is defined as follows:

(Def. 7) $\mathrm{len}\,\mathrm{LaplaceExpL}(M, i) = n$ and for every j such that $j \in \mathrm{dom}\,\mathrm{LaplaceExpL}(M, i)$ holds $(\mathrm{LaplaceExpL}(M, i))(j) = M_{i,j} \cdot \mathrm{Cofactor}(M, i, j)$.

Let us consider n, let j be a natural number, let us consider K, and let M be a matrix over K of dimension n. The functor $\mathrm{LaplaceExpC}(M, j)$ yields a finite sequence of elements of K and is defined by:

(Def. 8) $\mathrm{len}\,\mathrm{LaplaceExpC}(M, j) = n$ and for every i such that $i \in \mathrm{dom}\,\mathrm{LaplaceExpC}(M, j)$ holds $(\mathrm{LaplaceExpC}(M, j))(i) = M_{i,j} \cdot \mathrm{Cofactor}(M, i, j)$.

One can prove the following propositions:

(25) For every natural number i and for every matrix M over K of dimension n such that $i \in \mathrm{Seg}\,n$ holds $\mathrm{Det}\,M = \sum \mathrm{LaplaceExpL}(M, i)$.

(26) For every i such that $i \in \mathrm{Seg}\,n$ holds $\mathrm{LaplaceExpC}(M, i) = \mathrm{LaplaceExpL}(M^{\mathrm{T}}, i)$.

(27) For every natural number j and for every matrix M over K of dimension n such that $j \in \mathrm{Seg}\,n$ holds $\mathrm{Det}\,M = \sum \mathrm{LaplaceExpC}(M, j)$.

(28) For all p, i such that $\mathrm{len}\,f = n$ and $i \in \mathrm{Seg}\,n$ holds Line(the matrix of cofactor M, $i) \bullet f = \mathrm{LaplaceExpL}(\mathrm{RLine}(M, i, f), i)$.

(29) If $i \in \mathrm{Seg}\,n$, then $\mathrm{Line}(M, j) \cdot$ ((the matrix of cofactor $M)^{\mathrm{T}})_{\square,i} = \mathrm{Det}\,\mathrm{RLine}(M, i, \mathrm{Line}(M, j))$.

(30) If $\mathrm{Det}\,M \neq 0_K$, then $M \cdot (\mathrm{Det}\,M^{-1} \cdot$ (the matrix of cofactor $M)^{\mathrm{T}}) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}^{n\times n}_{K}$.

(31) For all f, i such that $\mathrm{len}\,f = n$ and $i \in \mathrm{Seg}\,n$ holds (the matrix of cofactor $M)_{\square,i} \bullet f = \mathrm{LaplaceExpL}(\mathrm{RLine}(M^{\mathrm{T}}, i, f), i)$.

(32) If $i \in \mathrm{Seg}\,n$ and $j \in \mathrm{Seg}\,n$, then Line((the matrix of cofactor $M)^{\mathrm{T}}, i) \cdot M_{\square,j} = \mathrm{Det}\,\mathrm{RLine}(M^{\mathrm{T}}, i, \mathrm{Line}(M^{\mathrm{T}}, j))$.

(33) If $\operatorname{Det} M \neq 0_K$, then $\operatorname{Det} M^{-1} \cdot (\text{the matrix of cofactor } M)^{\mathrm{T}} \cdot M = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

(34) M is invertible iff $\operatorname{Det} M \neq 0_K$.

(35) If $\operatorname{Det} M \neq 0_K$, then $M^{\smile} = \operatorname{Det} M^{-1} \cdot (\text{the matrix of cofactor } M)^{\mathrm{T}}$.

(36) Let M be a matrix over K of dimension n. Suppose M is invertible. Let given i, j. If $\langle i, j \rangle \in$ the indices of $M^{\smile}$, then $M^{\smile}{}_{i,j} = \operatorname{Det} M^{-1} \cdot \operatorname{power}_K(-\mathbf{1}_K, i+j) \cdot \operatorname{Minor}(M, j, i)$.

(37) Let A be a matrix over K of dimension n. Suppose $\operatorname{Det} A \neq 0_K$. Let x, b be matrices over K. Suppose $\operatorname{len} x = n$ and $A \cdot x = b$. Then $x = A^{\smile} \cdot b$ and for all i, j such that $\langle i, j \rangle \in$ the indices of x holds $x_{i,j} = \operatorname{Det} A^{-1} \cdot \operatorname{Det} \operatorname{ReplaceCol}(A, i, b_{\Box,j})$.

(38) Let A be a matrix over K of dimension n. Suppose $\operatorname{Det} A \neq 0_K$. Let x, b be matrices over K. Suppose $\operatorname{width} x = n$ and $x \cdot A = b$. Then $x = b \cdot A^{\smile}$ and for all i, j such that $\langle i, j \rangle \in$ the indices of x holds $x_{i,j} = \operatorname{Det} A^{-1} \cdot \operatorname{Det} \operatorname{ReplaceLine}(A, j, \operatorname{Line}(b, i))$.

6. Product by a Vector

Let D be a non empty set and let f be a finite sequence of elements of D. Then $\langle f \rangle$ is a matrix over D of dimension $1 \times \operatorname{len} f$.

Let us consider K, let M be a matrix over K, and let f be a finite sequence of elements of K. The functor $M \cdot f$ yielding a matrix over K is defined by:

(Def. 9) $M \cdot f = M \cdot \langle f \rangle^{\mathrm{T}}$.

The functor $f \cdot M$ yields a matrix over K and is defined by:

(Def. 10) $f \cdot M = \langle f \rangle \cdot M$.

Next we state two propositions:

(39) Let A be a matrix over K of dimension n. Suppose $\operatorname{Det} A \neq 0_K$. Let x, b be finite sequences of elements of K. Suppose $\operatorname{len} x = n$ and $A \cdot x = \langle b \rangle^{\mathrm{T}}$. Then $\langle x \rangle^{\mathrm{T}} = A^{\smile} \cdot b$ and for every i such that $i \in \operatorname{Seg} n$ holds $x(i) = \operatorname{Det} A^{-1} \cdot \operatorname{Det} \operatorname{ReplaceCol}(A, i, b)$.

(40) Let A be a matrix over K of dimension n. Suppose $\operatorname{Det} A \neq 0_K$. Let x, b be finite sequences of elements of K. Suppose $\operatorname{len} x = n$ and $x \cdot A = \langle b \rangle$. Then $\langle x \rangle = b \cdot A^{\smile}$ and for every i such that $i \in \operatorname{Seg} n$ holds $x(i) = \operatorname{Det} A^{-1} \cdot \operatorname{Det} \operatorname{ReplaceLine}(A, i, b)$.

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