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Alexandroff One Point Compactification

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Summary. In the article, I introduce the notions of the compactification of topological spaces and the Alexandroff one point compactification. Some properties of the locally compact spaces and one point compactification are proved.

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The articles [15], [5], [16], [17], [4], [18], [1], [8], [14], [13], [19], [7], [9], [10], [6], [12], [2], [3], and [11] provide the notation and terminology for this paper.

Let X be a topological space and let P be a family of subsets of X. We say that P is compact if and only if:

(Def. 1) For every subset U of X such that $U \in P$ holds U is compact.

Let X be a topological space and let U be a subset of X. We say that U is relatively-compact if and only if:

(Def. 2) $\overline{U}$ is compact.

Let X be a topological space. Note that $\emptyset_X$ is relatively-compact.

Let X be a topological space. Observe that there exists a subset of X which is relatively-compact.

Let X be a topological space and let U be a relatively-compact subset of X. Observe that $\overline{U}$ is compact.

Let X be a topological space and let U be a subset of X. We introduce U is pre-compact as a synonym of U is relatively-compact.

Let X be a non empty topological space. We introduce X is liminally-compact as a synonym of X is locally-compact.

Let X be a non empty topological space. Let us observe that X is liminally-compact if and only if:

(Def. 3) For every point x of X holds there exists a generalized basis of x which is compact.

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Let X be a non empty topological space. We say that X is locally-relatively-compact if and only if:

(Def. 4) For every point x of X holds there exists a neighbourhood of x which is relatively-compact.

Let X be a non empty topological space. We say that X is locally-closed/compact if and only if:

(Def. 5) For every point x of X holds there exists a neighbourhood of x which is closed and compact.

Let X be a non empty topological space. We say that X is locally-compact if and only if:

(Def. 6) For every point x of X holds there exists a neighbourhood of x which is compact.

Let us observe that every non empty topological space which is liminally-compact is also locally-compact.

Let us note that every non empty T_3 topological space which is locally-compact is also liminally-compact.

One can verify that every non empty topological space which is locally-relatively-compact is also locally-closed/compact.

Let us observe that every non empty topological space which is locally-closed/compact is also locally-relatively-compact.

Let us observe that every non empty topological space which is locally-relatively-compact is also locally-compact.

One can verify that every non empty Hausdorff topological space which is locally-compact is also locally-relatively-compact.

One can check that every non empty topological space which is compact is also locally-compact.

Let us observe that every non empty topological space which is discrete is also locally-compact.

Let us mention that there exists a topological space which is discrete and non empty.

Let X be a locally-compact non empty topological space and let C be a closed non empty subset of X. Note that $X \restriction C$ is locally-compact.

Let X be a locally-compact non empty T_3 topological space and let P be an open non empty subset of X. Note that $X \restriction P$ is locally-compact.

One can prove the following two propositions:

(1) Let X be a Hausdorff non empty topological space and E be a non empty subset of X. If $X \restriction E$ is dense and locally-compact, then $X \restriction E$ is open.

(2) For all topological spaces X, Y and for every subset A of X such that $\Omega_X \subseteq \Omega_Y$ holds $(\mathrm{incl}(X, Y))^\circ A = A$.

Let X, Y be topological spaces and let f be a function from X into Y. We say that f is embedding if and only if:

(Def. 7) There exists a function h from X into $Y \restriction \operatorname{rng} f$ such that $h = f$ and h is a homeomorphism.

The following proposition is true

(3) Let X, Y be topological spaces. Suppose $\Omega_X \subseteq \Omega_Y$ and there exists a subset X_1 of Y such that $X_1 = \Omega_X$ and the topology of $Y \restriction X_1 =$ the topology of X. Then $\operatorname{incl}(X, Y)$ is embedding.

Let X be a topological space, let Y be a topological space, and let h be a function from X into Y. We say that h is compactification if and only if:

(Def. 8) h is embedding and Y is compact and $h^\circ(\Omega_X)$ is dense.

Let X be a topological space and let Y be a topological space. Note that every function from X into Y which is compactification is also embedding.

Let X be a topological structure. The one-point compactification of X yields a strict topological structure and is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of the one-point compactification of $X = \operatorname{succ}(\Omega_X)$, and

(ii) the topology of the one-point compactification of $X =$ (the topology of $X) \cup \{U \cup \{\Omega_X\}; U$ ranges over subsets of X: U is open $\wedge$ $U^{\rm c}$ is compact$\}$.

Let X be a topological structure. Note that the one-point compactification of X is non empty.

We now state the proposition

(4) For every topological structure X holds $\Omega_X \subseteq \Omega_{\text{the one-point compactification of } X}$.

Let X be a topological space. Note that the one-point compactification of X is topological space-like.

Next we state the proposition

(5) Every topological structure X is a subspace of the one-point compactification of X.

Let X be a topological space. One can verify that the one-point compactification of X is compact.

One can prove the following propositions:

(6) Let X be a non empty topological space. Then X is Hausdorff and locally-compact if and only if the one-point compactification of X is Hausdorff.

(7) Let X be a non empty topological space. Then X is non compact if and only if there exists a subset X' of the one-point compactification of X such that $X' = \Omega_X$ and X' is dense.

(8) Let X be a non empty topological space. Suppose X is non compact. Then $\operatorname{incl}(X, \text{the one-point compactification of } X)$ is compactification.

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